Around some extensions of Casas-Alvero conjecture for non-polynomial functions

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Abstract. We show that two natural extensions of the real Casas-Alvero conjecture in the non-polynomial setting do not hold.

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1 Introduction and main results

The Casas-Alvero conjecture affirms that if a complex polynomial P of degree $n>1$ shares roots with all its derivatives, $P^{(k)}, k=1,2\ldots,n-1$, then there exist two complex numbers, a and $b\neq 0$, such that $P(z)=b(z-a)^n$. Notice that, in principle, the common root between P and each $P^{(k)}$ might depend on k. Casas-Alvero arrived to this problem at the turn of this century, when he was working in his paper [1] trying to obtain an irreducibility criterion for two variable power series with complex coefficients. See [2] for an explanation of the problem in his own words.

Although several authors have got partial answers, to the best of our knowledge the conjecture remains open. For $n\leq 4$ the conjecture is a simple consequence of the wonderful Gauss-Lucas Theorem ([6]). In 2006 it was proved in [5], by using Maple, that it is true for $n\leq 8$. Afterwards in [6, 7] it was proved that it holds when n is $p^m, 2p^m, 3p^m$ or $4p^m$, for some prime number p and $m\in\mathbb{N}$. The first cases left open are those where $n=24, 28$ or 30. See again [6] for a very interesting survey or [3, 8] for some recent contributions on this question.

Adding the hypotheses that P is a real polynomial and all its n roots, taking into account their multiplicities, are real, the conjecture has a real counterpart, that also remains open. It says that $P(x)=b(x-a)^n$ for some real numbers a and $b\neq 0$. For this real case, the conjecture can be proved easily for $n\leq 4$, simply by using Rolle's Theorem. This tool does not suffice for $n\geq 5$, see for instance [4] for more details, or next section.

Also in the real case, in [6] it is proved that if the condition for one of the derivatives of P is removed, then there exist polynomials satisfying the remaining $n-2$ conditions, different from $b(x-a)^n$. The construction of some of these polynomials presented in that paper is very nice and is a consequence of the Brouwer's fixed point Theorem in a suitable context.