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Non-existence, existence, and uniqueness of limit cycles for a generalization of the Van der Pol–Duffing and the Rayleigh–Duffing oscillators

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ABSTRACT

We study the limit cycles of some cubic family of differential equations, containing the well-known Van der Pol–Duffing and Rayleigh–Duffing oscillators. In particular, we characterize for the class of differential systems here studied the non-existence, existence and uniqueness of limit cycles. Moreover we provide their global phase portraits in the Poincaré disk.

1. Introduction and statement of the main results

In this paper we consider the planar system

$$x' = y, \quad y' = -a_1x - a_2x^3 + \mu(a_3 + a_4x^2 + a_5y^2)y, \tag{1}$$

where $a_1, a_2 > 0$, $a_3, a_4, a_5 \in \mathbb{R}$, $a_3 \neq 0$, $\mu > 0$ is a sufficiently small parameter, $a_5 < 0$ and the prime denotes derivative with respect to the time t.

The Van der Pol oscillator was discovered by engineer and physicist Balthasar Van der Pol while working at Philips company. Van der Pol [1] found in circuits that employ vacuum valves stable oscillations, which are now known as limit cycles. Van der Pol also found that at certain frequencies, some irregular noise appears near the coupling frequencies. It will be one of the first experimental discoveries of Chaos Theory [2,3]. The Van der Pol equation has a long history not only in physics but also in biology. Thus in biology FitzHugh [4] and Nagumo [5] applied the equation to a two-dimensional field in the FitzHugh–Nagumo model, as it is known now, to describe the potential action of neurons. It can also be used in seismology to model the behavior of plates in a failure [6]. We note that the Rayleigh equation introduced by Rayleigh in 1875 in his published book *The Theory of Sound* [7], where he has shown the first physical phenomenon modeled by a limit cycle. In fact the Rayleigh equation is more general than the Van der Pol equation. The Duffing equation introduced in [8] essentially adds a term x^3 to the Rayleigh equation.

The differential systems (1) contain as particular cases the so-called Van der Pol–Duffing oscillator and Rayleigh–Duffing oscillator both with positive linear damping and sufficiently small stiffness. Both models have been studied intensively because they are two essential oscillators in the nonlinear dynamical systems, see for instance the books of [5,9] and the references cited therein. Many researchers have investigated the existence of limit cycles for autonomous nonlinear systems depending on parameters and in special for system (1) due to the fact that they can be a mechanism for the creation of chaos, see [10,11]. This problem is also related with the well-known 16th Hilbert problem which asks for the number of limit cycles in polynomial differential equations in function of their degree, see for instance [12–14].

The main objective of this paper is to characterize the dynamics of the differential system (3) in an easy way using the qualitative theory of differential equations, and in particular we provide the non-existence, existence and uniqueness of limit cycles for the differential equation (3). Consequently we provide a new unified proof in the study of the limit cycles of the Van der Pol–Duffing equation and of the Rayleigh–Duffing equation.

The differential equation (1) when $a_5 = 0$ becomes a subclass of the classical polynomial Liénard differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{2}$$

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