ON THE AVERAGING THEORY FOR COMPUTING PERIODIC SOLUTIONS AND ITS APPLICATIONS

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Abstract. In this work we extend the averaging theory for computing periodic solutions of nonlinear T–periodic differential systems depending on a small parameter to some cases when the Jacobian of the averaged function is zero.

As an application we use the classical averaging theory and this new extension for studying the periodic solutions bifurcating from the zero–Hopf equilibrium points of three families of autonomous differential systems in $\mathbb{R}^3$.

1. Introduction and statement of the main result

For a general introduction to the averaging theory see the book of Sanders, Verhulst and Murdock [4], and the references quoted there. The averaging method is a classical theory for studying the solutions of the nonlinear dynamical systems, and in particular their periodic solutions. Following to Verhulst [5] the classical averaging method for computing periodic solutions can be summarized as follows.

We consider the initial value problem

$$\dot{\mathbf{x}} = \varepsilon \mathbf{F}_1(t, \mathbf{x}) + \varepsilon^2 \mathbf{F}_2(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{1}$$

and

$$\dot{\mathbf{y}} = \varepsilon f^0(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{2}$$

with $\mathbf{x}$, $\mathbf{y}$, and $\boldsymbol{x_0}$ in some open Ω of $\mathbb{R}^n$, $t \in [0, \infty)$, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. We assume $\mathbf{F}_1$, $\mathbf{F}_2$ are T–periodic in the variable t, and define the averaged function of $\mathbf{F}_1$ as usual

$$f^0(\mathbf{y}) = \frac{1}{T} \int_0^T \mathbf{F}_1(t, \mathbf{y}) dt. \tag{3}$$

Theorem 1. *Assume that $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{D_x F}_1$, $\mathbf{D_{xx} F}_1$ and $\mathbf{D_x F}_2$ are continuous and bounded by a constant M independent of ε in $[0, \infty) \times \Omega \times [-\varepsilon_0, \varepsilon_0]$, and that $\mathbf{y}(t) \in \Omega$ for $t \in [0, 1/|\varepsilon|]$. Then the following statements hold.*

(a) *For $t \in [0, 1/|\varepsilon|]$ we have $\mathbf{x}(t) - \mathbf{y}(t) = O(\varepsilon)$ as $\varepsilon \to 0$.*

(b) *If $\boldsymbol{s}$ is a singular point of system (2) and $\det \big(D_{\mathbf{y}} f^0(\boldsymbol{s})\big) \neq 0$, then there exists a T–periodic solution $\mathbf{x}(t, \varepsilon)$ for system (1) such that $\mathbf{x}(0, \varepsilon) \to \boldsymbol{s}$ as $\varepsilon \to 0$.*

(c) *If the singular point $\boldsymbol{s}$ is hyperbolic, then the stability of the periodic solution $\mathbf{x}(t, \varepsilon)$ is given by the stability of the singular point $\boldsymbol{s}$.*

For a proof of Theorem 1, see for instance Theorems 11.1, 11.5 and 11.6 of [5], where it is stated for $\varepsilon \in [0, \varepsilon_0)$ but in fact following the proof the same result works for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, as it is stated here.

We want to study the case when the hypothesis of statement (b), i.e. $\det \big(D_{\mathbf{y}} f^0(\boldsymbol{s})\big) \neq 0$, does not hold, as it is the case in several applications.

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