CENTERS AND LIMIT CYCLES OF VECTOR FIELDS DEFINED ON INVARIANT SPHERES

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ABSTRACT. The aim of this paper is the study of the center-focus and cyclicity problems inside the class $\mathfrak{X}$ of 3-dimensional vector fields that admit a first integral that leaves invariant any sphere centered at the origin. We classify the centers of linear, quadratic homogeneous and a family of quadratic vector fields $\mathcal{F} \subset \mathfrak{X}$, restricted to one of these spheres. Moreover, we show the existence of at least 4 limit cycles in family $\mathcal{F}$.

1. INTRODUCTION

Differential equations and dynamical systems appear naturally in the description of many phenomena for which local processes are known. The central problem is then to obtain global information on these phenomena. Once the local equations are formulated in a particular context, the next usual step is to solve them. But, as in general, the evolution of these process is governed by nonlinear differential equations, it is not always simple to solve them. The basic idea behind the first works in the 18th and 19th centuries was to seek solutions that are combinations of known functions. That is why it is imperative to search for new more geometric methods for a better understanding of the behavior of the solutions of a system of differential equations. Integrability is one of them.

The integrability is an intrinsic property of a given system that imposes strong constraints on the way solutions evolve in phase space. The notion of integrability was introduced to describe the property of equations for which all local and global information can be obtained either explicitly from the solutions or implicitly from invariants. The first class of invariants is the constants of motion, conserved quantities, or first integrals. Of course, there are also other invariants such as integral invariants, integrating factors, Jacobi multipliers, or symmetries which give rise to different techniques for integrating differential equations, see for instance [1, 4, 12, 19] and references therein. We have been motivated to consider the existence of first integrals.

The importance of the existence of a non-constant first integral lies in the fact that the trajectories of the vector field remain in the level sets of the function that defines the first integral, and hence this is a strong constraint on the dynamical behavior. In the theory of ordinary differential equations, the existence of first integrals is important not only because they allow decreasing the dimension where the differential system is defined but also because they simplify the characterization of the phase portrait. It is important to mention that if we are working in a space of dimension n and the system of equations has $n-1$ independent first integrals

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