THE LOCAL PERIOD FUNCTION FOR HAMILTONIAN SYSTEMS WITH APPLICATIONS.

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Abstract. In the first part of the paper we develop a constructive procedure to obtain the Taylor expansion, in terms of the energy, of the period function for a non-degenerated center of any planar analytic Hamiltonian system. We apply it to several examples, including the whirling pendulum and a cubic Hamiltonian system. The knowledge of this Taylor expansion of the period function for this system is one of the key points to study the number of zeroes of an Abelian integral that controls the number of limit cycles bifurcating from the periodic orbits of a planar Hamiltonian system that is inspired by a physical model on capillarity. Several other classical tools, like for instance Chebyshev systems are applied to study this number of zeroes. The approach introduced can also be applied in other situations.

1. Introduction and main results

Let γ_s, with $s \in I \in \mathbb{R}$, be a parameterized continua of periodic orbits of a planar autonomous differential system. In general, I is either an open interval or an interval of the form $[s_0, s_1)$. The function that assigns to each s the minimal period of γ_s is called *period function* and it is denoted by $T(s)$. Similarly, the function that assigns to each γ_s the area surrounded by this closed curve is denoted by $A(s)$ and called *area function.* The period function is important to study theoretical properties of planar ordinary differential equations and their perturbations, see for instance [9, pp. 369-370]; to understand some mathematical models in physics or ecology, see [14, 17, 39, 45] and the references therein; in the description of the dynamics of some discrete dynamical systems, see [6, 11, 12]; or for counting the solutions of some boundary value problems, see [7, 8]. When the system is Hamiltonian, with Hamiltonian function H and $\gamma_h \subset \{H = h\}$, it is natural to consider $s = h$ and write $T = T(h)$.

Given a planar analytic Hamiltonian system

$$\dot{x} = -H_y(x, y), \qquad \dot{y} = H_x(x, y), \tag{1}$$

with a non-degenerated center at the origin (that without loss of generality we will associated to $h = 0$ and then $I = [0, h_1) \subset \mathbb{R}$) it is known that $T(h)$, in a neighborhood of $h = 0$, is an analytic function of the energy h and it is given by the derivative with respect h of the area function $A(h)$, see [33]. There are several authors that compute the Taylor series of T at $h = 0$ for particular Hamiltonian systems but, to the best of our knowledge, most examples deal with Hamiltonian functions with separated variables $H(x, y) = F(x) + G(y)$, see for instance [5, 16] and their references. Our first result provides a systematic constructive approach for finding this Taylor series up to any order for any Hamiltonian system.

Theorem 1.1. *Let H be an analytic function with $H(0, 0) = 0$ and assume that the Hamiltonian system* (1) *has a non-degenerate center at the origin. Then:*

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