

LIMIT CYCLES FOR SOME FAMILIES OF SMOOTH AND NON-SMOOTH PLANAR SYSTEMS

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ABSTRACT. In this paper, we apply the averaging method via Brouwer degree in a class of planar systems given by a linear center perturbed by a sum of continuous homogeneous vector fields, to study lower bounds for their number of limit cycles. Our results can be applied to models where the smoothness is lost on the set $\Sigma = \{xy = 0\}$. We also apply them to present a variant of Hilbert 16th problem, where the goal is to bound the number of limit cycles in terms of the number of monomials of a family of polynomial vector fields, instead of doing this in terms of their degrees.

1. INTRODUCTION

A limit cycle is a periodic orbit isolated in the set of all periodic orbits in a differential system. The existence of limit cycles became important in the applications to the real world, because many phenomena are related with their existence, see for instance the Van der Pol oscillator [27, 28]. One of the useful tools to detect such objects is the averaging theory. We refer to the book of Sanders and Verhulst [25] and to the book of Verhulst [29] for an introduction of this subject. Buica and Llibre in [5], generalized the averaging theory for studying periodic solutions of continuous differential systems using mainly the Brouwer degree.

The theory of piecewise smooth differential system has been developing very fast and it has become certainly an important common frontier between Mathematics, Physics and Engineering for example. In many works on piecewise smooth differential system the set Σ , where the systems lose smoothness, is a regular manifold. But a few years ago it was increasing the study of the case where Σ can be the union of regular manifolds, which includes, the case when Σ is not regular, but it is an algebraic manifold. See for instance Panazzolo and Da Silva in [21]. Also there are works that deal with the search of limit cycles of discontinuous systems with Σ being an algebraic manifold, see for instance [16] and [19].

In this work we give some lower bounds for the number of limit cycles in some classes of continuous, non necessarily locally Lipschitz, piecewise smooth differential systems with $\Sigma = \{xy = 0\}$. The main technique will be the averaging theory via Brouwer degree developed in [5, 6].

In Section 2 we explain some of the problems that have motivated our study. They include systems that model the capillary rise, some population models and also some type of SIR models. All of them have in common that can be written as differential equations of the form

$$\dot{x} = f(x, y, \sqrt{x}, \sqrt{y}), \quad \dot{y} = g(x, y, \sqrt{x}, \sqrt{y}), \quad (1)$$

with f and g smooth or polynomial functions. Extending the function \sqrt{u} as $\text{sgn}(u)\sqrt{|u|}$ these systems can be considered in the full plane but they are non-smooth on the set

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$\Sigma = \{xy = 0\}$. Clearly, they fall into the category of systems described above. Notice also that on Σ , the corresponding vector fields are not Lipschitz functions.

In fact these systems could also be treated by introducing new variables u and v such that $u^2 = x$ and $v^2 = y$ and changing the time, but as we will see, our approach can be applied directly to the original system and can also be applied to more general systems involving simultaneously more non-differentiable functions. For instance functions like $\sqrt[k]{x}$, for different values of k , also fall in our point of view.

Recall that a continuous vector field $X(x, y)$ is called homogeneous with degree of homogeneity α , where $0 \leq \alpha \in \mathbb{R}$, if $X(rx, ry) = r^\alpha X(x, y)$ for all $(x, y) \in \mathbb{R}^2$ and all $0 \leq r \in \mathbb{R}$. For convenience we will write it as $X(x, y) = (f(x, y), g(x, y))$ instead of the more usual way $X(x, y) = f(x, y)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y}$. When $\alpha < 1$ this vector field is continuous but not Lipschitz. Its associated planar system of differential equations is $(\dot{x}, \dot{y}) = X(x, y)$, or equivalently, $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$. We prove:

Theorem 1.1. *Consider the class $\mathcal{F}_{\mathbf{a}}$ of planar vector fields*

$$X(x, y) = (-y, x) + \sum_{j=0}^n a_j X_j(x, y), \quad \mathbf{a} = (a_0, a_1, \dots, a_n) \in \mathbb{R}^{n+1}, \quad (2)$$

where for each j , $X_j = (f_j, g_j)$ is a fixed continuous homogeneous vector field with degree of homogeneity $0 \leq \alpha_j \in \mathbb{R}$ and $\alpha_0 < \alpha_1 < \dots < \alpha_n$. There exist values of \mathbf{a} such that the differential equation associated to X has at least m limit cycles, where $m + 1$ is the number of non-zero values among

$$I_j = \int_0^{2\pi} (f_j(\cos \theta, \sin \theta) \cos \theta + g_j(\cos \theta, \sin \theta) \sin \theta) d\theta, \quad j = 0, 1, \dots, n.$$

Moreover, if all the vector fields X_j are of class \mathcal{C}^1 , the m limit cycles obtained above are hyperbolic.

The proof of Theorem 1.1 is based on the averaging first order results for continuous differential equations via Browuer's degree given in [5, 6]. This result extends some of the results of [10] to the non-smooth case.

Notice that some simple examples of non-smooth X_j where our approach can be used are for instance

$$X_j(x, y) = (a_j \operatorname{sgn}(x)|x|^{\alpha_j} + b_j \operatorname{sgn}(y)|y|^{\alpha_j}, c_j \operatorname{sgn}(x)|x|^{\alpha_j} + d_j \operatorname{sgn}(y)|y|^{\alpha_j}),$$

where $0 < \alpha_j < 1$. They clearly include our goal functions.

The second part of the paper deals with polynomial vector fields. Recall that the second part of the Hilbert's 16th problem asks about the maximum number of limit cycles for planar polynomial vector fields in terms of their degrees. Usually, the maximum number of limit cycles of vector fields of degree n , is denoted as $\mathcal{H}(n)$ (admitting, in principle that this number could be infinity) and it is called Hilbert number. To prove its finiteness, and to know it, is one of the most famous and difficult open problems in mathematics, see [12, 26]. It is known that $\mathcal{H}(1) = 0$, $\mathcal{H}(2) \geq 4$, $\mathcal{H}(3) \geq 13$, see [24] for more lower bounds for small n and other related references. It is also known that there is a sequence of values n going to infinity such that $\mathcal{H}(n) \geq M(n)$ where $M(n) = \left(\frac{n^2 \log(n)}{2 \log 2}\right)(1 + o(1))$, see for instance [2] and their references. To the best of our knowledge the first result proving the existence of a lower bound of type $O(n^2 \log(n))$ for $\mathcal{H}(n)$ is due to Christopher and Lloyd ([9]).

From the statement of Theorem 1.1 we start to think into a different version of Hilbert sixteenth problem facing the question from a different point of view. Instead of trying to bound the number of limit cycles in terms of the degrees of the vector fields we start

wondering ourselves if it is not better to do this in terms of the number of homogeneous vector fields involved in a family. Very soon, we realize that this leads essentially to the same problem, because polynomial vector fields of degree n are the sum of $n + 1$ homogeneous vector fields. In fact, it is even a worst point of view in the light of the following family of polynomials vector fields studied in [11],

$$\dot{z} = Az + Bz|z|^{2(k-3)} + ic\bar{z}^{k-1},$$

where $z = x + iy$, $A = a_1 + ia_2$, $B = b_1 + ib_2 \in \mathbb{C}$, $c \in \mathbb{R}$ and $k \geq 3$. It has at least k limit cycles but it can also be written in real variables as

$$(\dot{x}, \dot{y}) = a_1X_1(x, y) + a_2X_2(x, y) + b_1X_3(x, y) + b_2X_4(x, y) + cX_5(x, y),$$

that is, involving only five homogeneous vector fields with only 3 different degrees.

Nevertheless this way of thinking the problem lead us to a new point of view that we hope that results interesting for the reader: Why do not try to study the number of limit cycles in terms of the number of homogeneous vector fields formed by single monomials?

Somehow this point of view tries to mimic the role of Descartes theorem for studying the number of real zeroes of a polynomial $P(x)$ of degree n , having m non-zero monomials. Recall that while the maximum number of real roots is n , the actual maximum number of real roots is $2m - 1$ and this bound is independent of the degree of P . In fact, P has at most $m - 1$ positive roots, $m - 1$ negative roots, and eventually the root 0.

To state more clearly our point of view and our results, for each $m \in \mathbb{N}$ fixed, we consider the following family of polynomials differential equations:

- Family \mathcal{M}_m given by

$$(\dot{x}, \dot{y}) = \sum_{j=1}^m a_j X_j(x, y), \quad \text{with} \quad X_j(x, y) = \begin{cases} (x^{n_j} y^{k_j}, 0), & \text{or,} \\ (0, x^{n_j} y^{k_j}), \end{cases}$$

where $\mathbf{a} \in \mathbb{R}^m$ and the couples $(n_j, k_j) \in \mathbb{N}^2$ vary among all the possible values. Varying m , this family covers all polynomial differential equations. The letter \mathcal{M} is chosen because the important point is to count the number of involved monomials.

We define $\mathcal{H}^M[m] \in \mathbb{N} \cup \{\infty\}$ to be the maximum number of limit cycles that systems of the family \mathcal{M}_m can have.

Next theorem includes our results about lowers bounds for this Hilbert type number. The proof of the first part for $m \geq 3$ is a straightforward consequence of Theorem 1.1 and also a consequence of other known results about classical Liénard systems. The second part is a direct corollary of the recent paper [2] and uses generalized Liénard systems.

Theorem 1.2. *With the notation introduced above it holds that $\mathcal{H}^M[m] = 0$ for $m = 1, 2, 3$ and for $m \geq 4$, $\mathcal{H}^M[m] \geq m - 3$. Moreover, there exists a sequence of values of m tending to infinity such that $\mathcal{H}^M[m] \geq N(m)$, where*

$$N(m) = \left(\frac{\binom{m-3}{2} \log\left(\frac{m-3}{2}\right)}{\log 2} \right) (1 + o(1)).$$

A similar result could be stated by using the lower bounds of $\mathcal{H}(n)$ of type $O(n^2 \log(n))$ because the systems of degree n involve $m = (n + 1)(n + 2)$ monomials. These systems and the ones of [2] are relevant because for m big enough they have more limit cycles than monomials.

It is not difficult to see that all results given in Theorem 1.2 also hold for the subclass of \mathcal{M}_m of second order differential equations $\ddot{x} = P(x, \dot{x})$ with P being a polynomial with

$m-1$ monomials because classical Liénard differential equations write as $\ddot{x} = f(x) + g(x)\dot{x}$, or equivalently, like the system $(\dot{x}, \dot{y}) = (y, -f(x) - g(x)y)$.

It is also worth to mention that the celebrated examples of quadratic systems that prove that $\mathcal{H}(2) \geq 4$ are given by systems with $m = 8$ monomials, see for instance [7, 22], and so they have $m - 4$ limit cycles. The cubic system given in [14] proving that $\mathcal{H}(3) \geq 13$ has $m = 9$ monomials and at least $m + 4$ limit cycles.

Under the light of the above results, a natural problem is to find the minimal m such that there exists a system with m monomials having at least $m + 1$ limit cycles.

2. SOME MOTIVATING MODELS

In this section we shortly explain some models that motivate the class of equations (1) that can be treated with the tools introduced in this paper.

2.1. Capillary rise. A first example is given by the equation that models the capillary rise. The capillary action is a physical property that the fluids have in to go down or up in extremely thin tubes. Sometimes this action to do the liquid to go up against the force of gravity or even to induce a magnetic field. This ability to rise or fall results from the ability of the liquid to “wet” or not the pipe surface (glass, plastic, metal, etc.). For instance in the case of water in a glass beaker, we have tendency of water to adhere to the glass, bending upward near the wall, forming a concave meniscus and rising to a certain height above water level, here we have a capillary rise. In the case of mercury the opposite happens, the tendency of mercury is to move away from the wall, forming a convex meniscus and descending at a certain height from the mercury level, here we have a capillary depression.

This phenomenon is described in more detail in [23] and can be modeled in an adimensional way by the planar system

$$\begin{cases} \dot{x} = y \\ \dot{y} = 1 - ay - \sqrt{2x} \end{cases},$$

where a is a positive parameter.

2.2. Some population models. Following [1] we introduce the herd behavior. If R represents the density of certain population, namely number of individuals per surface unit, with the herd occupying an area A , then the individuals who take the outermost positions in the herd are proportional to the perimeter of the region where the herd is located whose length depends on \sqrt{A} . They are therefore in number proportional to the square root of the density, that is to \sqrt{R} , with a proportionality constant that depend on the shape of the herd. Then, the interactions with the second population with density Q occur only via these peripheral individuals, so that instead of the standard RQ that appears in the usual predator-prey systems, there is a term proportional to $\sqrt{R}Q$. In a dimensional-less set of variables these type of models write as

$$\begin{cases} \dot{x} = x(1-x) - y\sqrt{x}, \\ \dot{y} = -xy + cy\sqrt{x}, \end{cases}$$

for $c \in \mathbb{R}$, see also [4]. For other population models, involving also square roots, see [3, Sec. 4.10].

2.3. A SIR type model. In [18] the author proposes a variation of the classical SIR model. Recall that it is a mathematical model of the spread of infectious diseases that classifies the population in three categories: Susceptible, Infectious, or Recovered. This model relate these categories by the differential system

$$\dot{S} = -\beta\sqrt{SI}, \quad \dot{I} = \beta\sqrt{SI} - \gamma\sqrt{I}, \quad \dot{R} = \gamma\sqrt{I}$$

where α, β and γ are real parameters. Notice that it can be studied via a planar system because $\dot{S} + \dot{I} + \dot{R} = 0$ and as a consequence $S(t) + I(t) + R(t) = S_0 + I_0 + R_0$.

3. DEFINITIONS AND PRELIMINARIES.

In this section we review some definitions and results that will be used in this paper. For the characterization of Chebyshev Systems in an open interval we will use the following results which can be found in [13] and [17].

Definition 3.1. *Let u_0, \dots, u_{n-1}, u_n be functions defined in an open interval L of \mathbb{R} . The ordered set $(u_i)_{i=0}^n$ forms an extended complete Chebyshev system, for short *ECT-system*, on L if any nontrivial linear combination $a_0u_0 + \dots + a_ku_k$ has at most k isolated roots in L counting multiplicity, for every $k = 0, 1, \dots, n$.*

The following result is a very useful characterization of smooth *ECT*-systems in terms of Wronskians.

Proposition 3.2. *The set of ordered C^n -functions (u_0, \dots, u_n) forms an *ECT-system* on L if, and only if, for every $k = 0, \dots, n$,*

$$W(u_0, \dots, u_k)(x) = \begin{vmatrix} u_0(x) & \cdots & u_k(x) \\ u_0'(x) & \cdots & u_k'(x) \\ \vdots & \ddots & \vdots \\ u_0^{(k)}(x) & \cdots & u_k^{(k)}(x) \end{vmatrix} \neq 0,$$

for every $x \in L$.

We will need the following lemma.

Lemma 3.3. *Consider $\beta_i \in \mathbb{R}$ such that $\beta_0 < \beta_1 < \dots < \beta_m$. Then the functions $(x^{\beta_0}, \dots, x^{\beta_m})$ form an *ECT-system* on $(0, \infty)$.*

Proof. We claim that

$$W = W(x^{\beta_0}, \dots, x^{\beta_k}) = x^S \left(\prod_{0 \leq i < j \leq k} (\beta_j - \beta_i) \right), \quad \text{where} \quad S = \sum_{i=0}^k \beta_i - \frac{k(k+1)}{2}. \quad (3)$$

Then, each $W(x^{\beta_0}, \dots, x^{\beta_k}) \neq 0$ in $(0, \infty)$, for $k = 0, \dots, m$, and by Proposition 3.2 the functions $(x^{\beta_j})_{j=0}^m$ form an *ECT* on $(0, \infty)$ as we wanted to prove.

Let us prove the claim. For $1 \leq k \in \mathbb{N}$, set $(\beta)_k = \beta(\beta - 1)(\beta - 2) \cdots (\beta - k)$. Then,

$$\begin{aligned}
 W &= \begin{vmatrix} x^{\beta_0} & \cdots & x^{\beta_k} \\ \beta_0 x^{\beta_0-1} & \cdots & \beta_k x^{\beta_k-1} \\ (\beta_0)_1 x^{\beta_0-2} & \cdots & (\beta_k)_1 x^{\beta_k-2} \\ \vdots & \ddots & \vdots \\ (\beta_0)_{k-1} x^{\beta_0-k} & \cdots & (\beta_k)_{k-1} x^{\beta_k-k} \end{vmatrix} = x^S \begin{vmatrix} 1 & \cdots & 1 \\ \beta_0 & \cdots & \beta_k \\ \beta_0(\beta_0 - 1) & \cdots & \beta_k(\beta_k - 1) \\ \vdots & \ddots & \vdots \\ (\beta_0)_{k-1} & \cdots & (\beta_k)_{k-1} \end{vmatrix} \\
 &= x^S \begin{vmatrix} 1 & \cdots & 1 \\ \beta_0 & \cdots & \beta_k \\ \beta_0^2 & \cdots & \beta_k^2 \\ \beta_0(\beta_0 - 1)(\beta_0 - 2) & \cdots & \beta_k(\beta_k - 1)(\beta_k - 2) \\ \vdots & \ddots & \vdots \\ (\beta_0)_{k-1} & \cdots & (\beta_k)_{k-1} \end{vmatrix} = \cdots = x^S \begin{vmatrix} 1 & \cdots & 1 \\ \beta_0 & \cdots & \beta_k \\ \beta_0^2 & \cdots & \beta_k^2 \\ \beta_0^3 & \cdots & \beta_k^3 \\ \vdots & \ddots & \vdots \\ \beta_0^k & \cdots & \beta_k^k \end{vmatrix},
 \end{aligned}$$

where this last determinant is the celebrated Vandermonde determinant and coincides with expression (3). Notice that in the first equality we have used that taking the product of $k + 1$ elements of the determinant, being each one of them elements of different rows and columns, always appears x^S as a factor. Moreover, in the first equality of the second line of equalities we have changed the third file by the sum of the second and third files of the previous determinant. Similarly, we change the fourth file of this new determinant by a suitable linear combinations of the second, third and fourth ones, and so on, until arriving to the final equality. So, the claim follows. \square

A second key tool for proving Theorem 1.1 will be next averaging type Theorem, proved in [5], which is applicable to continuous differential systems. See the Appendix for a short reminder about Brouwer topological degree.

Theorem 3.4. (Averaging theorem via Brouwer degree ([5])). *Consider the system of differential equations*

$$x'(t) = \varepsilon H(t, x) + \varepsilon^2 K(t, x, \varepsilon), \quad (4)$$

where $H : \mathbb{R} \times D \rightarrow \mathbb{R}^n$, $K : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^n$ are continuous functions, T -periodic in the first variable, D is an open subset of \mathbb{R}^n and $(-\varepsilon_0, \varepsilon_0)$ is a neighborhood of $0 \in \mathbb{R}$. We define the averaged function, $h : D \rightarrow \mathbb{R}^n$ as follow:

$$h(z) = \frac{1}{T} \int_0^T H(t, z) dt, \quad (5)$$

and we assume that each $a \in D$ with $h(a) = 0$, there is a neighborhood V of a such that $h(z) \neq 0$ for all $z \in \bar{V} \setminus \{a\}$ and Brouwer degree $d_B(h, V, 0) \neq 0$. Then for each $|\varepsilon| > 0$ small enough, there is a T -periodic solution $\varphi(\cdot, \varepsilon)$ of system (4) such that $\varphi(\cdot, \varepsilon) \rightarrow a$ when $\varepsilon \rightarrow 0$.

Remark 3.5. Theorem 3.4 shows that for each isolated solution a of $h(z) = 0$ in D , where h is given in (5), such that $d_B(h, V, 0) \neq 0$, there is, for ε small enough, a T -periodic orbit of system (4) tending to a when ε goes to 0. When h is of class \mathcal{C}^1 these hypotheses about the solution a can simply be replaced by $h'(a) \neq 0$.

The key result for proving the second part of Theorem 1.2 will be the following theorem.

Theorem 3.6. ([2]) *There is a sequence of natural numbers n , tending to infinity, such that for these values of n there exist generalized Liénard systems*

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = G(x), \end{cases}$$

with F and G are polynomials of degree at most n , having at least $K(n)$ limit cycles, where

$$K(n) = \left(\frac{n \log n}{\log 2} \right) (1 + o(1)).$$

We also will need the following result about non-existence of limit cycles.

Proposition 3.7. *Systems*

$$(\dot{x}, \dot{y}) = (ax^p y^q, bx^i y^j + cx^k y^l),$$

where $(a, b, c) \in \mathbb{R}^3$ and $(p, q, i, j, k, l) \in \mathbb{N}_0^6$, with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, have no limit cycle.

Proof. We will use the following well-known properties for proving non-existence of limit cycles:

- P_1 : Periodic orbits must surround some critical point. So systems without critical points have no periodic orbit.
- P_2 : If a system has an invariant line passing by all its critical points, if any, then it has no periodic orbits. This is so by property P_1 if the system has no critical points, or, otherwise, by uniqueness of solutions, because an eventual periodic orbit would surround some of the critical points and as a consequence, cut the line.
- P_3 : If one of the two differential equations only involves ones of the variables (for instance $\dot{x} = f(x)$) then the system has no periodic orbits. This is so, because autonomous one dimensional ordinary differential equations have no non-constant periodic solution.
- P_4 : If a planar system has a smooth first integral defined on an open set $\mathcal{U} \subset \mathbb{R}^2$, although it can have continua of periodic orbits, it can not have limit cycles entirely contained in \mathcal{U} .
- P_5 : If the divergence of a planar system $(\dot{x}, \dot{y}) = (P(x, y), Q(x, y))$, $\operatorname{div}(P, Q) = \frac{\partial P(x, y)}{\partial x} + \frac{\partial Q(x, y)}{\partial y}$ does not change sign and vanishes only on sets of zero Lebesgue measure, then the system does not have periodic orbits.
- P_6 : Let X be a planar vector field with a unique critical point, $(0, 0)$, and assume that it is reversible, that is, invariant by one of the two changes of variables and time:

$$(x, y, t) \longrightarrow (-x, y, -t) \quad \text{or} \quad (x, y, t) \longrightarrow (x, -y, -t).$$

If the system has a periodic orbit that crosses transversally the axes then it is in the interior of a continua of periodic orbits and it is not a limit cycle. This is so, because any of the described symmetries implies that if an orbit turns around the origin it is periodic. Sometimes this criterion is called *reversibility criterion of Poincaré*, because he was the first in using it for proving the existence of periodic orbits.

When $a = 0$, the system can not have periodic orbits because of property P_3 . When $bc = 0$, we assume, for instance, that $c = 0$ and $b \neq 0$, because when $c \neq 0$ and $b = 0$ the situation is the same and the case $b = c = 0$ is trivial. Then, two situations may happen: either $x = 0$ or $y = 0$ are a continuum of critical points and no other critical points appear or it writes as $(\dot{x}, \dot{y}) = (ay^q, bx^i)$. In the first case the only critical points belong to an invariant line full of critical points, so the system can not have periodic orbits by property P_2 . In the second case the system is integrable with $\mathcal{U} = \mathbb{R}^2$ and by property P_4 no limit cycle appears.

Hence, from now on, we will assume that $abc \neq 0$. Next step will use that the phase portraits of any two systems of the form

$$(\dot{x}, \dot{y}) = (P(x, y)R(x, y), Q(x, y)R(x, y)) \quad \text{and} \quad (\dot{x}, \dot{y}) = (P(x, y), Q(x, y)),$$

are the same (modulus a change of time orientation) in each connected component of $\mathbb{R}^2 \setminus \{R(x, y) = 0\}$. We will take R as some suitable polynomial of one of the forms x^n or y^n to reduce our study to simpler vector fields. Taking $R(x, y) = x^s$ for s to be the minimum of p, i and k we can reduce the situation to one of the next two differential systems:

$$(\dot{x}, \dot{y}) = (ay^q, bx^i y^j + cx^k y^l), \quad i \geq 1 \quad \text{or} \quad (\dot{x}, \dot{y}) = (ax^p y^q, by^j + cx^k y^l), \quad (6)$$

where for simplicity we keep the same notation for the new exponents and without loss of generality we have assumed that $i \leq k$. Next we take $R(x, y) = y^u$ with u being the minimum of q, j and l . Finally, we only need to study the following five cases:

$$\begin{aligned} (i) \quad (\dot{x}, \dot{y}) &= (a, bx^i y^j + cx^k y^l), \quad i \geq 1, & (ii) \quad (\dot{x}, \dot{y}) &= (ay^q, bx^i + cx^k y^l), \quad i \geq 1, \quad q \geq 1, \\ (iii) \quad (\dot{x}, \dot{y}) &= (ax^p, by^j + cx^k y^l), & (iv) \quad (\dot{x}, \dot{y}) &= (ax^p y^q, b + cx^k y^l), \\ (v) \quad (\dot{x}, \dot{y}) &= (ax^p y^q, by^j + cx^k), \end{aligned}$$

where we also keep the old notation for the new exponents. Notice that (i) and (ii) come from the first differential equations of (6) and the other three cases from the second one.

The case (i) has no critical point, so it has no periodic orbit by property P_1 .

In case (ii), when $l = 0$ we can apply property P_4 with $\mathcal{U} = \mathbb{R}^2$ because the system has a polynomial first integral.

When $l \neq 0$ the system has a unique critical point $(0, 0)$ and it writes as

$$(\dot{x}, \dot{y}) = (ay^q, bx^i + cx^k y^l), \quad i \geq 1, \quad q \geq 1, \quad l \geq 1. \quad (7)$$

Notice that studying the vector field on the axes we get

$$\dot{x}|_{x=0} = ay^q \quad \text{and} \quad \dot{y}|_{y=0} = bx^i.$$

Since a periodic orbit must surround the origin, the above conditions imply that this is only possible when q and i are both odd numbers and $ab < 0$. So, in this case we will assume that these conditions hold because otherwise the system has not periodic orbits.

If l is even the system is invariant by the change $(x, y, t) \rightarrow (x, -y, -t)$ and by property P_6 the system has no limit cycle and we are done. If l is odd, then the system is invariant by the change $(x, y, t) \rightarrow (-x, y, -t)$ and again by property P_6 we are done. Hence it only remains to consider the case l odd and k even. Notice that

$$\operatorname{div}(X) = clx^k y^{l-1},$$

and then it does not change sign and only vanishes on $\{xy = 0\}$, or on one subset of $\{xy = 0\}$. Hence by property P_5 the system has no periodic orbit.

In case (iii), we use property P_3 .

In case (iv) when $pqkl \neq 0$ we can apply property P_1 . Also, when $p = q = 0$ we can apply property P_1 . Next we split the study according one of the variables p, q, k or l vanishes and taking into account that $p^2 + q^2 \neq 0$.

Assume that $p = 0$. Then $q \neq 0$. When $l \neq 0$ we can apply again property P_1 . When $l = 0$ we can apply property P_4 because the system has a polynomial first integral.

Assume that $q = 0$. Then the first equation of the system is $\dot{x} = ax^p$ and we can apply property P_3 .

Assume that $k = 0$. Then the second equation of the system is $\dot{y} = b + cy^l$ and we can apply again property P_3 .

Finally, assume that $l = 0$. When $p = 0$ the system has a polynomial first integral and we can apply property P_4 with $\mathcal{U} = \mathbb{R}^2$. When $p \neq 0$, the system has the invariant line $\mathcal{L} = \{x = 0\}$, and it can be integrated by separating the variables, giving an smooth first integral in $\mathbb{R}^2 \setminus \mathcal{L}$. Then we can apply again property P_4 to each of the connected

components of $\mathbb{R}^2 \setminus \mathcal{L}$ and prove the non-existence of limit cycles because \mathcal{L} is also invariant and eventual limit cycles can not cut it.

Finally we study case (v). When $q = 0$ by property P_3 no periodic orbit appears. We consider four different subcases that cover all the situations.

When $q \neq 0, p = 0$ and $j = 0$ the system has a polynomial first integral and by property P_4 we are done.

When $q \neq 0, p = 0$ and $j \neq 0$ the system writes as

$$(\dot{x}, \dot{y}) = (ay^q, by^j + cx^k), \quad q \geq 1, j \geq 1. \quad (8)$$

If $k = 0$ in (8) then we use property P_3 . The case $k \neq 0$ in (8) we notice that, by changing the names of some of the parameters it coincides with the system (7) studied in case (ii) taking in that system $k = 0$. Hence, again this system has no limit cycle.

When $q \neq 0, p \neq 0$ and $j = 0$ the system has once more the invariant line $\mathcal{L} = \{x = 0\}$, and it can be integrated by separating the variables, giving an smooth first integral in $\mathbb{R}^2 \setminus \mathcal{L}$. As in the similar previous situation, we can prove that it has no limit cycles by using property P_4 .

In the remaining case $q \neq 0, p \neq 0$ and $j \neq 0$. Then the $(0, 0)$ is its unique critical point and $x = 0$ is an invariant line. By property P_2 it has no periodic orbit.

Hence we have proved that although sometimes the system has continua of periodic orbits it has not limit cycles, as is stated in the lemma. \square

4. PROOF OF THEOREM 1.1

In order to find periodic orbits for the continuous planar differential system

$$(\dot{x}, \dot{y}) = (-y, x) + \varepsilon \sum_{j=0}^n b_j X_j(x, y),$$

we will apply the averaging method via Brouwer degree given by the Theorem 3.4. Notice that we haven taken $\mathbf{a} = \varepsilon \mathbf{b}$ in the expression (2) and ε is a small parameter. As usual, we write the system in polar coordinates $x = r \cos \theta, y = r \sin \theta$, see for instance [5]. We get

$$\begin{aligned} \dot{r} &= \varepsilon \sum_{j=0}^n b_j (x f_j(x, y) + y g_j(x, y)) = \varepsilon \sum_{j=0}^n b_j F_j(\theta) r^{\alpha_j}, \\ \dot{\theta} &= 1 + \varepsilon \sum_{j=0}^n b_j (x g_j(x, y) - y f_j(x, y)) = 1 + \varepsilon \sum_{j=0}^n b_j G_j(\theta) r^{\alpha_j - 1}, \end{aligned}$$

where

$$\begin{aligned} F_j(\theta) &= f_j(\cos \theta, \sin \theta) \cos \theta + g_j(\cos \theta, \sin \theta) \sin \theta, \\ G_j(\theta) &= g_j(\cos \theta, \sin \theta) \cos \theta - f_j(\cos \theta, \sin \theta) \sin \theta. \end{aligned}$$

Finally, we have the differential equation

$$\frac{dr}{d\theta} = r' = \frac{\varepsilon \sum_{j=0}^n b_j F_j(\theta) r^{\alpha_j}}{1 + \varepsilon \sum_{j=0}^n b_j G_j(\theta) r^{\alpha_j - 1}} = \varepsilon \sum_{j=0}^n b_j F_j(\theta) r^{\alpha_j} + O(\varepsilon^2). \quad (9)$$

It is continuous for $(\theta, r) \in \mathbb{R} \times (0, R_0)$ for some $R_0 > 0$ and ε small enough. We can easily compute the averaged function h given in Theorem 3.4. We obtain

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=0}^n b_j F_j(\theta) z^{\alpha_j} d\theta = \sum_{j=0}^n \frac{b_j}{2\pi} \left(\int_0^{2\pi} F_j(\theta) d\theta \right) z^{\alpha_j} = \sum_{j=0}^n \frac{b_j I_j}{2\pi} z^{\alpha_j}$$

Since, from all $I_j, j = 0, 1, \dots, n$, only $m + 1$ values are non-zero, we rename the corresponding ordered α_j as $\beta_0, \beta_1, \dots, \beta_m$ and then

$$h(z) = \sum_{j=0}^m c_j z^{\beta_j},$$

with all c_j arbitrary real constants and $\beta_0 < \beta_1 < \dots < \beta_m$. By Lemma 3.3 they form an *ECT*-system on $(0, \infty)$. In particular, the maximum number of positive zeroes of h is m and there exist c_0, c_1, \dots, c_m such that h has exactly m simple zeroes. Notice that the upper bound of m zeroes for h is also a straightforward consequence of Descartes's rule of signs. Taking the corresponding values of \mathbf{b} we obtain a system with $|\varepsilon|$ small enough and at least m periodic orbits. In general we do not know yet that these periodic orbits are limit cycles, that is, isolated among all the existing periodic orbits. Nevertheless, because the right hand side of our differential equation (9) is continuous with respect to θ and diferenciable with respect to $r > 0$ we can apply Theorem 3 of [6] that asserts that the obtained periodic orbits are indeed limit cycles.

To prove the hyperbolicity in the smooth case it suffices to show that in this regular setting the positive zeroes of the averaged function h coincide with the ones of the first order Melnikov function. In [5] this fact is proved in several situations. We show this result again for the \mathcal{C}^1 planar differential equations of the form

$$(\dot{x}, \dot{y}) = (-y, x) + \varepsilon(P(x, y), Q(x, y)).$$

Recall that for this system, the Melnikov function M writes as

$$M(k) = \int_{x^2+y^2=k} P(x, y) dy - Q(x, y) dx, \quad 0 < k \in \mathbb{R},$$

see for instance [8]. By parameterizing the circles as $x = \sqrt{k} \cos \theta$, $y = \sqrt{k} \sin \theta$ we get that $M(k) = \sqrt{k}h(\sqrt{k})$ and, as a consequence, the positive simple zeroes of h give rise to hyperbolic limit cycles of our planar system for $|\varepsilon|$ small enough. Hence, the theorem is proved.

4.1. Examples of application. As a first application we prove that the simple differential system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} + \begin{pmatrix} q_{1,1} & q_{1,2} \\ q_{2,1} & q_{2,2} \end{pmatrix} \begin{pmatrix} \sqrt{|x|} \\ \sqrt{|y|} \end{pmatrix} + \begin{pmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad (10)$$

where $\sqrt{|z|} = \text{sgn}(z)\sqrt{|z|}$, has for some values of the parameters a limit cycle crossing $\Sigma = \{xy = 0\}$. This family includes for instance the one given in Subsection 2.1.

In the notation of the theorem, all systems of the form (10) can be written as

$$(\dot{x}, \dot{y}) = \sum_{j=0}^2 a_j X_j(x, y),$$

where $X_0(x, y) = (s_1, s_2)$, $X_1(x, y) = (q_{1,1}\sqrt{|x|} + q_{1,2}\sqrt{|y|}, q_{2,1}\sqrt{|x|} + q_{2,2}\sqrt{|y|})$ and $X_2(x, y) = (p_{1,1}x + p_{1,2}y, p_{2,1}x + p_{2,2}y)$. Moreover $(\alpha_0, \alpha_1, \alpha_2) = (0, 1/2, 1)$. Notice that for

simplicity we keep the same names for the constants although they have varied. Clearly,

$$\begin{aligned} I_0 &= \int_0^{2\pi} (s_1 \cos \theta + s_2 \sin \theta) d\theta = 0, \\ I_2 &= \int_0^{2\pi} (p_{1,1} \cos^2 \theta + (p_{1,2} + p_{2,1}) \sin \theta \cos \theta + p_{2,2} \sin^2 \theta) d\theta = (p_{1,1} + p_{2,2})\pi, \\ I_1 &= 4(q_{1,1} + q_{2,2}) \int_0^{\pi/2} \cos^{3/2} \theta d\theta, \end{aligned}$$

where in the last equality we have used that

$$\begin{aligned} \int_0^{2\pi} \sqrt{(\cos \theta) \cos \theta} d\theta &= 2 \int_{-\pi/2}^{\pi/2} \cos^{3/2} \theta d\theta = 4 \int_0^{\pi/2} \cos^{3/2} \theta d\theta > 0, \\ \int_0^{2\pi} \sqrt{(\sin \theta) \sin \theta} d\theta &= 4 \int_0^{\pi/2} \sin^{3/2} \theta d\theta = 4 \int_0^{\pi/2} \cos^{3/2} \theta d\theta, \end{aligned}$$

and by symmetry,

$$\int_0^{2\pi} \sqrt{(\sin \theta) \cos \theta} d\theta = \int_0^{2\pi} \sqrt{(\cos \theta) \sin \theta} d\theta = 0.$$

Thus when $(p_{1,1} + p_{2,2})(q_{1,1} + q_{2,2}) \neq 0$, the number of non-zero values in the list I_0, I_1, I_2 is 2 and by Theorem 1.1 we have a system of the form (10) with 1 limit cycle.

As a second example of application consider

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} + \begin{pmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} q_{1,1} & q_{1,2} \\ q_{2,1} & q_{2,2} \end{pmatrix} \begin{pmatrix} \sqrt[3]{x} \\ \sqrt[3]{y} \end{pmatrix}, \quad (11)$$

where recall that $\sqrt[3]{y} = \text{sgn}(y)\sqrt{|y|}$. We will prove that it has at least 2 limit cycles crossing $\Sigma = \{xy = 0\}$ for same values of the parameters.

Writing it in the notation of Theorem 1.1 we get

$$(\dot{x}, \dot{y}) = \sum_{j=0}^3 a_j X_j(x, y),$$

where $X_0(x, y) = (s_1, s_2)$, $X_1(x, y) = (q_{1,1}\sqrt[3]{x}, q_{2,1}\sqrt[3]{x})$, $X_2(x, y) = (q_{1,2}\sqrt[3]{y}, q_{2,2}\sqrt[3]{y})$ and $X_3(x, y) = (p_{1,1}x + p_{1,2}y, p_{2,1}x + p_{2,2}y)$. Moreover, $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0, 1/3, 1/2, 1)$. Notice that again, for simplicity, we keep the same names for the constants although they have varied. In this case,

$$I_0 = 0, \quad I_1 = 4q_{1,1} \int_0^{\pi/2} \cos^{4/3} \theta d\theta, \quad I_2 = 4q_{2,2} \int_0^{\pi/2} \sin^{3/2} \theta d\theta, \quad I_3 = (p_{1,1} + p_{2,2})\pi.$$

where I_0, I_2 and I_3 are obtained similarly that in the previous case and to get I_1 we have used that

$$\int_0^{2\pi} \sqrt[3]{\cos \theta} \cos \theta d\theta = 4 \int_0^{\pi/2} \cos^{4/3} \theta d\theta > 0 \quad \text{and} \quad \int_0^{2\pi} \sqrt[3]{\cos \theta} \sin \theta d\theta = 0.$$

Hence, when $q_{1,1}q_{2,2}(p_{1,1} + p_{2,2}) \neq 0$, the number of non-zero values in the list I_0, I_1, I_2, I_3 is 3 and by Theorem 1.1 we have an example of system (11) with at least 2 limit cycles.

5. PROOF OF THEOREM 1.2

That $\mathcal{H}^M[j] = 0$, for $j = 1, 2, 3$, is a straightforward consequence of Proposition 3.7. Notice that this proposition covers all cases except the trivial ones, where either $\dot{x} = 0$ or $\dot{y} = 0$, and the right-hand side of the other equation has j monomials.

Let us prove that for $m \geq 4$, $\mathcal{H}^M[m] \geq m - 3$. Consider the Liénard classic system in class \mathcal{M}_m ,

$$(\dot{x}, \dot{y}) = (y, -x + a_0y + a_1y^3 + \cdots + a_{m-3}y^{2m-5}).$$

With the notation of Theorem 1.1 we get that for all $j = 0, 1, \dots, m - 3$,

$$I_j = \int_0^{2\pi} \sin^{2j+2} \theta d\theta > 0,$$

and as a consequence we get examples with $m - 3$ limit cycles. In fact, this system includes the celebrated van der Pol system when $m = 4$ and coincides with the example of classical Liénard system studied in [15], where the author, with another notation, already proved the existence of $m - 3$ limit cycles.

Notice that there are many different families in \mathcal{M}_m with at least $m - 3$ limit cycles. For instance it suffices to consider systems of the form

$$(\dot{x}, \dot{y}) = (y, -x + a_0x^{2n_0}y^{2k_0+1} + a_1x^{2n_1}y^{2k_1+1} + \cdots + a_{m-3}x^{2n_{m-3}}y^{2k_{m-3}+1}),$$

with $n_j, k_j \in \mathbb{N}_0$ and all $2(n_j + k_j)$, $j = 0, 1, \dots, m - 3$, taking different values. Also similar terms could be added in the first differential equation, removing some other ones from the second one.

To prove that $\mathcal{H}^M[m] \geq N(m)$ we will use Theorem 3.6. For a sequence of values of n tending to infinity, the number of monomials of these generalized Liénard systems is $m = 2n + 3$ while their number of limit cycles is at least $K(n)$. Hence these systems are in \mathcal{M}_m and have at least $N(m) = K((m - 3)/2)$ limit cycles. This function is the one that appears in the statement of the theorem.

APPENDIX: THE BROUWER DEGREE

In this appendix we include a short introduction about the Brouwer degree. We will simply present some definitions and aspects about it. For more details we recommend [20]. From now on, V stands for a bounded open set in \mathbb{R}^n , and ∂V is the boundary of the V set. Our purpose here is to define the degree of a continuous mapping $f : \bar{V} \rightarrow \mathbb{R}^n$. Thus we will do by the usual method: first we consider smooth mappings, and then we extend the definition to any given continuous mapping.

Definition 5.1. *Consider V a non-empty, open and limited subset of \mathbb{R}^n . We define $\mathcal{C}^k(\bar{V}, \mathbb{R}^n)$ as the space of the k -times continuously differentiable functions into \bar{V} . If the function is continuously differentiable for any k then we said that the function is in $\mathcal{C}^\infty(\bar{V}, \mathbb{R}^n)$ or it is smooth.*

Note that, \bar{V} being compact and $f : \bar{V} \rightarrow \mathbb{R}^n$ a continuous mapping then we can define the norm

$$\|f\| = \max\{\|f(x)\| ; x \in \bar{V}\}.$$

As mentioned above, we will consider the case when f is smooth. In particular we will be dealing with the set $R_{f|_V} \subset \mathbb{R}^n$ of regular values of f in V . Recall that $b \in R_{f|_V}$ when the derivative $Df(x)$ is bijective at every point $x \in V$ with $f(x) = b$ (trivially true if $b \notin f(V)$), that is, $R_{f|_V} = f(R)$ where $R = \{x \in V ; \det(Df(x)) \neq 0\}$ and $\det(Df(x))$ is the determinant of the Jacobian matrix of the function f at the point x . We define

the set $S_{f|_V}$ of all critical points of f in V , that is, $S_{f|_V} = \{x \in V ; \det(Df(x)) = 0\}$ and therefore $R_{f|_V} = \mathbb{R}^n \setminus f(S_{f|_V})$.

Proposition 5.2. *Let $f \in \mathcal{C}^1(\overline{V}, \mathbb{R}^n)$ and consider $b \in R_{f|_V} \setminus f(\partial V)$ then the set $f^{-1}(\{b\})$ is finite.*

Proof. We have that f is continuous and the unitary set $\{b\}$ is closed, thus $f^{-1}(\{b\})$ is also a closed set on \overline{V} , consequently it is a closed set in \mathbb{R}^n . It is also a limited set, because $f^{-1}(\{b\}) \subset V$ and V is a limited set. Therefore $f^{-1}(\{b\})$ is closed and limited in \mathbb{R}^n , i.e. it is a compact set.

If $x \in f^{-1}(\{b\})$ we have $\det(Df(x)) \neq 0$, then by Inverse Map Theorem f is a diffeomorphism from a neighborhood U_x of x onto a neighborhood \widehat{U} of b . Observe that x is the unique point in U_x such that $f(x) = b$. We have

$$f^{-1}(\{b\}) \subset \bigcup_{x \in f^{-1}(\{b\})} U_x.$$

Using that $f^{-1}(\{b\})$ is compact and $\{U_x\}$ is an open cover for $f^{-1}(\{b\})$, then there is a finite set $\{x_1, \dots, x_k\} \subset f^{-1}(\{b\})$ such that

$$f^{-1}(\{b\}) \subset \bigcup_{j=1}^k U_{x_j}.$$

It implies that $f^{-1}(\{b\})$ is finite because $U_{x_j} \cap f^{-1}(\{b\}) = \{x_j\}$. □

Now we are able to define the Brouwer topological degree.

Definition 5.3. *Let $f \in \mathcal{C}^\infty(\overline{V}, \mathbb{R}^n)$ and consider $b \in R_{f|_V} \setminus f(\partial V)$ then we define the Brouwer topological degree of f relative to V at the point b as the integer*

$$d_B(f, V, b) = \sum_{x \in f^{-1}(\{b\})} \operatorname{sgn} \det(Df(x)),$$

where sgn denotes the sign function and the set $f^{-1}(\{b\})$ is finite. If $f^{-1}(\{b\}) = \emptyset$ then we define $d_B(f, V, b) = 0$.

Remark 5.4. *From the Definition 5.3 we can see that $d_B(f, V, b) = d_B(f - b, V, 0)$, because if we consider $g = f - b$ we have $g^{-1}(\{0\}) = f^{-1}(\{b\})$.*

The Sard-Brown Theorem says that $R_{f|_V}$ is dense \mathbb{R}^n , which allow us to find regular values in any neighborhood of a critical value. Furthermore in [20] is proved that the Brouwer topological degree is locally constant, that is, if $b \in R_{f|_V} \setminus f(\partial V)$ there is a neighborhood of b , $W \subset R_{f|_V} \setminus f(\partial V)$ such that $d_B(f, V, b) = d_B(f, V, a)$, for all $a \in W$. Now we can define Brouwer topological degree for critical values or regular values.

Definition 5.5. *Let $f \in \mathcal{C}^\infty(\overline{V}, \mathbb{R}^n)$ and consider $b \in \mathbb{R}^n \setminus f(\partial V)$ then we define the Brouwer topological degree of the f relative to V at the point b as the integer*

$$d_B(f, V, b) = d_B(f, V, a),$$

for any such regular value a (which exist by the Sard-Brown Theorem).

The next step is to define the Brouwer topological degree for continuous functions. Consider $f \in \mathcal{C}(\overline{V}, \mathbb{R}^n)$, $b \in \mathbb{R}^n \setminus f(\partial V)$ and $r = \rho(b, f(\partial V))$ being the distance between point b and set $f(\partial V)$. According to Weirstrass Approximation Theorem we find a polynomial (hence smooth) mapping g such that $\|g - f\| < \frac{r}{2}$. Next result ensures that we can define $d_B(f, V, b) = d_B(g, V, b)$.

Theorem 5.6. Fix the set $U = \{g \in C^\infty(\overline{V}, \mathbb{R}^n) ; \|g - f\| < \frac{\epsilon}{2}\}$ then $d_B(g_1, V, b) = d_B(g_2, V, b)$, for $g_1, g_2 \in U$ and $b \in \mathbb{R}^n \setminus f(\partial V)$.

Proof. For a proof and more details of this method we suggest [20]. □

Now we can define the Brouwer topological degree for continuous functions.

Definition 5.7. The Brouwer degree for $f \in C(\overline{V}, \mathbb{R}^n)$ is $d_B(f, V, b) = d_B(g, V, b)$, for $g \in U$ and $b \in \mathbb{R}^n \setminus f(\partial V)$.

This final result straightforward clarifies that the Brouwer degree for simple zeroes of C^1 -functions is non-zero.

Lemma 5.8. Consider $f \in C^1(\Omega, \mathbb{R}^n)$, where Ω is an open set of \mathbb{R}^n . If there is $a \in \Omega$ with $f(a) = 0$ and $\det(Df(a)) \neq 0$, then there is a neighborhood V of a such that $f(x) \neq 0$ for every $x \in \overline{V} \setminus \{a\}$ and $d_B(f, V, 0) \neq 0$.

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