BOUNDED POLYNOMIAL VECTOR FIELDS IN $\mathbb{R}^2$ AND $\mathbb{R}^n$

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Abstract. We characterize the bounded polynomial vector fields in $\mathbb{R}^2$. Additionally we provide a necessary condition but not sufficient which must be satisfied by bounded polynomial vector fields in $\mathbb{R}^n$.

1. Introduction

Many interesting problems coming from the physical and natural sciences can be modeled by polynomial vector fields in $\mathbb{R}^2$ as for instance the Lotka–Volterra systems, the Blausius equation, the van der Pol equation, ... [6]. But the polynomial vector fields started to be analyzed from a mathematical point of view in the works of Poincaré [15], Hilbert [11], Bendixson [1], Dulac [8], ... Since the general class of polynomial vector fields in $\mathbb{R}^2$ is very difficult to study, many authors put their attention to several subclasses. Here our main objective is to characterize the class of bounded polynomial vector fields in $\mathbb{R}^2$.

Bounded polynomial vector fields already have been studied by several authors. Thus the bounded quadratic vector fields have been studied by Coll, Dickson, Dumortier, Gasull, Herssens, Li, Llibre, Perko, Zhang, ... [4, 7, 9, 13]. Some results on the bounded polynomial vector fields of arbitrary degree in $\mathbb{R}^n$ for $n \geq 2$ can be found in Cima, Llibre, Mañosas, Villadelprat, ... [2, 3, 12]. Another work on bounded vector fields in $\mathbb{R}^2$ is due to Conti and Galeotti [5].

Let $X = (P^1, \dots, P^n)\colon \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial vector field. The *degree* of X is the maximum of the degrees of the polynomials P^k's for $k = 1, \dots, n$. Let $p \in \mathbb{R}^n$ and $\gamma(t) = \gamma(t, p)$ be the integral curve of X such that $\gamma(0) = p$ and let I_p be its maximal interval of definition. We say that X is *bounded* if for all $p \in \mathbb{R}^n$, there exists some compact set K such that $\gamma(t) \in K$ for each $t \in I_p \cap (0, +\infty)$.

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Given a vector field it is very difficult to know if it is bounded or not. In this setting it is very interesting to provide necessary and sufficient conditions (in terms only of the vector field) in order to ensure that it is bounded. The goal of this paper is precisely to provide these necessary and sufficient conditions in $\mathbb{R}^2$, and a necessary condition but not sufficient which must be satisfied by bounded polynomial vector fields in $\mathbb{R}^n$.

In order to state our results on the bounded polynomial vector fields we need to recall some basic notions of the Poincaré compactification, introduced by Poincaré in [16].

Roughly speaking the Poincaré compactification of a polynomial vector field $X = (P^1, \ldots, P^n)$ of degree d in $\mathbb{R}^n$ consists in extending it to an analytic vector field $p(X)$ in the closed unit ball $\mathbb{D}^n$ centered at the origin of $\mathbb{R}^n$, in such way that in the interior of that ball the vector field $p(X)$ is analytically equivalent to the vector field X, and consequently $p(X)$ restricted to the boundary sphere $\mathbb{S}^{n-1}$ of $\mathbb{D}^n$ provides the behavior of the vector field X at infinity. Note that in $\mathbb{R}^n$ there are as many directions for going or coming from infinity as points has the sphere $\mathbb{S}^{n-1}$.

In order to state our main results on the bounded polynomial vector fields we need to recall some basic notions of the Poincaré compactification, introduced by Poincaré in [16] in $\mathbb{R}^2$ and extended to $\mathbb{R}^n$ in [2]. All the details on this compactification can be found in Chapter 5 of [10] and in [2].

For studying the neighborhood of the boundary $\mathbb{S}^{n-1}$ of the ball $\mathbb{D}^n$ (i.e. the neighborhood of the infinity of $\mathbb{R}^n$) we consider the local charts (U_k, ϕ_k) and (V_k, ψ_k) for $k = 1, 2$ defined as follows

$$U_k = \{(x_1, \ldots, x_n) \in \mathbb{D}^n : x_k > 0\}, \quad V_k = \{(x_1, \ldots, x_n) \in \mathbb{D}^n : x_k < 0\},$$

the $\phi_k : U_k \to \mathbb{R}^n$ for $k = 1, \ldots, n$ are

$$\phi_k(x_1, \ldots, x_n) = \left(\frac{x_1}{x_k}, \ldots, \frac{x_{k-1}}{x_k}, \frac{1}{x_k}, \frac{x_{k+1}}{x_k}, \ldots, \frac{x_n}{x_k}\right) = (u_1, \ldots, u_n),$$

and $\psi_k(x_1, \ldots, x_n) = \phi_k(x_1, \ldots, x_n)$ for $k = 1, \ldots, n$.

Of course the coordinates $(u_1, \ldots, u_n)$ have different meaning in each local chart, but the points of the infinity, i.e. the points of the boundary $\mathbb{S}^{n-1}$ of $\mathbb{D}^n$ all have the coordinate $u_n = 0$. For polynomial vector fields in $\mathbb{R}^2$ we denote the coordinates in the local charts by (u, v) instead of (u_1, u_2).

We only provide the expression of the compactified analytical vector field $p(X)$ of the polynomial vector field X of degree d on the local chart U_1 of $\mathbb{D}^n$, because here we do not need the expressions in the other local charts. This expression is

$$(1) \quad u_n^d(-u_1P^1 + P^2, -u_2P^1 + P^3, \dots, -u_{n-1}P^1 + P^n, -u_nP^1),$$

where $P^k = P^k(1/u_n, u_1/u_n, \dots, u_{n-1}/u_n)$.

The singular points of $p(X)$ which are on the boundary $\mathbb{S}^{n-1}$ of $\mathbb{D}^n$ are called *infinite singular points*, and the ones which are in the interior of $\mathbb{D}^n$ are called *finite singular points*.

From (1) it follows that the infinity $\mathbb{S}^{n-1}$ of the Poincaré ball is invariant under the flow of the compactified vector field $p(X)$.

The expression for $p(X)$ in the local chart V_k is the same as in U_k multiplied by $(-1)^{d-1}$. Therefore the infinite singular points appear on pairs diametrally opposite on $\mathbb{S}^{n-1}$.

In [2] it has been proved that for a bounded polynomial vector field in $\mathbb{R}^2$ with all the singular points of $p(X)$ isolated, the sum of the indices of all its finite singular points is 1. This is a necessary condition in order that a polynomial vector field in the plane with all the singular points of $p(X)$ isolated be bounded, but it is not necessary. Here we prove this result without the assumptions that there are finitely many infinite singular points.

Theorem 1. *Let X be a bounded polynomial vector field in $\mathbb{R}^2$ with finitely many finite singular points. Then the sum of the indices of all its finite singular points is* 1.

In [2] it was also proved that a bounded polynomial vector field X in $\mathbb{R}^n$ such that $p(X)$ has finitely many finite and infinite singular points and satisfies a convenient generic condition, the sum of the topological indices of its finite singular points must be $(-1)^n$. Here we prove a similar result without the assumptions that there are finitely many infinite singular points and without the generic condition, but with a different assumption.

Theorem 2. *Let X be a polynomial vector field in $\mathbb{R}^n$ with finitely many finite singular points such that when the boundary $\mathbb{S}^{n-1}$ of $\mathbb{D}^n$ is collapsed to a point, this point is a repeller. Then the sum of the indices of all its finite singular points is $(-1)^n$.*

Theorems 1 and 2 are proved in section 2. Note that under the assumptions of Theorem 2 the polynomial vector field X is bounded.

The condition stated in Theorems 1 and 2 is only necessary due to the following result.

Proposition 3. *The polynomial differential system*

$$\dot{x}_1 = x_1, \quad \dot{x}_2 = x_2, \quad \dot{x}_k = -x_k \quad \text{for } k = 3, \ldots, n, \tag{2}$$

in $\mathbb{R}^n$ *for* $n \geq 2$ *(in the case* $n = 2$ *the system is simply* $\dot{x}_1 = x_1$, $\dot{x}_2 = x_2$*) satisfies that the sum of the indices of all its finite singular points is* $(-1)^n$*, but it is not bounded.*

Proposition 3 is proved in section 2.

Here a *degenerate hyperbolic sector* is a hyperbolic sector of an infinite singular point having its two separatrices at infinity, i.e. contained in the boundary $\mathbb{S}^1$ of $\mathbb{D}^2$.

In what follows we denote by P_k the homogeneous part of the polynomial P of degree k, and we state our main result for the bounded polynomial vector fields in $\mathbb{R}^2$, which provides a necessary and sufficient condition in order that a polynomial vector field X in $\mathbb{R}^2$ be bounded when it has all its finite and infinite singular points isolated, in other words, when $p(X)$ has finitely many singular points.

Theorem 4. *Let* $X = (P^1, P^2)$ *be a polynomial vector field of degree* d *in the plane such that its Poincaré compactification* $p(X)$ *has all its finite and infinite singular points isolated. Then* X *is bounded if and only if every infinite singular point* q *after putting* q *at the origin of the local chart* U_1*, doing a rotation of the vector field around the origin of coordinates if necessary,* q *satisfies one of the following two conditions for* $v > 0$ *sufficiently small:*

- (i) k *odd and* $-vP^1_d(1,0) - v^2P^1_{d-1}(1,0) - \cdots - v^{d+1}P^1_0(1,0) > 0$;
- (ii) k *even and*
  - (ii.1) *either* $-vP^1_d(1,0) - v^2P^1_{d-1}(1,0) - \cdots - v^{d+1}P^1_0(1,0) > 0$,
  - (ii.2) *or* $a_k[P^2_d(1,0) + vP^2_{d-1}(1,0)) + \cdots + v^dP^2_0(1,0)] > 0$*, where* $k > 0$ *is defined in the following expression*

$$F(u) = -uP^1_d(1,u) + P^2_d(1,u) = \sum_{j=k}^{d+1} a_j u^j, \;\; a_k \neq 0,$$

and there is a degenerate hyperbolic sector in $v > 0$.

Theorem 4 is proved in section 3.

Note that the zeros u^* of the polynomial $F(u)$ in the statement of Theorem 4 provides the first coordinate of the infinite singular points

$(u^*, 0)$ in the local chart U_1 of the Poincaré compactification, and that this polynomial in the statement of Theorem 4 has no independent term because we have that the origin of U_1 is a singular point.

2. Proof of Theorems 1, 2 and Proposition 3

In the proof of Theorems 1 and 2 we will need the following theorem, for a proof see for instance Theorem 6.30 of [10] for the case $n = 2$ and [14] for the case $n > 2$.

Poincaré–Hopf Theorem. *For every continuous tangent vector field on the sphere $\mathbb{S}^n$ with finitely many singular points, the sum of their indices is $1 + (-1)^n$.*

Proof of Theorem 1. Let X be a bounded polynomial vector field in $\mathbb{R}^2$ with finitely many finite singular points. Since its compactification $p(X)$ is defined in the closed unit disc $\mathbb{D}^2$, the maximal interval of definition of all the orbits of $p(X)$ is $(-\infty, +\infty)$, for more details see Theorem 1.2 of [10]. Then all the orbits of the vector field $p(X)$ have α–limit and ω–limit (for definitions, if necessary, see section 1.4 of [10]).

Note that to say that X is bounded is equivalent to say that any orbit contained in the interior of $\mathbb{D}^2$ has ω–limit outside $\mathbb{S}^1$. In other words if we identify $\mathbb{S}^1$ to one point, this point is locally a repeller or a center, whose index is one (we can do this identification in a continuous way). After this identification we have a 2–dimensional sphere, then it follows from the Poincaré–Hopf Theorem that the sum of the indices of all the finite singular points of X is one. This concludes the proof of the theorem. □

Proof of Theorem 2. Let X be a polynomial vector field in $\mathbb{R}^n$ with finitely many finite singular points. Since its compactification $p(X)$ is defined in the closed unit ball $\mathbb{D}^n$, the maximal interval of definition of all the orbits of $p(X)$ is $(-\infty, +\infty)$. By assumptions we have that the boundary $\mathbb{S}^{n-1}$ of $\mathbb{D}^n$ is collapsed to a point, this point is a repeller (note that in particular this implies that X is bounded). We can do this identification in a continuous way, and we obtain an n–dimensional topological sphere $\mathbb{S}^n$. Since the index of a repeller is the same as the index of the origin of the differential system

$$\dot{z}_1 = z_1, \quad \dots \quad \dot{z}_n = z_n.$$

Since the identity from $\mathbb{S}^n$ to $\mathbb{S}^n$ has a unique preimage and its determinant is one, the index of any repeller is one (see for more details [14].

Applying the Poincaré–Hopf Theorem on $\mathbb{S}^n$ and denoting by $\sum_f i_X$ the sum of the indices of all the finite singular points of $p(X)$, we obtain

$$\sum_f i_X + 1 = \chi(\mathbb{S}^n) = 1 + (-1)^n.$$

So $\sum_f i_X = (-1)^n$. This completes the proof of Theorem 2. □

Proof of Proposition 3. Note that the unique finite singular point of the differential system (2) is the origin. To compute its index we proceed as in the proof of Theorem 1 and since the map $\mathbb{S}^n \to \mathbb{S}^n$ defined by $(z_1, \dots, z_n) \mapsto (z_1, z_2, -z_3, \dots, -z_n)$ has a unique preimage whose determinant of the Jacobian matrix has determinant $(-1)^n$, the sum of the indices of all the finite singular points of (2) on $p(X)$ is indeed $(-1)^n$, but clearly the differential system (2) is not bounded. □

3. Proof of Theorem 4

Assume first that X is bounded. Let n be the degree of X. Without loss of generality we can assume that the infinite singular point q is at the origin $(0,0)$ in the local chart U_1 (because we can always make a rotation to place it at the origin if necessary). We separate the proof in two cases.

Case 1: n is even. Then from the proof of Proposition 2.2 of [2] we know that the local phase portrait of $p(X)$ at q is topologically equivalent to one of the local phase portraits of Figure 2.2 of [2].

We claim that all the local phase portraits of Figure 2.2 satisfy one of the conditions (i) or (ii). Now we prove the claim. In the local chart U_1 the expression of the vector field X is

$$\begin{aligned} \dot u = {} & P_d^2(1,u) - uP_d^1(1,u) + v(P_{d-1}^2(1,u) - uP_{d-1}^1(1,u)) + \cdots \\ & + v^d(P_0^2(1,u) - uP_0^1(1,u)), \\ \dot v = {} & -vP_d^1(1,u) - v^2P_{d-1}^1(1,u) - \cdots - v^{d+1}P_0^1(1,u). \end{aligned}$$

By looking at the phase portraits (a_1), (a_2), (b_1) and (b_2) in Figure 2.2 we can see that k odd and all the orbits in the local chart U_1 have $\dot v|_{u=0} > 0$ for $v > 0$ sufficiently small, which is in fact statement (i).

While for the local phase portraits (c_2), (c_4), (d_2) and (d_4) in Figure 2.2 k even and all the orbits in the local chart U_1 have $\dot v|_{u=0} > 0$ for $v > 0$ sufficiently small, which correspond to statement (ii.1)

Moreover, for the local phase portraits (c_1) and (c_3) in Figure 2.2 k even, $a_k > 0$, all the orbits in the local chart U_1 have $\dot u|_{u=0} > 0$ for

$v > 0$ sufficiently small and there is a degenerate hyperbolic sector in $v > 0$, while for the local phase portraits (d_1) and (d_3) in Figure 2.2 k even, $a_k < 0$, all the orbits in the local chart U_1 satisfy $\dot{u}|_{u=0} < 0$ for $v > 0$ sufficiently small and there is a degenerate hyperbolic sector in $v > 0$, which is in fact statement (ii.2).

Case 2: n is odd. Then from the proof of Proposition 2.4 of [2] we know that the local phase portrait of $p(X)$ at q is topologically equivalent to one of the local phase portraits of Figure 2.4 of [2].

We claim that all the local phase portraits of Figure 2.4 satisfy one of the conditions (i) or (ii). Now we prove the claim. By looking at the local phase portrait (a) and at all the local phase portraits (b_1) in Figure 2.4 we see that that k is odd and all the orbits in the local chart U_1 are such that $\dot{v}|_{u=0} > 0$ for $v > 0$ sufficiently small, which is in fact statement (i).

While for the second and fourth local phase portraits (c_1), the second local phase portraits (d_2), and the local phase portrait (d_4) in Figure 2.4 k even and all the orbits in the local chart U_1 have $\dot{v}|_{u=0} > 0$ for $v > 0$ sufficiently small, which correspond to statement (ii.1)

Moreover, for the first and third local phase portraits (c_1) in Figure 2.4 k even, $a_k > 0$, all the orbits in the local chart U_1 have $\dot{u}|_{u=0} > 0$ for $v > 0$ sufficiently small and there is a degenerate hyperbolic sector in $v > 0$, while for the first local phase portrait (d_2) and the local phase portrait (d_3) in Figure 2.4 k even, $a_k < 0$, all the orbits in the local chart U_1 satisfy $\dot{u}|_{u=0} < 0$ for $v > 0$ sufficiently small and there is a degenerate hyperbolic sector in $v > 0$, which is in fact statement (ii.2).

We shall prove the converse implication. As before we can consider the infinite singular point q at the origin of the local chart U_1, doing a rotation of the vector field with respect to the origin of coordinates if necessary. Now we claim that there are no solutions in a neighborhood of $(0,0) \in U_1$ contained in $\{v > 0\}$ whose ω–limit is q. Consequently the polynomial vector field will be bounded.

We prove the claim. Since

$$\dot{v}|_{u=0} = -vP_d^1(1,0) - v^2P_{d-1}^1(1,0) - \cdots - v^{d+1}P_0^1(1,0),$$

If condition (i) or (ii.1) holds then $\dot{v}|_{u=0} > 0$ for $v > 0$ sufficiently small, so in a neighborhood of the $(0,0) \in U_1$ we have that $\dot{v} > 0$, and consequently a finite orbit contained in the interior of $\mathbb{D}^2$ can not have as ω–limit the singular point $(0,0)$.

Since

$$\dot{u}|_{u=0} = P_d^2(1,0) + vP_{d-1}^2(1,0) + \cdots + v^d P_0^2(1,0),$$

and $a_k \neq 0$, if condition (ii.2) holds then $\dot{u}|_{u=0} \neq 0$ for $v > 0$ sufficiently small and there is a degenerate hyperbolic sector in $v > 0$, and again a finite orbit contained in the interior of $\mathbb{D}^2$ can not have as ω–limit the singular point $(0,0)$. In summary the claim is proved, and therefore Theorem 4.

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