TIPS OF TONGUES IN THE DOUBLE STANDARD FAMILY

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Abstract. We answer a question raised by Misiurewicz and Rodrigues concerning the family of degree 2 circle maps $F_\lambda : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ defined by

$$F_\lambda(x) := 2x + a + \frac{b}{\pi}\sin(2\pi x) \quad \text{with} \quad \lambda := (a,b) \in \mathbb{R}/\mathbb{Z} \times (0,1).$$

We prove that if $F_\lambda^{\circ n} - \mathrm{id}$ has a zero of multiplicity 3 in $\mathbb{R}/\mathbb{Z}$, then there is a system of local coordinates $(\alpha,\beta) : W \to \mathbb{R}^2$ defined in a neighborhood W of λ, such that $\alpha(\lambda) = \beta(\lambda) = 0$ and $F_\mu^{\circ n} - \mathrm{id}$ has a multiple zero with $\mu \in W$ if and only if $\beta^3(\mu) = \alpha^2(\mu)$. This shows that the tips of tongues are regular cusps.

Introduction

Following Misiurewicz and Rodrigues [MR07], we consider the family of double standard maps of the circle $F_\lambda : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ defined by

$$F_\lambda(x) := 2x + a + \frac{b}{\pi}\sin(2\pi x) \quad \text{with} \quad \lambda := (a,b) \in \mathbb{R}/\mathbb{Z} \times [0,1].$$

If $b \in [0,1/2)$, then $F_\lambda : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is expanding and all periodic cycles of F_λ in $\mathbb{R}/\mathbb{Z}$ are repelling. If $b \in [1/2,1]$, it may happen that $F_\lambda : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has a non-repelling cycle. The multiplier of such a cycle belongs to $[0,1]$. There is at most one such cycle. Connected components of the open sets of parameters $\lambda \in (a,b) \in \mathbb{R}/\mathbb{Z} \times [0,1]$ for which F_λ has an attracting cycle are called *tongues* (see [MR07] and [D10]). The period of the attracting cycle remains constant in each tongue, and is called the period of the tongue

Let T be a tongue of period $p \geq 1$. The boundary of T consists of two smooth curves which are graphs with respect to b and intersect tangentially at the tip $\lambda_T \in \mathbb{R}/\mathbb{Z} \times (0,1)$ (see [MR07, MR08] and Figure 1). If $\lambda \in \partial T$ then F_λ has a cycle of period p and multiplier 1. On the one hand, if $\lambda \in \partial T \smallsetminus \{\lambda_T\}$, then the points of the cycle are double zeros of $F_\lambda^{\circ p} - \mathrm{id}$. On the other hand, for the tip parameter λ_T the points of the cycle are triple zeros of $F_{\lambda_T}^{\circ p} - \mathrm{id}$. Moreover, there is a cusp bifurcation which takes place around λ_T (see [HK91] for an introduction to cusp bifurcations).

It arises as a relevant topic to understand the shape of a given tongue T near its tip λ_T. This can be studied in terms of the order of contact of its boundary curves. Let $B_1(b)$ and $B_2(b)$ be the parametrizations of the boundary curves of T

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