LIMIT CYCLES OF PIECEWISE DIFFERENTIAL EQUATIONS ON THE CYLINDER

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Abstract. We consider the piecewise differential equations of the form

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{cases} a_0(t) + a_1(t)x_1 + ... + a_n(t)x_1^n, & \text{if} \quad 0 \leqslant t \leqslant \pi, \\ b_0(t) + b_1(t)x_2 + ... + b_m(t)x_2^m, & \text{if} \quad \pi \leqslant t \leqslant 2\pi, \end{cases}$$

where $a_0(t), a_1(t), ..., a_n(t)$ and $b_0(t), b_1(t), ..., b_m(t)$ are 2π-periodic functions in the variable t, and we study the number of limit cycles of such equations on the cylinder. In this way we give exact bounds for the maximum number of limit cycles that the piecewise differential equations have in function of n and m.

1. Introduction and statements of the main results

Pugh proposed the following problem (see [7]). Let $a_0, a_1, ..., a_n : [0, 2\pi] \to \mathbb{R}$ be analytic functions and consider the differential equation

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + ... + a_n(t)x^n, \qquad 0 \leqslant t \leqslant 2\pi. \tag{1}$$

A solution $x(t)$ of (1) is called a *closed solution* or a *periodic solution* if it is defined in the interval $[0, 2\pi]$ and $x(0) = x(2\pi)$. The adjectives closed and periodic are motivated by the case where $a_0, ..., a_n$ are 2π-periodic, in which (1) can be considered in the cylinder and the closed solutions really correspond to closed orbits in the cylinder. Closed orbits in polynomial planar differential systems can be isolated or belong to an annulus of periodic orbits. In the isolated case they are called *limit cycles*. So the problem is this: *Is there a bound on the number of limit cycles of* (1)*?*

We note that the differential equation (1) with $n = 1$ (resp. $n = 2$) is a *linear equation* (resp. a *Riccati equation*), and when $n = 3$, (1) is called an *Abel equation*. It is well known that linear (resp. Riccati) equations have either a continuum of periodic solutions or at most 1 (resp. 2) periodic solutions. For the Abel equation it was proved that for any k there exist equations (1) with $a_i(t)$ trigonometric 2π-periodic polynomials, having at least k limit cycles. A similar result holds for $n > 3$, see for more details [7, 9, 1].

In this paper we deal with the piecewise differential equation

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{cases} a_0(t) + a_1(t)x_1 + ... + a_n(t)x_1^n, & \text{if} \quad 0 \leqslant t \leqslant \pi, \\ b_0(t) + b_1(t)x_2 + ... + b_m(t)x_2^m, & \text{if} \quad \pi \leqslant t \leqslant 2\pi, \end{cases} \tag{2}$$

2010 *Mathematics Subject Classification.* 37G15.
Key words and phrases. Limit cycles, piecewise smooth system, Hilbert number, differential equations on the cylinder.