PHASE PORTRAITS OF A CLASS OF CUBIC SYSTEMS WITH AN ELLIPSE AND A STRAIGHT LINE AS INVARIANT ALGEBRAIC CURVES

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Abstract. In this paper we classify the phase portraits in the Poincaré disc of a class of cubic polynomial differential systems having an invariant ellipse and an invariant straight line. We prove that a such class of cubic polynomial differential systems have exactly 49 different topological phase portraits in the Poincaré disc.

1. Introduction and statements of the main results

The *phase portrait* of a differential system defined in the plane $\mathbb{R}^2$ consists in describing $\mathbb{R}^2$ as union of all the orbits of the differential system. To provide the phase portrait of a differential system is to provide the maximal qualitative information about its dynamics. This is the best information which can be given for a differential system whose orbits cannot be given explicitly in function of the time.

A *polynomial differential system* in the plane $\mathbb{R}^2$ is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

where P and Q are real polynomials in the variables x and y, and the independent variable t usually is called the *time*. The maximum degree of the polynomials P and Q is called the *degree* of the polynomial differential system. The vector field $\mathcal{X}$ associated to system (1) is

$$\mathcal{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y},$$

or simply $\mathcal{X} = (P, Q)$.

The phase portrait of a polynomial differential system in the plane usually is presented in the Poincaré disc because then we can control the orbits which go to or come from the infinity. Roughly speaking the Poincaré disc is the closed unit disc centered at the origin of $\mathbb{R}^2$, its interior is identified with $\mathbb{R}^2$, and its boundary the circle $\mathbb{S}^1$ is identified with the infinity of $\mathbb{R}^2$. Note that in the plane $\mathbb{R}^2$ we can go to infinity in as many as directions as points has the circle $\mathbb{S}^1$. See subsection 2.2 for more details on the Poincaré compactification and the Poincaré disc.

The polynomial differential systems of degree one are the *linear differential systems*, and it is well known that these differential systems can be solved explicitly and their phase portraits are known. So the next polynomial differential systems are the ones of degree two usually called *quadratic systems* there are more than one thousand of papers about these systems, see for instance the books of Ye Yanquian [21], Reyn [17] and Artés et al. [2], and the references cited in these books. Many subclasses of quadratic systems have been studied, thus for instance the phase portraits of all quadratic systems having centers (see [3, 4, 9, 10, 18, 19, 22]), or the phase portraits of all quadratic Hamiltonian systems (see [1, 3, 8]), or the phase portraits of all quadratic systems having an invariant ellipse (see [11, 12]), or the phase portraits of all quadratic systems having an invariant ellipse and an invariant straight line (see [13]), ...

After the quadratic systems come the *cubic systems*, i.e. the polynomial differential systems of degree three. Very few things are known for the cubic systems if we compare these with the quadratic ones. Thus for instance are unknown the phase portraits of all cubic systems having centers, or the cubic Hamiltonian systems, or of the cubic systems having an ellipse, ... Here start the classification of the phase portraits of the cubic systems having an invariant ellipse and an invariant straight line.

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