QUADRATIC PLANAR DIFFERENTIAL SYSTEMS WITH ALGEBRAIC LIMIT CYCLES VIA QUADRATIC PLANE CREMONA MAPS

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Abstract. In this paper we show how we can transform quadratic systems into new quadratic systems after some kind of birational transformations, the quadratic plane Cremona maps. We afterwards apply these transformations to the families of quadratic differential systems having an algebraic limit cycle. As a consequence, we provide a new family of quadratic systems having an algebraic limit cycle of degree 5. Moreover we show how the known families of quadratic differential systems having an algebraic limit cycle of degree greater than four are obtained using these transformations. We also provide the phase portraits on the Poincaré disk of all the families of quadratic differential systems having algebraic limit cycles.

1. Introduction

We consider the quadratic planar differential system

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y), \tag{1}$$

where $p(x, y)$ and $q(x, y)$ are real coprime polynomials of degree two. Let $f \in \mathbb{R}[x, y]$. We say that $f = f(x, y) = 0$ is an *invariant algebraic curve* of system (1) if it satisfies

$$p(x, y)\frac{\partial f}{\partial x}(x, y) + q(x, y)\frac{\partial f}{\partial y}(x, y) = k(x, y)f(x, y), \tag{2}$$

for some $k(x, y)$ polynomial of degree at most 1 called the *cofactor* of $f(x, y) = 0$. If $f \in \mathbb{R}[x, y]$ has degree n, it is irreducible in $\mathbb{R}[x, y]$ and $f = 0$ is an invariant algebraic curve, then we say that $f = 0$ is an *irreducible invariant algebraic curve of degree n*.

A *limit cycle* of system (1) is an isolated periodic solution in the set of all periodic solutions of the system. If a limit cycle is contained into the set of points of an invariant algebraic curve, then it is called an *algebraic limit cycle*. We say that an algebraic limit cycle has *degree n* if it is contained into the set of points of an irreducible invariant algebraic curve of degree n.

One of the most interesting questions on limit cycles was proposed by Hilbert [19] in 1900 in the second part of 16^{th} Hilbert's Problem: *Compute $H(m)$ such that the number of limit cycles of any polynomial differential system of degree m is less than or equal to $H(m)$.*

Hilbert's Problem remains unsolved even for $m = 2$. It is known that a quadratic system with an invariant straight line has at most one limit cycle (see [11] or [12]).

The paper is structured as follows. In section 2 we provide the known families of planar quadratic differential systems having an algebraic limit cycle. We also provide their phase portrait in the Poincaré disk, which was never done before. In section 3 we first introduce the plane Cremona maps, in particular the quadratic ones. Afterwards we state and prove some results connecting local and global behavior that allow us to know *a priori* whether a Cremona transformation can be applied to obtain a new quadratic system, according to the local behavior of the base points. The degree of the transformed algebraic curve is also computed. To finish this section we study the particular case of quadratic Cremona maps applied to quadratic differential systems, which is the main aim of this work. The

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