LOWER BOUNDS FOR THE LOCAL CYCLICITY OF CENTERS USING HIGHER ORDER DEVELOPMENTS AND PARALLELIZATION

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Abstract. We are interested in small-amplitude isolated periodic orbits, so called limit cycles, surrounding only one equilibrium point, that we locate at the origin. We develop a parallelization technique to study higher order developments, with respect to the parameters, of the return map near the origin. This technique is useful to study lower bounds for the local cyclicity of centers. We denote by $M(n)$ the maximum number of limit cycles bifurcating from the origin via a degenerate Hopf bifurcation for a polynomial vector field of degree n. We get lower bounds for the local cyclicity of some known cubic centers and we prove that $M(4) \geq 20$, $M(5) \geq 33$, $M(7) \geq 61$, $M(8) \geq 76$, and $M(9) \geq 88$.

1. Introduction

Hilbert early last century presented a list of problems that almost all of them are solved. One problem that remains opened is the second part of the 16th Hilbert's problem: It consists in determine the maximal number $H(n)$ of limit cycles, and their relative positions, of planar polynomial vector fields of degree n. In last years have been proposed other related problems. In 1977, Arnold in [2] proposed a weakened version, focused on the study of the number of limit cycles bifurcating from the period annulus of Hamiltonians systems. We are interested here in another local version, that consists in to provide the maximum number $M(n)$ of small-amplitude limit cycles bifurcating from an elementary center or an elementary focus, clearly $M(n) \leq H(n)$. In other words, $M(n)$ is an upper bound of the cyclicity of such equilibrium points. For more details, we refer to [25]. For $n = 2$, Bautin proved in [4] that $M(2) = 3$. Sibirskiĭ in [26] proved that for cubic systems without quadratic terms there are no more than five limit cycles bifurcating from one critical point. In fact, these are the unique general families for which this local number is completely determined. The first evidence that $M(3) \geq 11$ was presented by Żołądek in [30]. Providing a center with very high local cyclicity. This problem was recently revisited by himself in [32]. The first proof of this fact was done by Christopher in [10], studying first order perturbations of another cubic center also provided by Żołądek in [31]. Basically the used technique consists in to choose a point on the center variety and at this point consider the linear terms, $L_k^{(1)}$, of the *Lyapunov constants*. If the point is chosen on a component of the center variety of codimension r, then the first r linear terms of the Lyapunov constants are independent, that is, there exist perturbations which can produce $r - 1$ limit cycles, and this number is generically the maximum. Apart from the fact that the solution of the center's problem for vector fields of degree n is unknown, the main problem is how to compute the codimension of each component of the center variety. Usually, technique has been used to provide lower bounds for $M(n)$. The idea to study only linear developments, with respect to the parameters, near centers appear previously in [8] and also in [20].

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