# The flow map parameterization method for invariant tori of autonomous Hamiltonian systems

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# Motivation & goals (I)

- A well-established approach for the computation of an invariant curve of a flow (and therefore a 2D torus of a vector field, Gomez, M. 2001 and later Baresi, Olikara, Scheeres) is: to solve its invariance equation,

$$\text{find } K : \mathbb{T}^1 \to \mathbb{R}^{2n} \text{ such that } \varphi_T\big(K(\theta)\big) = K(\theta+\rho) \quad \forall \theta \in \mathbb{T}^1,$$

  by discretizing it (e.g. by collocation) and expanding $K$ as a Fourier series. This is, if $K(\theta) = A_0 + \sum_{k=1}^{N_f} \big(A_l \cos(2\pi k\theta) + B_l \sin(2\pi k\theta)\big)$, then

$$\{\varphi_T\big(K(\tfrac{j}{1+2N_f})\big) - K(\tfrac{j}{1+2N_f} + \rho) = 0\}_{j=0}^{N_f}$$

  is a finite system of non–linear equations in $\{A_k\}_{k=0}^{N_f}$, $\{B_k\}_{k=1}^{N_f}$, that can be solved by Newton iterations.
- This approach suffers from the *large-matrix problem*: for $N_f$ (not very) large, the solution of the linear system of each Newton iteration ($O\big((1+2N_f)^3\big)$ ops) outweighs the remaining computational effort.

- The parameterization method for the computation of invariant tori (Haro, de la Llave 2006-2007 and, later, Anderson, Calleja, Canadell, Celletti, Figueras, Gimeno, González, Kumar, Luque. . . ), inspired in the parameterization method for fixed points of flows (Cabré, Fontich, de la Llave 2003-2005 and, later, many more people), avoids this problem.
- Idea:
  1. Do **not** discretize the invariance eqs. $\varphi_T(K(\theta)) = K(\theta + \rho)\ \forall \theta \in \mathbb{T}^1$. Instead:
  2. Perform the Newton corrections **at the functional level**.
  3. Use simplectic geometry to find a coordinate change that simplifies the linear system to be solved (block upper-triangular with diagonal diagonal blocks).
- As of 2017, it still had not been attempted in the context of lower-dimensional (non–Lagrangian), partially hyperbolic invariant tori of Hamiltonian flows.

  This is the case of the tori around the collinear points of the circular, spatial RTBP (3-dof). The tori are 2D and have 1D stable and unstable manifolds.
- Our goal is to develop algorithms for this case and test them.
- As of now, a philosophically similar development but that differs in formulation and applications can be found in Kumar, Anderson, de la Llave, 2021-2022.

# Outline

# Invariance equations

- Assume we are given a Hamiltonian flow in $\mathbb{R}^{2n}$ with the standard symplectic structure (everything works with an arbitrary exact symplectic form).
  Denote the Hamiltonian as $H$, the flow as $\varphi_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$,
  the system of ODE as $\dot{x} = X_H(x)$.
- We want to compute **lower-dimensional** ($d < n$), **partially hyperbolic invariant tori**, with rank 1 stable and unstable bundles.
- Invariance equation for a $d$-torus parameterized by $\hat{K} : \mathbb{T}^d \to \mathbb{R}^{2n}$:

$$\varphi_t\big(\hat{K}(\hat{\theta})\big) = \hat{K}(\hat{\theta} + t\hat{\omega}), \quad \hat{\omega} \text{ non-resonant}$$

- Invariance equation for a rank-one bundle over $\mathbb{T}^d$ with exponent $\chi$ parameterized by $\hat{W} : \mathbb{T}^d \to \mathbb{R}^{2n}$:

$$\mathrm{D}\varphi_t\big(\hat{K}(\hat{\theta})\big)\, \hat{W}(\hat{\theta}) = e^{t\chi}\, \hat{W}(\hat{\theta} + t\hat{\omega}),$$

- Set $d = n - 1$.

# Reduce dimension by one (flow-map trick)

In order to reduce the dimension of the objects to be computed by one:

- Define $\omega \in \mathbb{R}^{n-2}$ by $\hat{\omega} = \frac{1}{T}(\omega, 1)$.
- Look for $K : \mathbb{T}^{n-2} \to \mathbb{R}^{2n}$ satisfying

$$\varphi_T\big(K(\theta)\big) = K(\theta + \omega).$$

  $\hat{K}$ can be recovered as ($\mathbb{T}^d := \mathbb{R}^d/[0,1]^d$)

$$\hat{K}(\hat{\theta}) = \varphi_{\theta_n T}\big(K(\theta - \theta_n \omega)\big).$$

- Look for $W : \mathbb{T}^{n-2} \to \mathbb{R}^{2n}$ satisfying

$$\mathrm{D}\varphi_T\big(K(\theta)\big)\, W(\theta) = \underbrace{e^{T\chi}}_{:=\lambda}\, W(\theta + \omega).$$

  $\hat{W}$ can be recovered as

$$\hat{W}(\hat{\theta}) = \lambda^{-\theta_n} \mathrm{D}\varphi_{\theta_n T}\big(K(\theta - \theta_n \omega)\big)\, W(\theta - \theta_n \omega).$$

# Invariance equations to solve

For fixed $\omega \in \mathbb{R}^{n-2}$, solve

$$\int_{\mathbb{T}^{n-1}} H(K(\theta))\, d\theta - h = 0,$$
$$\varphi_T\big(K(\theta)\big) - K(\theta+\omega) = 0,$$
$$\mathrm{D}\varphi_T\big(K(\theta)\big)\, W(\theta) - \lambda\, W(\theta+\omega) = 0.$$

It can be done:

- For fixed $T$, find $K$, $h$, $W$, $\lambda$ (**isochronous case**). The first equation can be ignored in this case.
- For fixed $h$, find $K$, $T$, $W$, $\lambda$ (**iso-energetic case**).

For shortness, in all the following we will focus in the isochronous case.

$\lambda \approx 1000$ requires **multiple shooting**: find $\{K_i\}_{i=0}^{m-1}$, $\{W_i\}_{i=0}^{m-1}$, all parameterizations $\mathbb{T}^{n-2} \to \mathbb{R}^{2n}$, satisfying

$$\varphi_{T/m}\big(K_i(\theta)\big) - K_{i+1}\big(\theta + \tfrac{\omega}{m}\big) = 0,$$
$$\mathrm{D}\varphi_{T/m}\big(K_i(\theta)\big)\, W_i(\theta) - \lambda\, W_{i+1}\big(\theta + \tfrac{\omega}{m}\big) = 0.$$

where $K_m := K_0$, $W_m := W_0$.

For shortness, in all the following we will assume $m = 1$.

# Outline

# Construction of the frame

- We want to perform **Newton corrections** and **continuation predictions** on the invariance equations at the functional level.
  This is, **before discretization**, in order to avoid large matrices.
- We want to find an **adapted frame** in order to simplify the equations obtained (from Newton corrections or continuation predictions).
- By differentiating the invariance equations,

$$L(\theta) := \begin{pmatrix} DK(\theta), \; X_H(K(\theta)), \; W(\theta) \end{pmatrix} \implies D\varphi_T(K(\theta))L(\theta) = L(\theta+\omega)\begin{pmatrix} I_{n-1} & \\ & \lambda \end{pmatrix}.$$

- By symplectic geometry, for

$$\hat{N}(\theta) := JL(\theta)\left(L(\theta)^\top L(\theta)\right)^{-1} \in \mathbb{R}^{2n\times n}, \quad \hat{P}(\theta) := \begin{pmatrix} L(\theta), \; \hat{N}(\theta) \end{pmatrix} \in \mathbb{R}^{2n\times 2n},$$

  we have

$$D\varphi_T(K(\theta))\hat{P}(\theta) = \hat{P}(\theta+\omega)\left(\begin{array}{cc|cc} I_{n-1} & & \multicolumn{2}{c}{\hat{S}(\theta)} \\ & \lambda & & \\ \hline & & I_{n-1} & \\ & & & \lambda^{-1} \end{array}\right),$$

# Construction of the frame (II)

- In the last expression,

$$\hat{S}(\theta) = \hat{N}(\theta+\omega)^\top \left(\begin{array}{c|c} 0 & -I_n \\ \hline I_n & 0 \end{array}\right) \boxed{D\varphi_T(K(\theta))\hat{N}(\theta)}.$$

  The $\boxed{\text{squared term}}$ requires numerical integration ($(1+n)\cdot 2n$ ODE).

- A further simplification, that amounts to solve $n$ non-small divisors cohomological equations (see later), allows to obtain a symplectic matrix $Q(\theta)$ such that, for $P(\theta) := \hat{P}(\theta)Q(\theta)$, we have

$$D\varphi_T(K(\theta))P(\theta) = P(\theta+\omega) \left(\begin{array}{cc|cc} I_{n-1} & & S^1(\theta) & 0 \\ & \lambda & 0 & 0 \\ \hline & & I_{n-1} & \\ & & & \lambda^{-1} \end{array}\right),$$

- As a byproduct, the last column of $P(\theta)$ is the rank-one invariant bundle over $\mathbb{T}^{n-2}$ with multiplier $\lambda^{-1}$.

# Solving cohomological equations (I)

Performing Newton and continuation steps will amount to:

- solving small divisors and "non-small-divisors" cohomological equations , and
- performing back-substitutions.

A "non-small divisors" cohomological equation for $\eta, \xi : \mathbb{T}^l \to \mathbb{R}$ ($\eta$ data, $\xi$ unknown) is,

$$\lambda \xi(\theta) - \mu \xi(\theta + \omega) = \eta(\theta),$$

for $\lambda, \mu \in \mathbb{R}$, $|\lambda| \neq |\mu|$. It is solved by Fourier series: by defining

$$\xi(\theta) = \sum_{k \in \mathbb{Z}^l} \hat{\xi}_k e^{\mathbf{i} 2\pi k \theta}, \quad \eta(\theta) = \sum_{k \in \mathbb{Z}^l} \hat{\eta}_k e^{\mathbf{i} 2\pi k \theta},$$

the solution is given by

$$\hat{\xi}_k = \frac{\hat{\eta}_k}{\lambda - \mu e^{\mathbf{i} 2\pi k \omega}}.$$

The denominators stay away from zero.

# Solving cohomological equations (II)

A small divisors cohomological equation for $\eta, \xi : \mathbb{T}^l \to \mathbb{R}$ ($\eta$ data, $\xi$ unknown),

$$\xi(\theta) - \xi(\theta + \omega) = \eta(\theta),$$

is analogously solved: by denoting

$$\mathscr{R}\eta(\theta) = \sum_{k \neq 0} \hat{\xi}_k e^{i2\pi k\theta},$$

for

$$\hat{\xi}_k = \frac{\hat{\eta}_k}{1 - e^{i2\pi k\omega}},$$

then, **as long as** $\langle \eta \rangle = 0$, the general solution is

$$\xi(\theta) = \xi_0 + \mathscr{R}\eta(\theta),$$

for $\xi_0 \in \mathbb{R}$ free.
($\omega$ needs to be Diophantine to have convergence)

# Outline

# Newton step in the torus (I)

Assume $K(\theta)$ satisfies the invariance equation approximately:

$$E(\theta) = \boxed{\varphi_T(K(\theta))} - K(\theta+\omega) \qquad \text{is small.}$$

(The squared term requires numerical integration.)

- Compute the frame $P : \mathbb{T}^{n-2} \to \mathbb{R}^{2n\times 2n}$, consider a correction of the form $\Delta K(\theta) = P(\theta)\xi(\theta)$, and substitute above:

$$\begin{aligned}
0 &\overset{!}{=} \varphi_T\big(K(\theta)+P(\theta)\xi(\theta)\big) - \big(K(\theta+\omega)+P(\theta+\omega)\xi(\theta+\omega)\big) \\
&= \varphi_T\big(K(\theta)\big) + D\varphi_T\big(K(\theta)\big)P(\theta)\xi(\theta) + O\big((P(\theta)\xi(\theta))^2\big) \\
&\qquad -K(\theta+\omega) - P(\theta+\omega)\xi(\theta+\omega) \\
&\approx E(\theta) + D\varphi_T\big(K(\theta)\big)P(\theta)\xi(\theta) - P(\theta+\omega)\xi(\theta+\omega).
\end{aligned}$$

- Multiply by $P(\theta+\omega)^{-1}$:

$$P(\theta+\omega)^{-1}D\varphi_T\big(K(\theta)\big)P(\theta)\xi(\theta) - \xi(\theta+\omega) = -P(\theta+\omega)^{-1}E(\theta) =: \eta(\theta).$$

# Newton step in the torus (II)

- Now, by construction of the frame,

$$P(\theta+\omega)^{-1} D\varphi_T\big(K(\theta)\big) P(\theta) = \left(\begin{array}{cc|cc} I_{n-1} & & S(\theta) & 0 \\ & \lambda & 0 & 0 \\ \hline & & I_{n-1} & \\ & & & \lambda^{-1} \end{array}\right) + O(E).$$

- The $O(E)$ is multiplied by $\xi(\theta) \Rightarrow$ quadratically small $\Rightarrow$ we neglect it
  The equations become

$$\left(\begin{array}{cc|cc} I_{n-1} & & S(\theta) & 0 \\ & \lambda & 0 & 0 \\ \hline & & I_{n-1} & \\ & & & \lambda^{-1} \end{array}\right) \begin{pmatrix} \xi_1(\theta) \\ \xi_2(\theta) \\ \xi_3(\theta) \\ \xi_4(\theta) \end{pmatrix} - \begin{pmatrix} \xi_1(\theta+\omega) \\ \xi_2(\theta+\omega) \\ \xi_3(\theta+\omega) \\ \xi_4(\theta+\omega) \end{pmatrix} = \eta(\theta)$$

# Newton step in the torus (III)

We need to solve

$$\xi_1(\theta) - \xi_1(\theta+\omega) = \eta_1(\theta) - S(\theta)\xi_3(\theta) \quad (1)$$
$$\lambda\xi_2(\theta) - \xi_2(\theta+\omega) = \eta_2(\theta) \quad (2)$$
$$\xi_3(\theta) - \xi_3(\theta+\omega) = \eta_3(\theta) \quad (3)$$
$$\lambda^{-1}\xi_4(\theta) - \xi_4(\theta+\omega) = \eta_4(\theta) \quad (4)$$

- (2), (4) are solved as non-small divisors cohomological equations.
- (3) is solved as small divisors cohomological equation, $\xi_{3,0} := \langle\xi_3\rangle$ free ($\langle\eta_3\rangle$ quadratically small by symplectic geometry).
- Adjust $\xi_{3,0}$ for RHS of (1) to have zero average,

  $$\langle S\rangle\xi_{3,0} = \langle\eta_1 - S\mathscr{R}\eta_3\rangle,$$

  and solve (1) as small divisors cohomological equation.
- $\xi_{1,0} := \langle\xi_1\rangle$ is free (arbitrary phase origin of $K$, arbitrary choice of $(n-2)$-torus inside the $n-1$ one). **Can be taken as zero.**

# Newton step on the bundle, and continuation steps

- A Newton step in the bundle is deduced analogously, by taking frame coordinates on
$$E^W(\theta) = \boxed{D\varphi_T\big(K(\theta)\big)\,W(\theta)} - \lambda\, W(\theta+\omega).$$
- A continuation step in the torus is deduced by considering $K_T(\theta)$ a function of $T$ defined implicitly by the invariance equation
$$\varphi_T\big(K(\theta)\big) - K(\theta+\omega) = 0,$$
differentiating w.r.t $T$ and solving for $\partial_T K(\theta)$ in frame coordinates.
- A continuation step in the bundle is deduced by proceeding analogously on
$$D\varphi_T\big(K(\theta)\big)\,W(\theta) - \lambda\, W(\theta+\omega) = 0.$$
In this case, the RHS of the equations to solve for $\partial_T W_T(\theta)$ in frame coordinates is
$$\eta(\theta) = -P(\theta+\omega)^{-1}\,\boxed{\partial_T\Big(D\varphi_T\big(K(\theta)\big)\Big)\,W(\theta)},$$
The $\boxed{\text{squared term}}$ requires numerical integration of the 2nd variational equations in order to find $\mathrm{D}^2\varphi_T\big(K(\theta)\big)\big[W(\theta),\partial_T K(\theta)\big]$ ($4\cdot 2n$ ODE).

# Outline

# Fourier-grid representation

Given $\zeta : \mathbb{T} = \mathbb{R}/[0,1] \to \mathbb{R}$, we choose $N$ and represent it numerically as

$$\textbf{grid:} \quad \{\zeta_j\}_{j=0}^{N-1}, \qquad \textbf{Fourier:} \quad \{\tilde{\zeta}_k\}_{k=0}^{[N/2]},$$

where $\zeta_j = \zeta(j/N)$, and $\{\tilde{\zeta}_k\}_{k=0}^{[N/2]}$ are approximate Fourier coefficients:

$$\zeta(\theta) \approx \sum_{k=-[N/2]}^{[N/2]} \tilde{\zeta}_k e^{\boldsymbol{i} 2\pi k \theta}.$$

Both representations are related by the (one-dimensional) Discrete Fourier Transform,

$$\tilde{\zeta}_k = \frac{1}{N} \sum_{j=0}^{N-1} \zeta_j e^{-\boldsymbol{i} 2\pi k \frac{j}{N}}, \qquad \zeta_j = \sum_{k=0}^{N-1} \tilde{\zeta}_k e^{\boldsymbol{i} 2\pi k \frac{j}{N}},$$

which is evaluated in $O(N \log N)$ operations through FFT.

# Performing the computations

The evaluation of all the formulae that have appeared can be done (formally) in $O(N)$ operations, either in grid or Fourier form.
(Going from grid to Fourier and vice-versa is $O(N \log N)$ ops.)
Consider e.g. all the formulae involved in computing a frame:

$$L(\theta) = \begin{pmatrix} DK(\theta) & X_H(K(\theta)) & W(\theta) \end{pmatrix}$$

$$\hat{N}(\theta) = JL(\theta)\left(L(\theta)^\top L(\theta)\right)^{-1}$$

$$\hat{S}(\theta) = \hat{N}(\theta+\omega)^\top \left(\begin{array}{c|c} 0 & -I_n \\ \hline I_n & 0 \end{array}\right) \boxed{D\varphi_T(K(\theta))\hat{N}(\theta)}$$

$$\hat{P}(\theta) = \begin{pmatrix} L(\theta) & \hat{N}(\theta) \end{pmatrix}$$

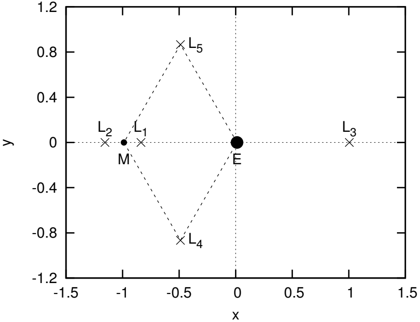
Compute $Q(\theta)$ (by solving $n$ non-small-div. cohomol. eq.)

$$P(\theta) = \hat{P}(\theta)Q(\theta)$$

Nevertheless, for feasible $N$, numerical integration outweighs everything.
(It can be parallelized.)

# Outline

# The $L_1$ point of the RTBP




Hamiltonian of the spatial, circular Restricted Three–Body problem: for $x, p \in \mathbb{R}^3$,

$$
\begin{aligned}
H(x,p) &= \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) \\
&\quad - x_1 p_2 + x_2 p_1 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}
\end{aligned}
$$

with $r_1^2 = (x-\mu)^2 + y^2 + z^2$
$r_2^2 = (x-\mu+1)^2 + y^2 + z^2$

For the Earth-Moon mass parameter, $L_1 = \left(x_{L_1}, 0, 0, 0, x_{L_1}, 0\right)$ with

$$
\operatorname{Spec} \mathrm{D}X_H(L_1) = \{\boldsymbol{i}2\pi\omega_p^0, -\boldsymbol{i}2\pi\omega_p^0, \boldsymbol{i}2\pi\omega_v^0, -\boldsymbol{i}2\pi\omega_v^0, \lambda^0, -\lambda^0\}
$$

and $x_{L_1} \approx -0.83692$, $\omega_p^0 \approx 0.371529, \omega_v^0 \approx 0.361096, \lambda^0 \approx 2.932056$.

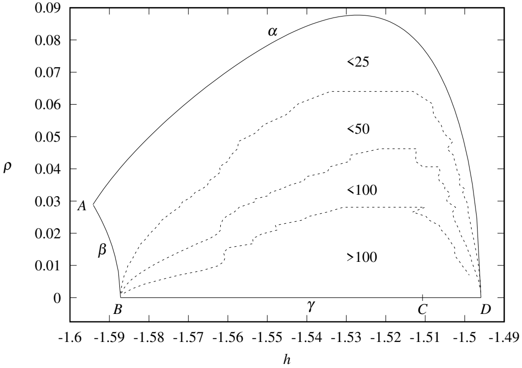
# Periodic orbits and tori around $L_1$

- Lyapunov's center theorem $\Longrightarrow \exists$ 2 families of p.o. parameterized by $h$:
  - $\{\mathrm{lpo}_p^h\}_h$ tangent to $\pm \boldsymbol{i} 2\pi\omega_p^h$ eigenvectors (**planar Lyapunov family**),
  - $\{\mathrm{lpo}_v^h\}_h$ tangent to $\pm \boldsymbol{i} 2\pi\omega_v^h$ eigenvectors (**vertical Lyapunov family**),

  both center $\times$ saddle.
- Also from Lyapunov's center theorem,
  if we denote $e^{\pm \boldsymbol{i} 2\pi\nu_p^h} \in \mathrm{Spec}\big(\mathrm{Monodr}(\mathrm{lpo}_p^h)\big)$, $e^{\pm \boldsymbol{i} 2\pi\nu_v^h} \in \mathrm{Spec}\big(\mathrm{Monodr}(\mathrm{lpo}_v^h)\big)$,
  $$\text{then } e^{\pm \boldsymbol{i} 2\pi\nu_p^h} \xrightarrow{h\to h_0} e^{\pm \boldsymbol{i} 2\pi\omega_v^0/\omega_p^0}, \qquad e^{\pm \boldsymbol{i} 2\pi\nu_v^h} \xrightarrow{h\to h_0} e^{\pm \boldsymbol{i} 2\pi\omega_p^0/\omega_v^0}$$
- $\nu_p^h$, $\nu_v^h$ are taken in $[0,1/2]$, as numerically found.
  It is observed: $\nu_v^h \xrightarrow{h\to h_0} \omega_p^0/\omega_v^0 - 1$
- There also are KAM tori ("Lissajous tori") with "natural" frequencies $\omega_p, \omega_v$,
  taken as to have $(\omega_p, \omega_v) \xrightarrow{\text{torus}\to L_1} (\omega_p^0, \omega_v^0)$.
- The KAM tori can be parameterized by
  - either $(\omega_p, \omega_v)$, or
  - $(h, \nu_v)$, with $\nu_v(\omega_p, \omega_v) := \omega_p/\omega_v - 1$.
- We will represent everything (KAM tori, $\{\mathrm{lpo}_p^h\}_h$, $\{\mathrm{lpo}_v^h\}_h$) in terms of $(h, \nu_v)$.
- We will denote $\nu_v$ as $\rho$, which will also be $\omega$ when computing the KAM tori via the flow map parameterization method.

# Energy-rotation number representation

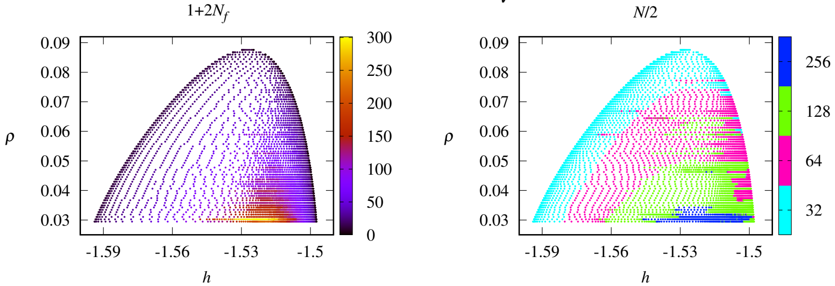
- $\alpha$ curve (from $A$ to $D$): $(h,\rho)$ of the vertical Lyapunov family of p.o.
- $A$ point: $(h,\rho)$ of $L_1$.
- $D$ point: 1st 1:1 bifurcation of the vertical Lyap. family.
- $\beta$ curve (from $A$ to $B$): $(h,\rho)$ of the vertical Lyapunov family of p.o.
- $B$ point: first 1:1 bif. of the planar Lyap. fam. (to the halo one).




- $\gamma$ curve (from $B$ to $D$): separatrices between Lissajous tori and other families.
- $C$ point: 2nd 1:1 bifurcation of the planar Lyap. fam.

# Comparison vs. large matrix method

74 constant-$\omega$ families for noble values of $\omega \geq v_v^0$

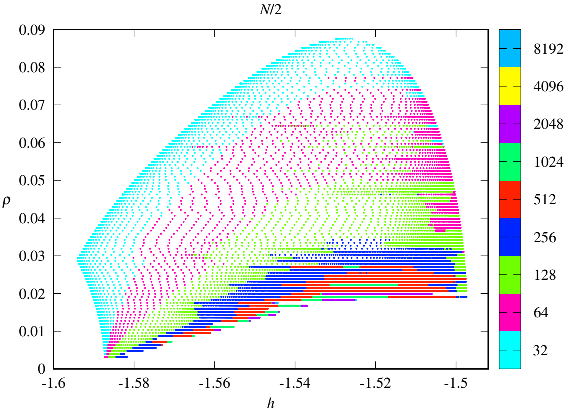



Large matrix approximation: $K(\theta) = A_0 + \sum_{k=1}^{N_f} \left(A_k \cos(2\pi k\theta) + B_k \sin(2\pi k\theta)\right)$

Execution data:

- Large matrix time: 22793 s tori + 45515 s bundles = 68308 s total.
- Parameterization time (tori & bundles): 5992 s
- Factors: $22793/5992 = 3.80$, $68308/5992 = 11.40$,
  (wall-clock on 16-core workstation: $68308/16 = 4269.25$, $5992/16 = 374.5$)
- Number of tori: 4141 large matrix, 7008 parameterization.

# Full exploration of the Lissajous family

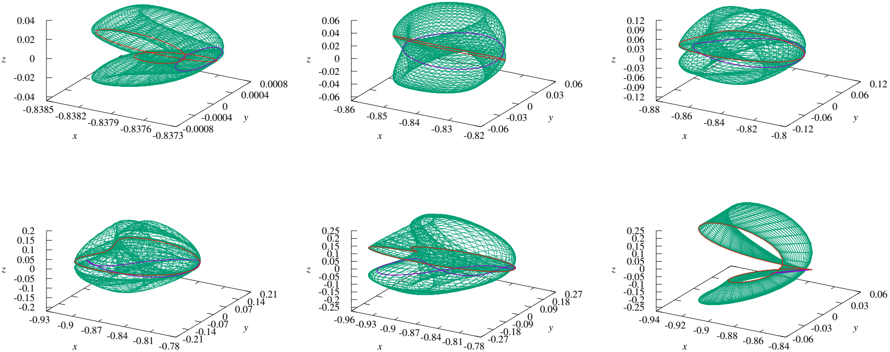



Execution data for $\omega < \nu_v^0$: 31 constant-$\omega$ families, 130574 tori (`.txt.bz2` files), 127.51 GiB, 27.0816 days.

$(27.0816/16 = 1.69)$

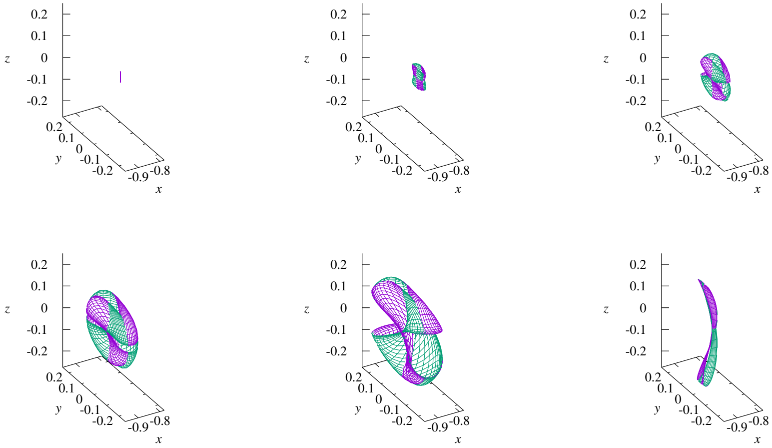
# Sample tori (I)

For $\omega = 0.031865 > \nu_v^0$



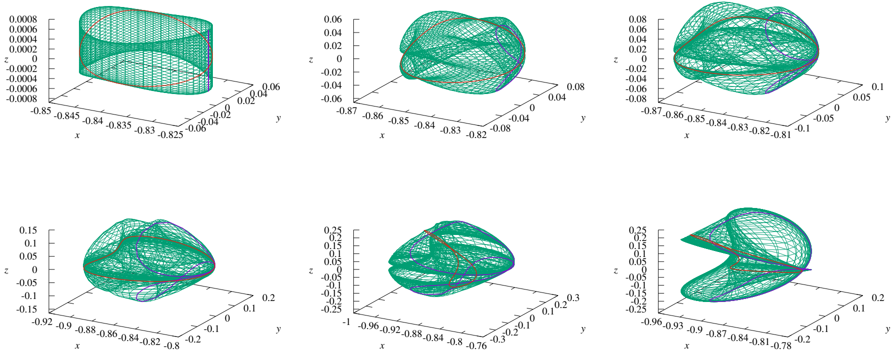
# Sample tori (II)

Same tori ($\omega = 0.031865 > \nu_V^0$), fixed scale:



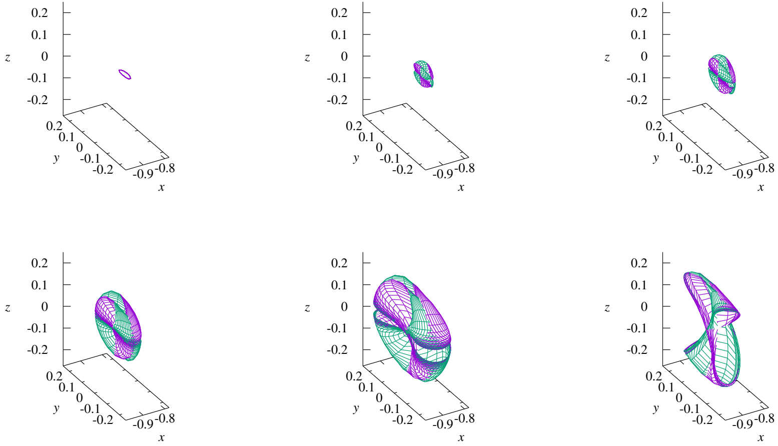
# Sample tori (III)

For $\omega = 0.019091 < \nu_v^0$

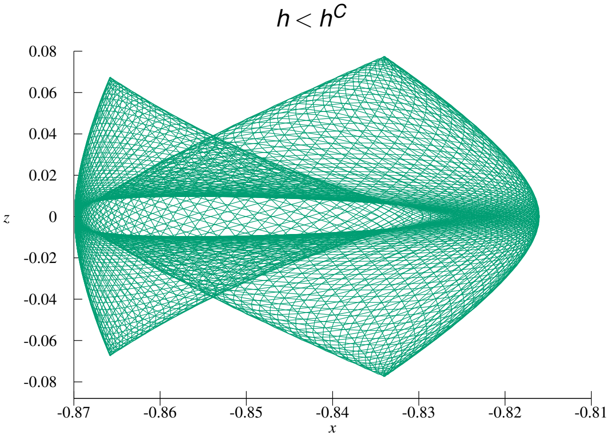


# Sample tori (IV)

Same tori ($\omega = 0.019091 < \nu_v^0$), fixed scale:

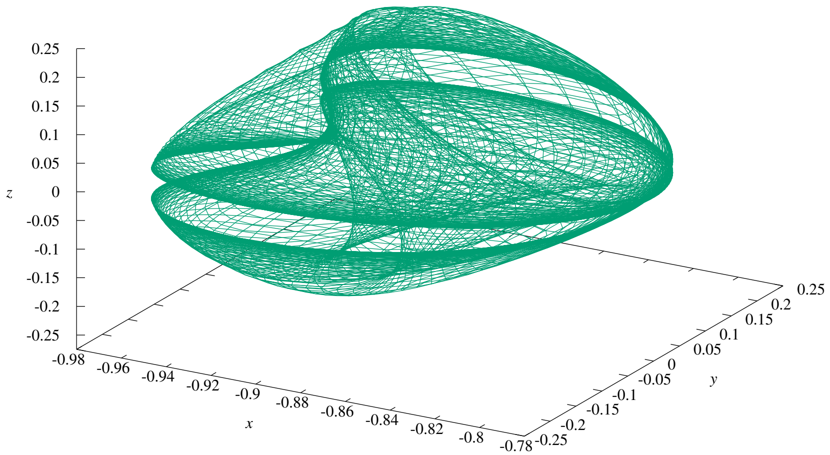


# Sample tori (V)



# Sample tori (VI)

$h > h^C$



# Outline

## Conclusion & Future work

We're happy that the method is fast.
Future work:

- Analysis of the effect of approximating Fourier coefficients through DFT.
  $\longrightarrow$ still no candidate
- Start and end at periodic orbits (the frame becomes singular).
  $\longrightarrow$ Alex working on it.
- Improve the continuation strategy.
  $\longrightarrow$ Miquel Barcelona?
- Extension to periodic and quasi–periodic systems.
  $\longrightarrow$ Alvaro Fernandez (done)
- Proof of convergence.
  $\longrightarrow$ Alvaro Fernandez (quasi-periodic case, close to finished)
- Taylor expansions of invariant manifolds instead of bundles.
  $\longrightarrow$ Alvaro Fernandez (in writing)
- An alternative to flow map would be to impose the invariance equation directly on the vector field
  $\longrightarrow$ Pedro Porras (in progress, Lagrangian case)

# Some references & Thank You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