Detecting nilpotent centers on center manifolds of three-dimensional systems

Lucas Queiroz Arakaki

March 13, 2023

We consider analytical vector fields X in $\mathbb{R}^3$ having a singular point p such that $DX(p)$ has two zero eigenvalues and one real eigenvalue $\lambda \neq 0$ and the rank of $DX(p)$ is 2. The differential system associated to these vector fields can always be written, by an affine change of variables, in the form:

$$\begin{aligned} \dot{x} &= y + P(x,y,z), \\ \dot{y} &= \qquad Q(x,y,z), \\ \dot{z} &= -\lambda z + R(x,y,z), \end{aligned} \tag{1}$$

where P, Q, R are analytic and $j^1P(0) = j^1Q(0) = j^1R(0) = 0$.

By the Center Manifold Theorem, for every $r \geqslant 1$, there exists a bidimensional C^r-manifold invariant by system (1) tangent to the xy-plane at the origin (see [10, 12]).

The restriction of system (1) to a center manifold is a bidimensional system that can be written in the form

$$\begin{aligned} \dot{x} &= y + P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{2}$$

where $j^1P(0) = j^1Q(0) = 0$. Since the restricted system has a nilpotent singular point at the origin, we say that the origin of system (1) is a *nilpotent singular point*.

In the plane, nilpotent singular points are widely studied in the literature [1, 5, 9]. One of the most important problems in the study of these types of singularities is the so-called *Nilpotent Center Problem* [1, 14] which consists of distinguishing whether the origin of (2) is a center or not.

Our goal in this work is to study the Nilpotent Center Problem for three-dimensional system (1), i.e., to determine conditions for the origin to be a nilpotent center on a center manifold.

As far as we know, this is a barely explored problem in the literature. The only two works on this subject [13, 15] study particular families of system (1) by restricting the system to a polynomial approximation of the center manifold.

We seek to study the nilpotent center problem without going through the restriction of the system to a center manifold.

The Monodromy Problem

Theorem 1 (C^r-Andreev's Theorem)

Let X be the vector field associated to the C^r-system, $r \geqslant 3$, given by

$$\dot{x} = y + X_2(x,y), \quad \dot{y} = Y_2(x,y), \tag{3}$$

where $X_2, Y_2 \in C^r$, $j^1X_2(0) = j^1Y_2(0) = 0$ and such that the origin is an isolated singular point. Let $y = F(x)$ be the solution of the equation $y + X_2(x,y) = 0$ through $(0,0)$ and consider $f(x) = Y_2(x, F(x))$ and $\Phi(x) = \mathrm{div}X|_{(x,F(x))}$. We can write

$$f(x) = ax^{\alpha} + O(x^{\alpha+1}), \quad \Phi(x) = bx^{\beta} + O(x^{\beta+1}).$$

for $\alpha < r$. Suppose that $a \neq 0$, then the origin is monodromic if and only if $a < 0, \alpha = 2n-1$ and one of the following conditions holds:

i) $\beta > n-1$ *or* $j^r\Phi(0) \equiv 0$;

ii) $\beta = n-1$ *and* $b^2 + 4an < 0$.

The positive integer n in the statement of Theorem 1 plays an important role in the study of nilpotent monodromic singular points. So we define the *Andreev number* of a nilpotent singular point by the number n in function $f(x) = ax^{2n-1} + O(x^{2n})$ described in Theorem 1.

The positive integer n in the statement of Theorem 1 plays an important role in the study of nilpotent monodromic singular points. So we define the *Andreev number* of a nilpotent singular point by the number n in function $f(x) = ax^{2n-1} + O(x^{2n})$ described in Theorem 1.

In [7] the author proves that the Andreev number is invariant by analytical and formal orbital equivalence, i.e. via analytical and formal diffeomorphisms and time rescalings.

Consider the following representation of analytic system (1):

$$\begin{aligned} \dot{x} &= \quad y + \sum_{j+k+l\geqslant 2} a_{jkl} x^j y^k z^l, \\ \dot{y} &= \sum_{j+k+l\geqslant 2} b_{jkl} x^j y^k z^l, \\ \dot{z} &= -\lambda z + \sum_{j+k+l\geqslant 2} c_{jkl} x^j y^k z^l. \end{aligned} \tag{4}$$

Proposition 2.1

The origin is a nilpotent monodromic singular point with Andreev number 2 on a center manifold of system (4) if and only if $b_{200} = 0$ and

$$\frac{b_{101} c_{200}}{\lambda} < -\frac{(2a_{200} - b_{110})^2}{8} - b_{300}. \tag{5}$$

Moreover, if $2a_{200} + b_{110} \neq 0$, the restricted system satisfies the monodromy condition $\beta = n - 1$ in Theorem 1.

We highlight the singular points with Andreev number 2 since, in the plane, there are interesting tools to study the Center Problem that only hold for this case.

We highlight the singular points with Andreev number 2 since, in the plane, there are interesting tools to study the Center Problem that only hold for this case.

For instance, in [7], the *inverse integrating factor method* is presented which consists in finding a formal inverse integrating factor for such systems.

We now prove some results for planar C^r-systems which are useful for the Nilpotent Center Problem in $\mathbb{R}^3$.

We now prove some results for planar C^r-systems which are useful for the Nilpotent Center Problem in $\mathbb{R}^3$.

Proposition 2.2

Consider a C^r-system (2). Let n be a positive integer such that $2n-1<r$. The origin is a monodromic isolated singular point with Andreev number n if and only if there exists a local analytical change of variables that transforms (2) into the following form

$$\begin{aligned} \dot{x} &= y + \mu x^n + \sum_{i+nj \geqslant n+1} \tilde{a}_{ij} x^i y^j + O(\|x,y\|^r), \\ \dot{y} &= -nx^{2n-1} + n\mu x^{n-1} y + \sum_{i+nj \geqslant 2n} \tilde{b}_{ij} x^i y^j + O(\|x,y\|^r). \end{aligned} \tag{6}$$

Proposition 2.3

For system (6) with monodromic nilpotent singular points having Andreev number n, the first non-zero focal value is the coefficient of a power of ρ_0 whose parity is the same as the parity of n.

Proposition 2.3

For system (6) with monodromic nilpotent singular points having Andreev number n, the first non-zero focal value is the coefficient of a power of ρ_0 whose parity is the same as the parity of n.

Theorem 2

Suppose that the C^r system (2) has nilpotent monodromic singular point with odd Andreev number n. If the system satisfies the monodromy condition $\beta = n - 1$ then the origin is a nilpotent focus.

Using the definition of quasi-homogeneous maps and the expansion of formal functions in quasi-homogeneous terms, we were able to obtain the following result.

Using the definition of quasi-homogeneous maps and the expansion of formal functions in quasi-homogeneous terms, we were able to obtain the following result.

Proposition 2.4

For C^r system (2) having an isolated monodromic nilpotent singular point at the origin, with Andreev number n such that $2n - 1 < r$, the monodromy conditions (i) and (ii) in Theorem 1 are invariant by local diffeomorphisms.

Normal Form

An important step in the study of vector fields having nilpotent singular points is to find normal forms for its associated differential system.

The following theorem, which is proven in [2] gives us a starting point.

Theorem 3 (Belitskii Normal Form)

Consider a formal system of differential equations of form

$$\dot{\mathbf{x}} = A\mathbf{x} + X(\mathbf{x}),$$

where $\mathbf{x} \in \mathbb{R}^n$, $A \in M_n(\mathbb{R})$ *and* $j^1X(0) = 0$. *Then there exists a germ at* 0 *of diffeomorphism* $\mathbf{x} \to \mathbf{x} + \varphi(\mathbf{x})$, $j^1\varphi(0) = 0$, *that transforms the above system into the following:*

$$\dot{\mathbf{x}} = A\mathbf{x} + f(\mathbf{x}),$$

where $j^1 f(0) = 0$ *and*

$$A^t f(\mathbf{x}) - df(\mathbf{x})A^t\mathbf{x} = 0. \qquad (7)$$

Theorem 4 (Nilpotent Normal Form in $\mathbb{R}^3$)

For system (1) *having a nilpotent singular point at the origin, there exist a formal change of variables that transforms it into the formal normal form*

$$\begin{aligned} \dot{x} &= y + xP_1(x), \\ \dot{y} &= Q_2(x) + yP_1(x), \\ \dot{z} &= -\lambda z + zR_1(x). \end{aligned} \tag{8}$$

for which $P_1(0) = j^1 Q_2(0) = R_1(0) = 0$.

Theorem 4 (Nilpotent Normal Form in $\mathbb{R}^3$)

For system (1) *having a nilpotent singular point at the origin, there exist a formal change of variables that transforms it into the formal normal form*

$$\begin{aligned} \dot{x} &= y + xP_1(x), \\ \dot{y} &= Q_2(x) + yP_1(x), \\ \dot{z} &= -\lambda z + zR_1(x). \end{aligned} \tag{8}$$

for which $P_1(0) = j^1 Q_2(0) = R_1(0) = 0$.

We were not able to prove that the normal form (8) is analytic, i.e. that the series P_1, Q_2 and R_1 are convergent. That does not mean that this normal form is not useful. We remark that the above normal form has $z = 0$ as an invariant surface which is a center manifold and the first two components are decoupled from the third.

The Belitskii normal form for planar systems (2) having a nilpotent singular point is

$$\begin{aligned} \dot{x} &= y + xP_1(x), \\ \dot{y} &= Q_2(x) + yP_1(x), \end{aligned} \tag{9}$$

where $j^1 Q_2(0) = P_1(0) = 0$ (see Theorem 1.8.6, page 37 in [16]), which is very similar to its three-dimensional counterpart. From this form it is possible to arrive at the Liénard Canonical Form

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= Q_2(x) + yP_1(x), \end{aligned} \tag{10}$$

where $j^1 Q_2(0) = P_1(0) = 0$. In [14] it is proven that this form is actually analytic.

First Integrals and Nilpotent singular points

It is well-known that the center problem for systems having a Hopf singularity is equivalent to the existence of a first integral at the point [3, 6, 11]. In this section we search for some link between integrability and nilpotent centers on center manifolds.

Definição 4.1

Let X be a vector field defined on an open set $U \subset \mathbb{R}^k$ and $H : U \subset \mathbb{R}^k \to \mathbb{R}$ be a C^1-function not locally constant. If H satisfies

$$XH \equiv 0,$$

then H is a first integral for X and in this case X is integrable.

Our first result regarding first integrals and nilpotent singular points is the following:

Our first result regarding first integrals and nilpotent singular points is the following:

Theorem 5

Consider a vector field X associated to system (1) having a nilpotent singular point. Then there exists a formal series $H(x, y, z) = y^2 + \sum_{n \geqslant 3} H_n(x, y, z)$ such that $XH = \sum_{n \geqslant 4} \omega_n x^n$.

Lema 4.1

If H is a formal first integral for the normal form (8), then $\dfrac{\partial H}{\partial z} \equiv 0$, that is $H = H(x, y)$.

This lemma lets us conclude that formal integrability of the normal form (8) is essentially formal integrability of the planar normal form (9). Hence, we proceed to study the integrability of the formal system (9).

One of the consequences of Lemma 4.1 is that if system (1) is formally integrable, then it has a formal first integral H such that $j^2 H(0) = y^2$.

One of the consequences of Lemma 4.1 is that if system (1) is formally integrable, then it has a formal first integral H such that $j^2H(0) = y^2$.

Therefore, the quantities ω_n in Theorem 5 present obstructions for the system (1) to be analytically or formally integrable.

One of the consequences of Lemma 4.1 is that if system (1) is formally integrable, then it has a formal first integral H such that $j^2H(0) = y^2$.

Therefore, the quantities ω_n in Theorem 5 present obstructions for the system (1) to be analytically or formally integrable.

For integrable systems, the monodromic singular point must be a center, since in this case the restricted system is also integrable.

However not all nilpotent centers are formally integrable. For instance, consider the following system:

$$\begin{aligned} \dot{x} &= y + x^2, \\ \dot{y} &= -x^3, \\ \dot{z} &= -\lambda z. \end{aligned} \tag{11}$$

which has $z = 0$ as a center manifold and its restriction is a time-reversible system.

However not all nilpotent centers are formally integrable. For instance, consider the following system:

$$\begin{aligned} \dot{x} &= y + x^2, \\ \dot{y} &= -x^3, \\ \dot{z} &= -\lambda z. \end{aligned} \tag{11}$$

which has $z = 0$ as a center manifold and its restriction is a time-reversible system.

If we try to construct a formal first integral $H(x, y, z)$ for system we obtatin $\omega_5 = 2$.

However not all nilpotent centers are formally integrable. For instance, consider the following system:

$$\begin{aligned} \dot{x} &= y + x^2, \\ \dot{y} &= -x^3, \\ \dot{z} &= -\lambda z. \end{aligned} \tag{11}$$

which has $z = 0$ as a center manifold and its restriction is a time-reversible system.

If we try to construct a formal first integral $H(x, y, z)$ for system we obtatin $\omega_5 = 2$.

Note that the obstruction is a coefficient of odd power of x in XH. If it was an even power, we would not have a nilpotent center, as the next result holds.

Theorem 6

Let X be the vector field associated to system (1) having a nilpotent singular point and H be a formal series as in Theorem 5. If there exists $n \in \mathbb{N}$ such that $j^{2n}XH(0) = \omega_{2n}x^{2n}$ with $\omega_{2n} \neq 0$, then the origin cannot be a center on the center manifold.

Studying the integrability of the normal form (8) and using Proposition 2.4 we were able to prove the following results.

Studying the integrability of the normal form (8) and using Proposition 2.4 we were able to prove the following results.

Theorem 7

Consider system (1) *having a monodromic singular point such that its restriction to a center manifold satisfies monodromy condition* $\beta = n - 1$ *in Theorem 1. Then it cannot admit formal first integral.*

Studying the integrability of the normal form (8) and using Proposition 2.4 we were able to prove the following results.

Theorem 7

Consider system (1) *having a monodromic singular point such that its restriction to a center manifold satisfies monodromy condition* $\beta = n - 1$ *in Theorem 1. Then it cannot admit formal first integral.*

Theorem 8

Suppose that the origin of system (1) *is monodromic with odd Andreev number n and satisfies the monodromy condition* $\beta = n - 1$*. Then the origin cannot be a center on a center manifold.*

We also propose a formal normal form for integrable systems (1).

Theorem 9

Consider system (9) having a monodromic singular point. If it admits formal first integral $H(x, y)$, then either $P_1(x) \equiv 0$ or $m = 2sn - 1$ for some $s \in \mathbb{N}$.

The same result holds for planar systems and it was proven by a different method in [9, Theorem 4].

Inverse Jacobi Multipliers

In [4], the authors explored the relationship between the center problem for Hopf points and the existence of an inverse Jacobi multiplier. Inspired by the ideas from that paper, in this section we study the properties of inverse Jacobi multipliers applied to system (1), i.e. for the nilpotent case.

Definição 5.1

Let X be a vector field defined on an open set $U \subset \mathbb{R}^k$ and $V : U \subset \mathbb{R}^k \to \mathbb{R}$ be a C^1-function not locally null such that

$$XV - V\,\mathrm{div}X \equiv 0.$$

We say that V is an inverse Jacobi multiplier of X.

Proposition 5.1

Let X be the vector field associated to system (1) having a nilpotent singular point in $\mathbb{R}^3$. Any non-flat C^∞ inverse Jacobi multiplier $V(x,y,z)$ for vector field X has the form $j^{m+1}V(0) = y^m z$ for $m \geqslant 0$ up to multiplication by constant.

Proposition 5.1

Let X be the vector field associated to system (1) having a nilpotent singular point in $\mathbb{R}^3$. Any non-flat C^∞ inverse Jacobi multiplier $V(x, y, z)$ for vector field X has the form $j^{m+1}V(0) = y^m z$ for $m \geqslant 0$ up to multiplication by constant.

Theorem 10

Consider a vector field X associated to system (1) having a nilpotent singular point. Then there exists a formal series $V(x, y, z)$ such that $XV - V\mathrm{div}X = \sum_{n\geqslant 1} \Lambda_n x^{n-1} z$.

Inspired by the *Inverse Integrating Factor Method* [7] we connect the results regarding Inverse Jacobi Multipliers and the Nilpotent Center Problem.

Inspired by the *Inverse Integrating Factor Method* [7] we connect the results regarding Inverse Jacobi Multipliers and the Nilpotent Center Problem.

Theorem 11

Consider system (1) *having a monodromic singular point with Andreev number* 2 *at the origin. If the origin is an analytic nilpotent center on a center manifold, then there exists a formal inverse Jacobi multiplier* $V(x, y, z)$ *for system* (1) *such that* $j^{m+1}V(0) = y^m z$ *for* $m \geqslant 0$.

For planar systems, the following results hold

For planar systems, the following results hold

Proposition 5.2

Consider system (2) having monodromic singular point at the origin with Andreev number n, satisfying monodromy condition $\beta = n - 1$*. If there is a formal inverse integrating factor* $V(x, y)$ *for (2), then it has the form* $j^2 V(0) = y^2$*.*

Proposition 5.3

Consider system (2) having monodromic singular point at the origin with Andreev number n, satisfying monodromy condition $\beta > n - 1$*. If the system admits formal inverse integrating factor* $V(x, y)$*, then either* $V(0, 0) \neq 0$ *or* $j^2 V(0) = y^2$*.*

For planar systems, the following results hold

Proposition 5.2

Consider system (2) having monodromic singular point at the origin with Andreev number n, satisfying monodromy condition $\beta = n-1$. If there is a formal inverse integrating factor $V(x,y)$ for (2), then it has the form $j^2V(0) = y^2$.

Proposition 5.3

Consider system (2) having monodromic singular point at the origin with Andreev number n, satisfying monodromy condition $\beta > n-1$. If the system admits formal inverse integrating factor $V(x,y)$, then either $V(0,0) \neq 0$ or $j^2V(0) = y^2$.

For the three-dimensional case, any inverse Jacobi multiplier must have either $j^1V(0) = z$ or $j^3V(0) = zy^2$.

Three-dimensional nilpotent centers as limits of non-degenerate centers

For nilpotent singular points in the plane, the authors of [8] (and its previous versions) proved that any analytical nilpotent center can be studied as a limit of non-degenerate centers.

Three-dimensional nilpotent centers as limits of non-degenerate centers

For nilpotent singular points in the plane, the authors of [8] (and its previous versions) proved that any analytical nilpotent center can be studied as a limit of non-degenerate centers.

Using this approach the authors were able to use the Poincaré-Lyapunov algorithm to investigate the nilpotent center problem for planar systems (2).

Theorem 12

Suppose the origin of the following system

$$\begin{aligned}\dot{x} &= y + P(x,y),\\ \dot{y} &= Q(x,y),\end{aligned}$$

is a nilpotent center. Then there are two functions $F_1(x,y)$, $F_2(x,y)$ analytical at the origin with $j^1F_1(0) = j^1F_2(0) = 0$ such that the 1-parameter family

$$\begin{aligned}\dot{x} &= y + P(x,y) + \varepsilon F_1(x,y),\\ \dot{y} &= -\varepsilon x + Q(x,y) + \varepsilon F_2(x,y).\end{aligned}$$

has a non-degenerate center at the origin for any $\varepsilon > 0$. Also there is an analytic function $f(x,y)$ at the origin with $j^1 f(0) = 0$ such that
$$(x - F_1(x,y))\frac{\partial f}{\partial y} = F_2(x,y)(1 + \frac{\partial f}{\partial x}).$$

Theorem 13

Suppose the origin of the following system

$$\begin{aligned} \dot{x} &= y + P(x,y,z), \\ \dot{y} &= Q(x,y,z), \\ \dot{z} &= -\lambda z + R(x,y,z), \end{aligned} \tag{12}$$

is a nilpotent center on an analytical center manifold. Then there are two functions $F_1(x,y)$, $F_2(x,y)$ analytical at the origin with $j^1F_1(0) = j^1F_2(0) = 0$ such that the 1-parameter family

$$\begin{aligned} \dot{x} &= y + P(x,y,z) + \varepsilon F_1(x,y), \\ \dot{y} &= -\varepsilon x + Q(x,y,z) + \varepsilon F_2(x,y), \\ \dot{z} &= -\lambda z + R(x,y,z). \end{aligned} \tag{13}$$

has a non-degenerate center at the origin for any $\varepsilon > 0$. Also there is an analytic function $f(x,y)$ at the origin with $j^1f(0) = 0$ such that
$$(x - F_1(x,y))\frac{\partial f}{\partial y} = F_2(x,y)(1 + \frac{\partial f}{\partial x}).$$

Examples

The Generalized Lorenz system is one of the most studied three-dimensional systems in the literature since its dynamics are very rich.

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= bx + cy - xz, \\ \dot{z} &= dz + xy. \end{aligned} \tag{14}$$

with $ad \neq 0$. The origin is a nilpotent singular point for $b + c = 0$ and $c = a$.

By means of the coordinate change $\bar{x} = y$, $\bar{y} = a(y - x)$, $\bar{z} = z$, dropping the bars, system (14) becomes

$$\begin{aligned} \dot{x} &= y - xz + \tfrac{1}{a} yz, \\ \dot{y} &= -axz + yz, \\ \dot{z} &= dz + x^2 - \tfrac{1}{a} xy. \end{aligned} \tag{15}$$

Theorem 6 is powerful enough to solve the Nilpotent Center Problem for the above system. We compute the quantities ω_n (Theorem 5) for system (15) and obtain the first non-zero one:

$$\omega_6 = -\frac{2\,a\,(2\,a + d)}{3d^3}.$$

For $d = -2a$, the function $V(x, y, z) = x^2 - \frac{2xy}{a} + \frac{y^2}{a^2} - 2az$ defines a center manifold for system (15). The restriction of the system to this center manifold is given by

$$\dot{x} = y - xz + \frac{y}{2a^2}\left(x - \frac{y}{a}\right)^2,$$

$$\dot{y} = -axz + \frac{y}{2a}\left(x - \frac{y}{a}\right)^2,$$

which is a Hamiltonian system with Hamiltonian function $H(x, y) = y^2 + \frac{x^4}{4} - \frac{x^3 y}{a} + \frac{3x^2y^2}{2a^2} - \frac{xy^3}{a^3} + \frac{y^4}{4a^4}$.

The following system was first considered in [13].

$$\begin{aligned} \dot{x} &= y - 2xy + axz, \\ \dot{y} &= -2x^3 + y^2 + byz, \\ \dot{z} &= -z + dxy. \end{aligned} \tag{16}$$

By Proposition 2.1, the origin of (16) is monodromic with Andreev number $n = 2$ and satisfies the monodromy condition $\beta > n - 1$.

We compute the Lyapunov quantities of the perturbation (13) for system (16). We have:

$$\eta_1 = -\frac{4\varepsilon^2 d(a-b)}{12\varepsilon + 3}.$$

By Theorem 13, $d(a-b) = 0$ is a necessary condition for the origin to be an analytic nilpotent center.

Now we search for obstructions for system (16) to have a formal inverse Jacobi multiplier. We consider both possibilities of $V(x,y,z)$. First, if $j^1V(0) = z$. We have $\Lambda_1 = \cdots = \Lambda_4 = 0$ and:

$$\Lambda_5 = -4d\,(2a-b),\ \Lambda_6 = -a\,d.$$

Now for $j^3V(0) = zy^2$, we obtain $\Lambda_1 = \cdots = \Lambda_8 = 0$ and:

$$\Lambda_9 = -\frac{12\,d\,(2\,a-b)}{5},\ \Lambda_{10} = -\frac{2\,d\,(9\,a-2\,b)}{15}.$$

Thus, by Theorem 11, the conditions $a = b = 0$ or $d = 0$ are necessary for the origin to be an analytic nilpotent center on a center manifold.

Under $d = 0$ or $a = b = 0$, the restricted system is the Hamiltonian system given by

$$\begin{aligned} \dot{x} &= y - 2xy, \\ \dot{y} &= -2x^3 + y^2. \end{aligned} \tag{17}$$

Under $d = 0$ or $a = b = 0$, the restricted system is the Hamiltonian system given by

$$\begin{aligned} \dot{x} &= y - 2xy, \\ \dot{y} &= -2x^3 + y^2. \end{aligned} \tag{17}$$

Theorem 14

The origin of system (16) is an analytic nilpotent center on a center manifold if and only if $a = b = 0$ or $d = 0$.

Bibliografia I

M. J. Álvarez and A. Gasull, *Monodromy and stability for nilpotent critical points*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 15 (2005), pp. 1253–1265.

G. Belitskii, *C^∞-normal forms of local vector fields*, vol. 70, 2002, pp. 23–41.
Symmetry and perturbation theory.

Y. N. Bibikov, *Local theory of nonlinear analytic ordinary differential equations*, vol. 702 of Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1979.

A. Buică, I. A. García, and S. Maza, *Existence of inverse Jacobi multipliers around Hopf points in $\mathbb{R}^3$: emphasis on the center problem*, J. Differential Equations, 252 (2012), pp. 6324–6336.

F. Dumortier, J. Llibre, and J. C. Artés, *Qualitative theory of planar differential systems*, Universitext, Springer-Verlag, Berlin, 2006.

V. F. Edneral, A. Mahdi, V. G. Romanovski, and D. S. Shafer, *The center problem on a center manifold in $\mathbb{R}^3$*, Nonlinear Anal., 75 (2012), pp. 2614–2622.

I. A. García, *Formal inverse integrating factor and the nilpotent center problem*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 26 (2016), pp. 1650015, 13.

I. A. García, H. Giacomini, J. Giné, and J. Llibre, *Analytic nilpotent centers as limits of nondegenerate centers revisited*, J. Math. Anal. Appl., 441 (2016), pp. 893–899.

Bibliografia II

I. A. García and J. Giné, *Analytic nilpotent centers with analytic first integral*, Nonlinear Anal., 72 (2010), pp. 3732–3738.

A. Kelley, *The stable, center-stable, center, center-unstable, unstable manifolds*, J. Differential Equations, 3 (1967), pp. 546–570.

L. Queiroz, *Teoria dos centros e ciclicidade de pontos de hopf para campos de vetores planares e tridimensionais*, Master's thesis, Universidade Estadual Paulista (UNESP), IBILCE, 2019.

J. Sijbrand, *Properties of center manifolds*, Trans. Amer. Math. Soc., 289 (1985), pp. 431–469.

S. Song, J. Feng, and Q. Wang, *The center conditions and bifurcation of limit cycles at the degenerate singularity of a three-dimensional system*, J. Appl. Math., (2014), pp. Art. ID 546243, 8.

E. Stróżyna and H. Żoładek, *The analytic and formal normal form for the nilpotent singularity*, J. Differential Equations, 179 (2002), pp. 479–537.

Q.-l. Wang, W.-t. Huang, and Y.-r. Liu, *Multiple limit cycles bifurcation from the degenerate singularity for a class of three-dimensional systems*, Acta Math. Appl. Sin. Engl. Ser., 32 (2016), pp. 73–80.

M. Zhitomirskii, *Smooth local normal forms for vector fields and diffeomorphisms*, 1992.

Obrigado pela atenção!