Border collision bifurcations in a PWL stock market model

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in collaboration with

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Introduction

The applied model: Stock markets with heterogeneous interacting agents

We consider a new behavioral stock market model recently developed:
- GRSSW, Causes of fragile stock market stability (JEBO, 2022) (first model)
- GRSSW, Perception of fundamental values and financial market dynamics: Mathematical insights from a 2D piecewise linear map (SIAM J. Applied Dynamical Systems 2022 (in press) the map is the same as in the first model)
- GRSSW, The role of sentiment traders in the fragility of stock markets (second model, work in progress)

Several scholars have proposed models with heterogeneous interacting agents (mainly chartists and fundamentalists) to explain the intricate dynamics of stock markets.

We recall the pioneering contributions by Day & Huang (1990), Chiarella (1992), Lux (1995) and Brock & Hommes (1998),
the surveys provided by Westerhoff (2009), Hommes (2013), Dieci & He (2018), and the papers Dieci & Westerhoff (2013) and Schmitt & Westerhoff (2021)

Introduction

Piecewise smooth models

Many of the papers cited above are modeling the stock markets with piecewise smooth systems in discrete time, **with continuous maps**.
However, in the one-dimensional (1D) case, there are some models in which the financial market is described by systems with piecewise linear and discontinuos functions. A few works have been published a few years ago by F.Tramontana, F.Westerhoff, and me:

TWG Journal of Economic Behavior and Organization (2010); TGW: Computational Economics (2011); TWG: Decisions in Economics and Finance (2014); TWG: Mathematics and Computers in Simulation (2015).

This stimulated our studies on 1D discontinuous systems, and the BCBs which were not well known, whose results are also summarized in our book:
Avrutin, V., Gardini, L., Schanz, M., Sushko, I. and Tramontana, F. WS (2019).

Introduction

Piecewise smooth models

Also **1D maps with two discontinuities** have been studied:
Panchuk, A., Sushko, I., Schenke, B. & Avrutin, V. IJBC (2013);
V. Avrutin, I. Sushko, F. Tramontana (2014) Abstract and Applied Analysis;
A. Panchuk, I. Sushko, V. Avrutin IJBC (2016).

Recently, Frank and Noemi involved us in the analysis of a model that is a improvement of the previous ones. **The new system is 2D, discontinuous, and described by PWL functions.**

There are nowadays several papers related to the 2D PWL border-collision normal form map, that is **continuous**, and still with dynamics and bifurcation structures that need to be investigated, while the discontinuous case is at its early stage of analysis.
This kind of 2D PWL discontinuous models is indeed not new in financial markets, it was already used by some authors, but with the 2D PWL BCNF map: Gu, E.G. IJBC (2017) and (2018), Anufriev, M., Gardini, L. and Radi, D. (2020) Nonlinear Dynamics.

Introduction

2D Piecewise smooth models

So, the **discontinuity in 2D systems** leads to a new field of research. Besides the mentioned economic applications, only a few papers have been published, with the discontinuos 2D BCNF map. One of the first is due to C. Mira: (2013) Embedding of a Dim1 Piecewise Continuous and Linear Leonov Map into a Dim2 Invertible Map, pp. 337–368. Springer Book,
also Rakshit, B., Apratim, M., Banerjee, S. in Chaos (2010) and recently by Simpson, D. in IJBC (2020)

Our analysis of the dynamics of the 2D applied system (with one discontinuity line and with two lines of discontinuity) is different from the BCNF, although PWL, and contributes with some more steps towards the understanding of the new kinds of bifurcations that may occur, evidencing the changes with respect to the bifurcations occurring in continuous systems.

Stock markets with heterogeneous interacting agents, Basic model

Consider a *basic linear model*: a market maker quotes the stock price for the next period with respect to traders' current excess demand, using the price-adjustment rule

$$\text{Market Maker:} \quad P_{t+1} = P_t + a(D_t^C + D_t^F), \quad a > 0$$

a is a positive price adjustment parameter and D_t^C, D_t^F represent the orders submitted by chartists, fundamentalists. We express

$$\text{Chartists' orders:} \quad D_t^C = b(P_t - P_{t-1}), \quad b > 0$$

assuming them as trend followes, and b indicates their aggressiveness, while fundamentalists believe that stock prices revert towards their fundamental values

$$\text{Fundamentalists' orders:} \quad D_t^F = c(F - P_t), \quad c > 0$$

c controls their market impact, F denotes the stock market's fundamental value.

New model

Stock markets with sentiment traders

This basic model seems too simple to represent the market's dynamics. The system is just a 2D continuous PWL map. Let us add the action of sentiment traders, modeled as traders who optimistically (pessimistically) buy (sell) a certain amount of stocks (d units) in rising (falling) markets. The price-adjustment rule becomes

$$\text{Market Maker:} \quad P_{t+1} = P_t + a(D_t^C + D_t^F + D_t^S), \quad a > 0$$

where D_t^S captures the orders by

$$\text{Sentiment traders :} \quad D_t^S = \begin{cases} d & \text{if} \quad P_t - P_{t-1} > 0 \\ 0 & \text{if} \quad P_t - P_{t-1} = 0 \\ -d & \text{if} \quad P_t - P_{t-1} < 0 \end{cases}, \quad d > 0$$

In the new system, with $P_t = P_{t-1}$ as discontinuity line, the attracting fixed point is substituted by attracting cycles or new kinds of virtual attractors.

New model

Stock markets with sentiment traders

Then, in a second model we assume that sentiment traders are either optimistic (sufficiently rising prices), pessimistic (sufficiently falling prices) or neutral ("relatively" stable prices), leading to a map with two relevant parameters, d and h, having two discontinuity lines $P_t = P_{t-1} + h$ and $P_t = P_{t-1} - h$:

$$\text{Sentiment traders' orders: } D_t^S = \begin{cases} d & \text{if} \quad P_t - P_{t-1} > h \\ 0 & \text{if} \quad |P_t - P_{t-1}| < h \\ -d & \text{if} \quad P_t - P_{t-1} < -h \end{cases}, \quad d > h > 0$$

Now the stable fundamental equilibrium exists, and coexists with different attractors.

New model

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Now the stable fundamental equilibrium exists, and coexists with different attractors.

- We have considered parameters with $d > h > 0$, but also $0 < d < h$ is plausible, and a comment related to this case will be given.
- The same buying/selling constant d is only a simplifying assumption. Similar results hold also for different buying/selling fixed amounts.

New model

Stock markets with sentiment traders

The more general model is thus represented as follows:

$$P_{t+1} = \begin{cases} P_t + a(b(P_t - P_{t-1}) + c(F - P_t) + d) & \text{if} \quad P_t - P_{t-1} > h \\ P_t + a(b(P_t - P_{t-1}) + c(F - P_t)) & \text{if} \quad -h \leq P_t - P_{t-1} \leq h \\ P_t + a(b(P_t - P_{t-1}) + c(F - P_t) - d) & \text{if} \quad P_t - P_{t-1} < -h \end{cases}$$

New model

Stock markets with sentiment traders

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Renaming the parameters ab, ac, ad as b, c, d, we can set $a = 1$

With the change of coordinate of the state variable as $P := P - F$, the origin represents the fundamental equilibrium

Introducing the auxiliary variable $X_t = P_{t-1}$, we get the two-dimensional (PWL and discontinuous) map

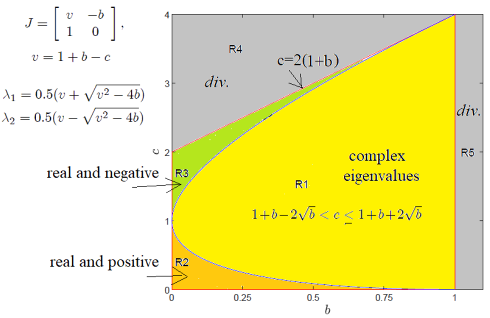
$$T : \begin{cases} P_{t+1} = \begin{cases} (T_L) \ (1 + b - c)P_t - bX_t + d & \text{if } P_t - X_t > h \\ (T_O) \ (1 + b - c)P_t - bX_t & \text{if } -h \leq P_t - X_t \leq h \\ (T_U) \ (1 + b - c)P_t - bX_t - d & \text{if } P_t - X_t < -h \end{cases} \\ X_{t+1} = P_t \end{cases}$$

Basic Model

Basic model: d=0, h=0

When sentiment traders are not acting, then the map simplifies to the basic linear model with only chartists and fundamentalists:

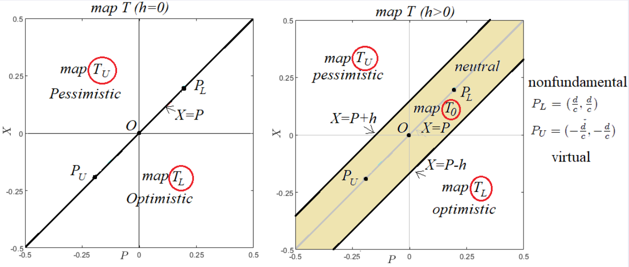
$$T_O: \begin{cases} P_{t+1} = (1+b-c)P_t - bX_t \\ X_{t+1} = P_t \end{cases}$$



d>0, h=0, h>0

The new system is given by the two-dimensional (PWL and discontinuous) map:

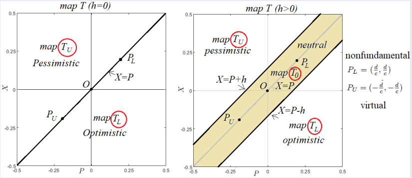
$$T : \begin{cases} P_{t+1} = & \begin{cases} (T_L)\ (1+b-c)P_t - bX_t + d & \text{if } P_t - X_t > h \\ (T_O)\ (1+b-c)P_t - bX_t & \text{if } -h \le P_t - X_t \le h \\ (T_U)\ (1+b-c)P_t - bX_t - d & \text{if } P_t - X_t < -h \end{cases} \\ X_{t+1} = & P_t \end{cases}$$



New Models

d>0, h=0, h>0

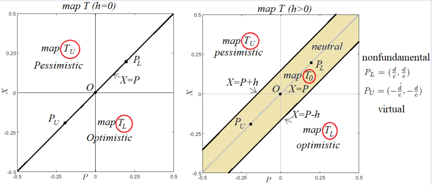
Generic properties



- When $h = 0$, $d \neq 0$ is a scale parameter. Defining $P_t := P_t/d$ we could set $d = 1$, however, we prefer to keep d as a parameter using $d = 0.02$ in our numerics, the aggressiveness of sentiment traders merely increases the amplitude of stock price fluctuations.
- When $h \neq 0$, $d \neq 0$ one of them is a scale parameter, we use $d = 0.02$ and $h = 0.01$ in our numerics.

d>0, h=0, h>0

Generic properties

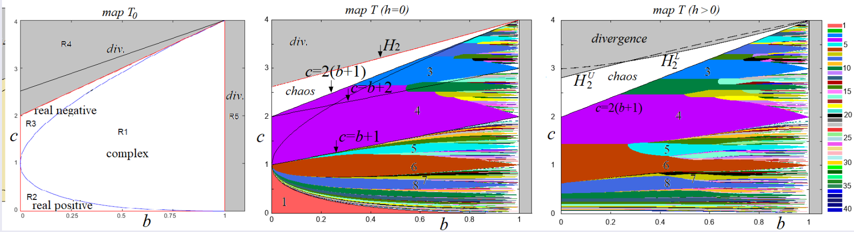


- From $T(-P,-X) = -T(P,X)$ it follows that the map is symmetric with respect to the fundamental fixed point $(0,0)$. This symmetry property is lost when the demands of the sentiment traders are not of the same value, say d_L/d_U, but we shall see that the results are similar.
- As already mentioned, PWL, 2D discontinuous map is a new research area. Characterized by BCBs. The behavior of sentiment traders (discontinuities) leads to endogenous-driven boom-bust cycles.

d>0, h=0, h>0

Generic properties: stability/instability

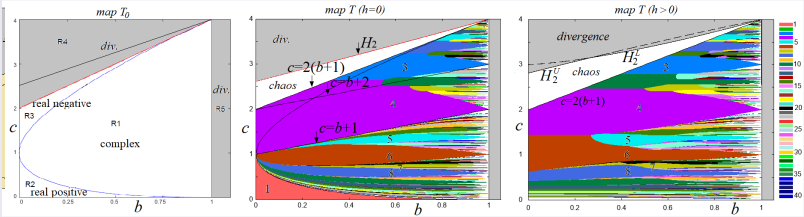
The Jacobian matrix J is the same for all the partitions leading to an immediate relevant property of the stability/instability of the possible existing cycles. A cycle of period n is a fixed point of the $n-$th iterate of the map, T^n, and the eigenvalues of the matrix J^n are given by $(\lambda_1)^n$ and $(\lambda_2)^n$ where λ_1 and λ_2 are the eigenvalues of J, explicitly known.



d>0, h=0, h>0

Generic properties: stability/instability

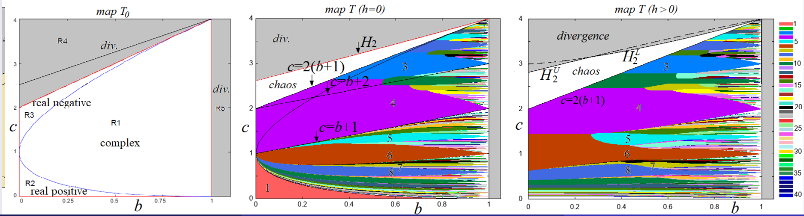
It follows that for parameters (b, c) belonging to the stability box S, map T (for $h \geq 0$) cannot have unstable cycles, and thus no chaotic sets. So we can still define that region as stability region for the dynamics of the model: divergent trajectories do not exist, and the trajectories must have some bounded limit set (the $\omega-$limit set of a trajectory), and generically we have convergence to a cycle of period n, $n \geq 1$.



New Models

d>0, h=0, h>0

Generic properties: outside the stability box chaos may exist.
When a parameter point (b, c) crosses the upper boundary for $b < 1$, then the situation is opposite: no existing cycle can be attracting, all the existing cycles are saddles.
However, the region in the (b, c) parameter plane labelled R_4, when the basic map T_O has divergent trajectories, may become meaningful in the model T.
Even if all the existing cycles are unstable (saddle) the saddles may be homoclinic and bounded chaotic dynamics may exist, oscillating around the fundamental, and we shall define the proper parameter range.

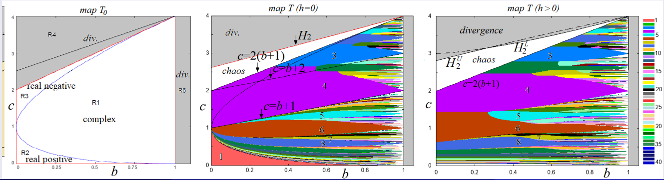


New Models

d>0, h=0, h>0

Generic properties: Arnold tongues

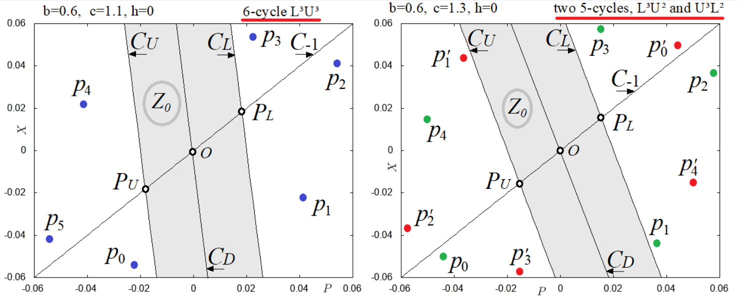
At the boundary $b = 1$ the map has det(J)= 1 everywhere, except a the discontinuity line, and for $c \in$[0,4] the three fixed points (real or virtual) are centers. Periodicity regions of attracting cycles with rotation number m/n issue from the point $c = c_{m/n} = 2(1 - cos(2\pi m/n))$, and we have the same dynamic behavior occurring in the KAM (Kolmogorov, Arnold, Moser) theorem: breaking an area preserving system, considering $b < 1$ and close to 1, Arnold tongues in the parameter plane appear, related to periodicity regions of cycles, and the periods seem to follow the *Farey summation rule*, we can represent the periods obtaining the famous *devil staircase*.



First New Model

d>0, h=0

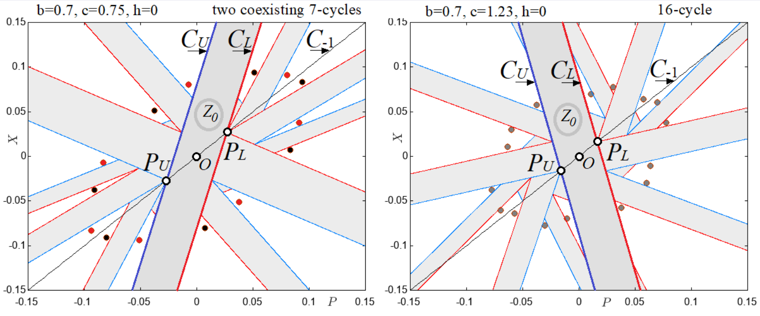
Property of the "gap-maps"
The role of the discontinuity line, crossing which the map changes its definition, and called critical line, is particularly important (not only for the BCBs). The images of the discontinuity line $X = P$ via the two maps T_L and T_U lead to the critical curves $C_L = T_L(C_{-1})$ and $C_U = T_U(C_{-1})$ bounding a strip Z_0 which cannot include any periodic point. Two examples of the existing cycles, when the parameters belong to region R_1:



d>0, h=0

Property of the "gap-maps"

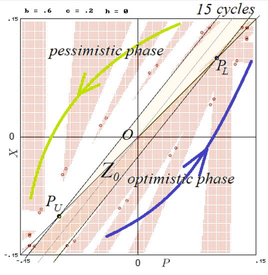
The existing periodic points only belong to the residual set of the plane deprived by the regions $U_{n\geq 0} T^n(Z_0)$.Two examples showing few images of the strip Z_0 when the parameters belong to region R_1:

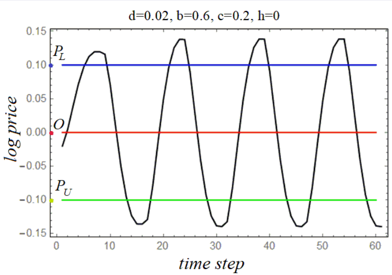


First New Model

d>0, h=0

Economic behavior of our financial market model.
We present an example of the price dynamics when the parameters belong to region R_1 (complex eigenvalues). Considering T_O the trajectory would converge to the fundamental, spiraling around it. The presence of sentiment traders, buying in the optimistic phase and selling in the pessimistic one, keeps the prices oscillating around the fundamental and around the virtual nonfundamental points, which are responsible of the change of direction.



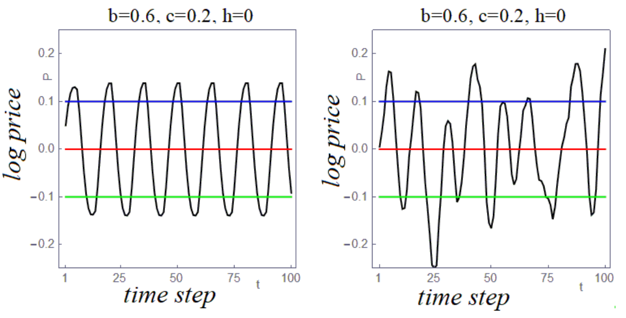


First New Model

d>0, h=0

Economic behavior of our financial market model.

The dynamic behavior is robust with respect to weak exogenous noise, and of course it becomes more irregular. Exogenous noise may lead to more frequent and more erratic regime changes, although the stock market's general boom-bust nature remains intact. Here the same example is subject to normally distributed additive noise with a mean of zero and standard deviation $\sigma = 0.02$



First New Model

d>0, h=0

Existence of cycles and BCB.

When a cycle exists, with periodic points in the proper partitions of the phase plane, a bifurcation occurs when a periodic point of the cycle has a contact with the diagonal, i.e. merges with the discontinuity line $X = P$ leading to the appearance or disappearance of a cycle.

A trajectory $\{p_i\}_{i\geq 0} = \{(P_i, X_i)\}_{i\geq 0}$ of map T can be represented with a symbolic sequence $\sigma_0\sigma_1...$, where

$$\sigma_i = L/D/U \quad \text{if} \quad P_i \gtreqless X_i$$

We have determined the BCB curves related to cycles of even periods $n = 2m$: $L^m U^m$, $m \geq 1$, and pairs of cycles with odd periods $n = 2m - 1$: $L^m U^{m-1}$, $U^m L^{m-1}$, $m \geq 2$.

First New Model

d>0, h=0

Existence of cycles and BCB.

We consider $p_0 = (P_0, X_0)$ as the leftmost point in the lower partition, and we solve the equation

$$(P_0, X_0) = T_U^m \circ T_L^m (P_0, X_0)$$

for $m > 1$, a BCB occurs when $P_0 = X_0$ and when $(P_{m-1}, X_{m-1}) = T_L^{m-1}(P_0, X_0)$ belongs to the diagonal, that is, $P_{m-1} = X_{m-1}$.

A relevant help comes from the expression of the matrix J^m, $m \geq 1$, which can be written as follows:

$$J^m = \begin{bmatrix} a_m & -ba_{m-1} \\ a_{m-1} & -ba_{m-2} \end{bmatrix}$$

where the entries are determined via the second-order linear difference equation:

$$a_m = (1 + b - c)a_{m-1} - ba_{m-2} \quad , \quad a_{-1} = 0, \ a_0 = 1$$

First New Model

d>0, h=0

Existence of cycles and BCB.

For a cycle of even period, from $(P_0, X_0) = T_U^m \circ T_L^m (P_0, X_0)$ we get

$$\begin{bmatrix} P_0 \\ X_0 \end{bmatrix} = \frac{d/c}{1 + a_m - ba_{m-2} + b^m} \begin{bmatrix} b^m - 1 + a_{m-1}(1 - b - c) \\ b^m - 1 + a_{m-1}(1 - b + c) \end{bmatrix}$$

The condition $P_0 = X_0$ leads to the BCB curve of equation:

$$BC_{2m}^L : \qquad a_{m-1} = 0$$

while the condition $P_{m-1} = X_{m-1}$ leads to the other BCB:

$$BC_{2m}^U : \qquad b^{m-1} - a_{m-2} = 0$$

First New Model

d>0, h=0

Existence of cycles and BCB.

For a cycle of odd period $L^m U^{m-1}$ the BCB condition $P_0 = X_0$ leads to:

$$BC^L_{2m-1}: \quad a_{m-2}(1-a_{2m-1}) - a_{2m-2}(1-a_{m-1}) = 0$$

while the condition $P_{m-1} = X_{m-1}$ leads to:

$$BC^U_{2m-1}: \quad a_{m-2}(ba_{2m-3}+1)(1-a_{m-1}+ba_{m-3})+$$

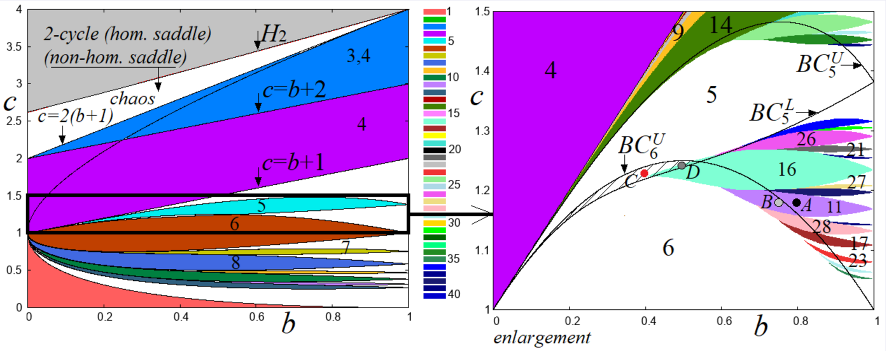
$$+ba_{2m-2}(a^2_{m-2}-a_{m-3}(1+ba_{m-3})) = 0$$

and the two bifurcations also correspond to the BCBs of the symmetric cycle $U^m L^{m-1}$.

First New Model

d>0, h=0; Existence of cycles and BCB

Examples of periodicity regions and overlapping regions.

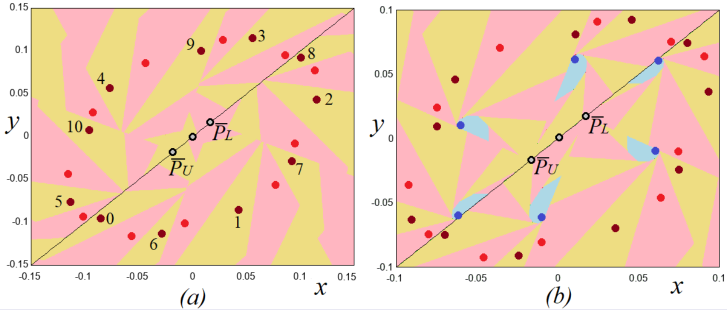


Notice that, since in the "stability regions" saddle cycles cannot exist, the boundaries of the basins consist in segments of the line $X = P$ and preim.

First New Model

d>0, h=0

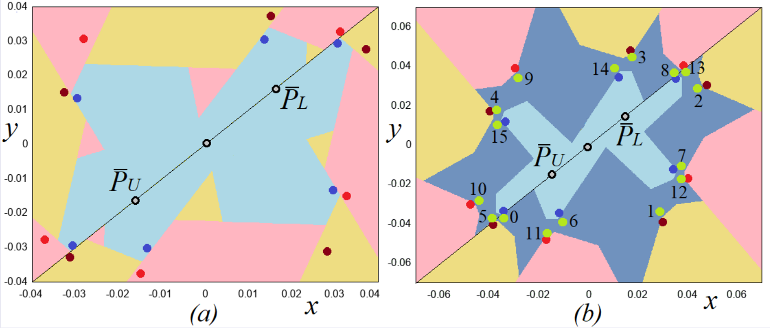
Coexistence of cycles (parameter points A and B), and disconnected basins



(a) (A: $c = 1.18$, $b = 0.8$) 11-cycles $L^3U^3L^3U^2$ (brown circles) and $U^3L^3U^3L^2$ (red circles) and their basins;
(b) (B: $c = 1.18$, $b = 0.75$) the same 11-cycles plus a 6-cycle L^3U^3 (blue circles) born after crossing the BCB curve BC_6^U.

d>0, h=0

Coexistence of cycles (parameter points C and D), and disconnected basins

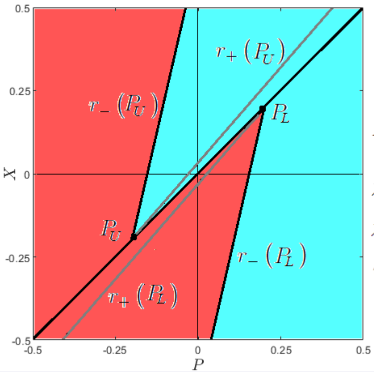


(a) (C: $b = 0.4$, $c = 1.23$) Cycles L^3U^3 (blue circles), L^3U^2 (brown circles) and U^3L^2 (red circles) together with their basins;
(b) (D: $b = 0.5$, $c = 1.24$) the same cycles plus 16-cycle $L^3U^3L^2U^3L^3U^2$ (light green circles), whose points are numbered.

d>0, h=0

Region R_2, nonfundamental fixed points as Milnor attractors.

A second new mechanism is evidenced when the parameters belong to region R_2 (eigenvalues real and positive), which leads to a particular kind of dynamics, a trajectory is convergent to a nonfundamental virtual fixed point P_L or P_U.



eigenvectors

$$X = \frac{1}{\lambda_\pm}(P - P_{L/U}) + P_{L/U}$$

$$\lambda_+ = \lambda_1 = 0.5(v + \sqrt{v^2 - 4b})$$

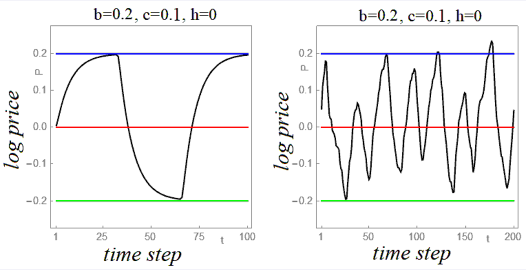
$$\lambda_- = \lambda_2 = 0.5(v - \sqrt{v^2 - 4b})$$

$$v = 1 + b - c$$

First New Model

d>0, h=0

Region R_2, nonfundamental fixed points as Milnor attractors.
This property has important implications for the behavior of stock prices: boom-bust dynamics can emerge when the market is perturbed by occasional random shocks, example:

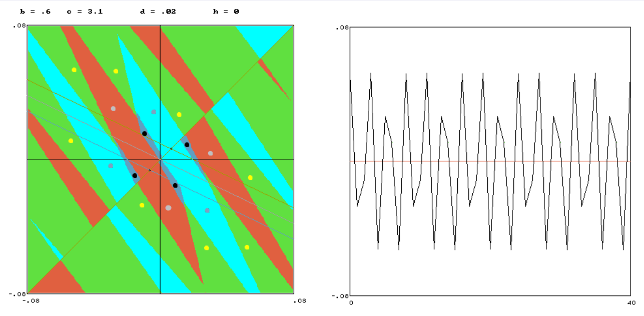


without any shock the trajectory converges to P_U, (a) shows the effect of sporadic shocks while (b) of a normally distributed additive shocks with zero mean and standard deviation $\sigma = 0.01$.

First New Model

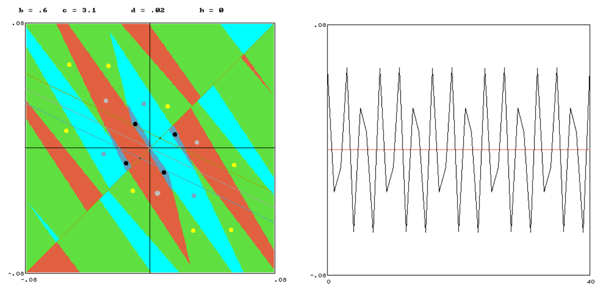
d>0, h=0

Region R_3. Nonfundamental fixed points with real and negative eigenvalues. There are overlapping regions related to cycles of period 4 (L^2U^2) and to a pair of 3-cycles (LU^2, UL^2), and for b larger than 0.5 other periodicity regions may be observed. An example is given where there is also an attracting 8-cycle (L^2ULU^2LU):



First New Model

d>0, h=0

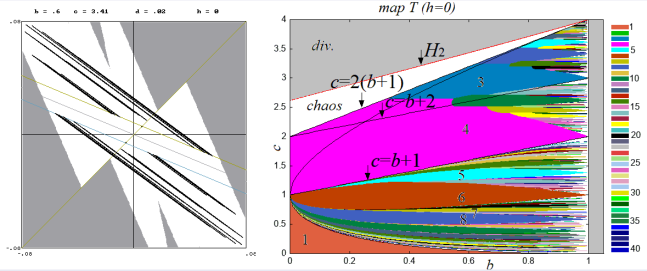


The behavior of our stock market model in region R_3 offers a third new mechanism for the emergence of endogenous stock price fluctuations. The characteristic here is that in one or at most two iterations the periodic points of a cycle change partition. This means that the path of the stock price frequently changes its direction, due to the rather aggressive trading behavior of fundamentalists, which causes so-called overshooting dynamics.

First New Model

d>0, h=0

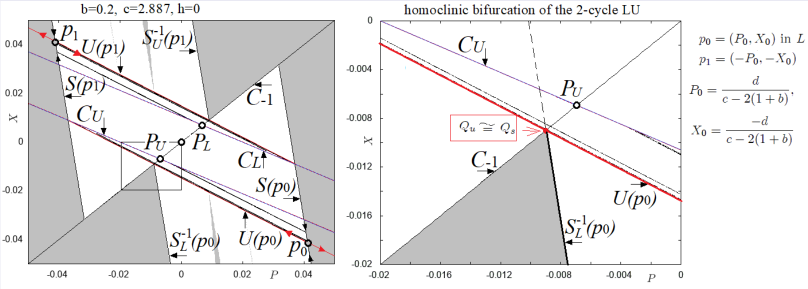
Region R_4. Chaotic attractors and high volatility.
The dynamics of the basic model are unbounded above the upper curve of the stability box S, while when sentiment traders become active in the stock market the dynamics of the model's complete map T may be still bounded in a portion of region R_4, where there can be only chaotic sets. A 2-cycle saddle LU appears at the bifurcation curve $c = 2(1 + b)$ existing above it, via a degenerate transcritical bifurcation (SG-IJBC-2010). The stable set of the saddle 2-cycle belongs to the border of the region of divergent trajectories.



First New Model

d>0, h=0

Mechanism of the homoclinic bifurcation of the 2-cycle in region R_4



The unstable, resp. stable, set of the periodic point p_0 intersects the diagonal in a point $Q_u = (q_u, q_u)$, resp. $Q_s = (q_s, q_s)$. As long as it is $q_s < q_u$ the 2-cycle is not homoclinic. The contact occurs for $q_s = q_u$ (homoclinic bifurcation leading to the curve in the parameter plane), and for $q_s > q_u$ the 2-cycle is homoclinic.

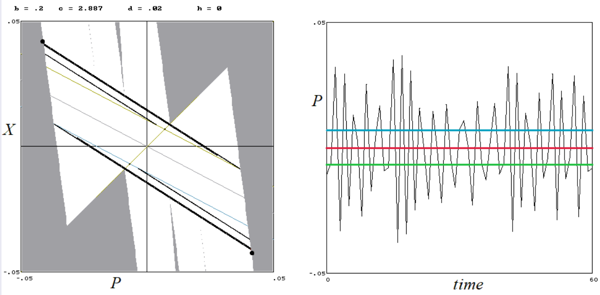
First New Model

d>0, h=0

Chaotic attractors and high volatility in region R_4

This suggests that there are situations where sentiment traders are beneficial for the overall stability of stock markets.

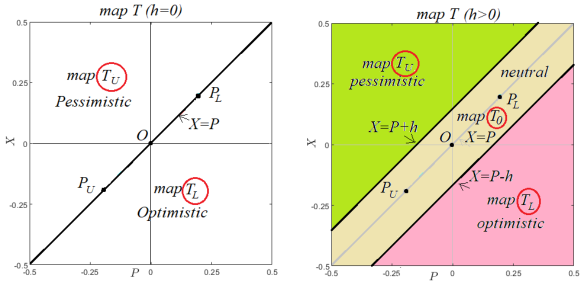
The two real and negative eigenvalues lead to strong unpredictable volatility.



In the figure we show an example of chaotic attractor of the stock price, which manifest high volatility as in the case of the 8-cycle shown before.

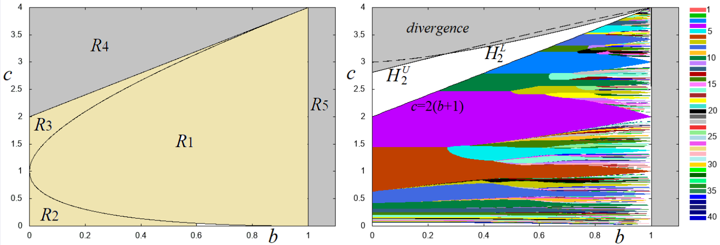
d>0, h>0

The introduction of an interval around the fundamental value, inside which sentiment traders are neutral, non active, leads to a more general model. So, let us assume now $h > 0$ (and $d > h$). There exists a region in which sentiment traders are neutral ("relatively" stable prices, for $|P_t - P_{t-1}| < h$), or optimistic (sufficiently rising prices, $P_t - P_{t-1} > h$), pessimistic (sufficiently falling prices, $P_t - P_{t-1} < -h$)



d>0, h>0

The main difference with respect to the dynamic behavior that we have described before is that the fundamental fixed point, attracting for parameters belonging to the box S, always coexists with other attracting cycles



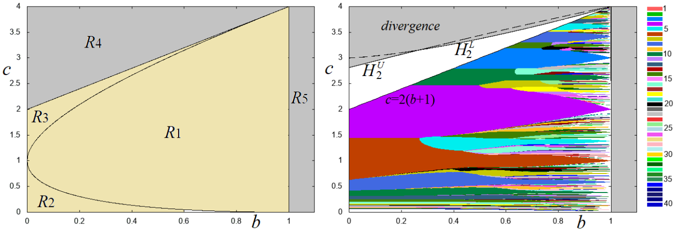
now a trajectory $\{p_i\}_{i\geq 0} = \{(P_i, X_i)\}_{i\geq 0}$ of map T can be represented with a symbolic sequence with three symbols $\sigma_0\sigma_1...$, where

$$\sigma_i = L/O/U \quad \text{if} \quad P_i - X_i > h, \quad |P_i - X_i| < h, \quad P_i - X_i < -h$$

Second New Model

d>0, h>0

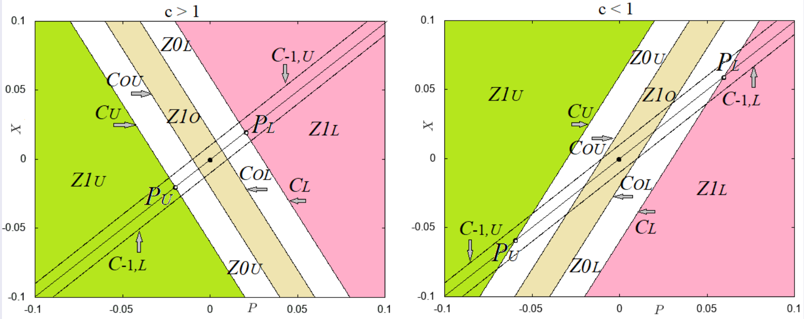
The second relevant difference is that also for parameter points belonging to region R_2 (eigenvalues real and positive), the map has the attracting node in the fundamental coexisting with other attracting cycles (also nodes) around it, which necessarily include points in the middle partition, and thus whose symbolic sequences necessarily include all three symbols $L/O/U$.



The third relevant difference is that also in region R_1 the fundamental coexists with attracting cycles which may have points belonging to two partitions, with the only symbols L/U or to all the three partitions, with all three symbols $L/O/U$.

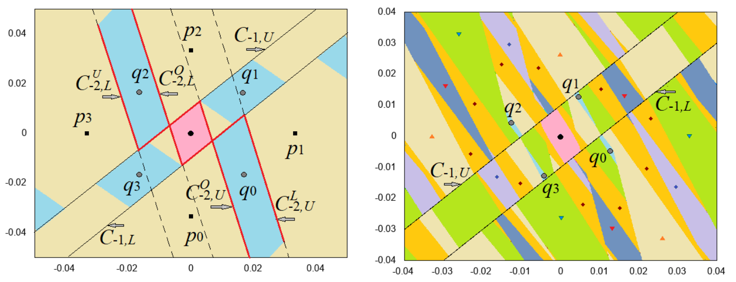
d>0, h>0

The increase in the number of discontinuity lines (critical lines) leads now to two regions Z_0 whose points have no rank-1 preimage with the three linear maps $T_L\,/\,T_O\,/\,T_U$, denoted Z_{0L} and Z_{0U}. The existing periodic points only belong to the residual set of the plane deprived by the regions $U_{n\geq 0}\,T^n(Z_{0L})$ and $U_{n\geq 0}\,T^n(Z_{0U})$



d>0, h>0

For parameter points in S, the coexistence of several cycles, besides the fundamental fixed point, is a characteristic of the model, and the boundaries of the basins of attraction of the existing cycles consist of segments belonging to the discontinuity lines and their preimages of different ranks. Examples:



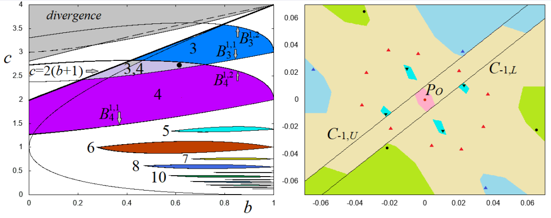
(a) two 4-cycles; $(h = 0.01,\ d = 0.02)$ $b = 0.4$, $c = 2$;

(b) one 4-cycle, four 3-cycles and one 10-cycle, $b = 0.5$, $c = 2.9$

d>0, h>0; BCBs of cycles in two partitions.

Now the collisions take place with the lines $X = P - h$ and $X = P + h$. BCB curves of some families of cycles:

- those of even periods having symbolic sequence $L^m U^m$, $m \geq 1$
- those of odd periods having symbolic sequence $L^m U^{m-1}$, $U^m L^{m-1}$, $m \geq 2$

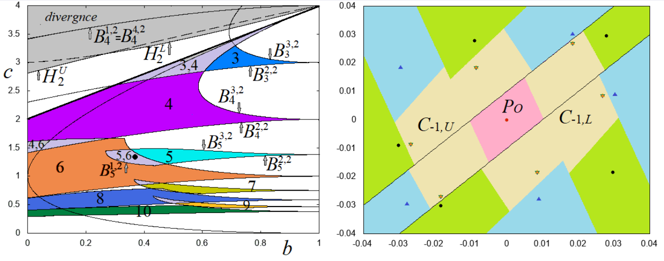


(a) some examples for $m = 2, ..., 8$;(b) basins of coexisting attracting cycles, the fundamental P_O, 4-cycle L^2U^2, 3-cycles L^2U, U^2L and a 10-cycle L^2UOLU^2LOU at $b = 0.61$, $c = 2.67$ (black point in (a)).

d>0, h>0

BCBs of cycles in three partitions

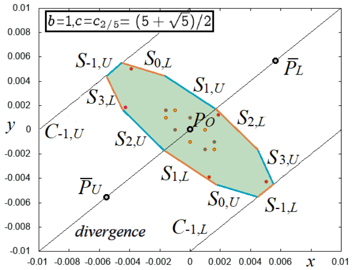
- those of even periods having symbolic sequence $L^m OU^m O$, $m \geq 1$
- those of odd periods having symbolic sequence $L^m OU^m$, $U^m OL^m$, $m \geq 1$



(a) examples of BCB boundaries for $m = 1, ..., 4$; (b) basins of coexisting attracting cycles fundamental P_O, 6-cycle $L^2 OU^2 O$ and 5-cycles $L^2 OU^2$, $U^2 OL^2$ at $b = 0.35$, $c = 1.3$ (black point in (a)).

d>0, h>0 : b=1 Center bifurcation

As already mentioned, at $b = 1$ a center bifurcation occurs, and we can apply the results similar to those in SG-IJBC-2008:
A periodicity region of attracting cycles with rotation number m/n issues from the point $(b, c) = (1, c_{m/n})$, $c_{m/n} = 2(1 - cos(2\pi m/n))$, for $h = 0$ are issuing points without any related property at this bifurcation value, while for h>0 a polygon $P_{m/n}$ filled by $n-$cycles exists, with n edges if n is even, and $2n$ edges if n is odd, given by the generating segments $S_{-1,L} \subset LC_{-1,L}$ and $S_{-1,U} \subset LC_{-1,U}$ and their images.



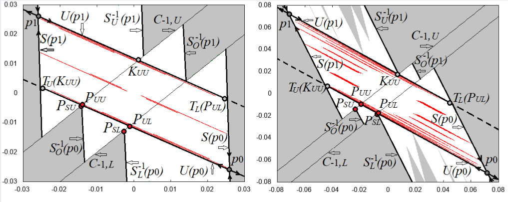
Second New Model

d>0, h>0

Chaos and 2-cycle in region R_4

At $c = 2(b+1)$, $0 < b < 1$, the 2-cycle saddle LU appears, but increasing c it disappears colliding with the critical lines, and it exists for $2(b+1) < c < c_B$, where $c_B = 2(1+b) + \frac{2d}{h}$.

The role of the stable set of the saddle 2-cycle is the same as for $h = 0$, bounded trajectories exist up to its homoclinic bifurcation, which now may occur in two different ways: when $P_{SU} = P_{UU}$ or when $P_{SL} = P_{UL}$.



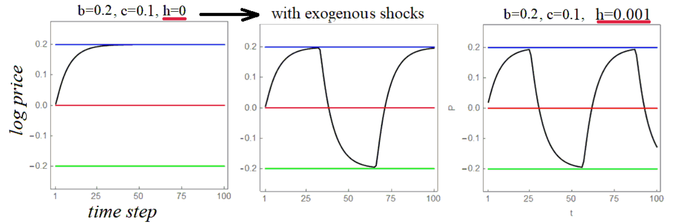
(a) $P_{SU} \approx P_{UU}$ at $b = 0.05$, $c = 2.87$; (b) $P_{SL} \approx P_{UL}$ at $b = 0.6$, $c = 3.48$

Second New Model

d>0, h>0

Emergence of endogenous boom-bust dynamics which do not exist in the basic model. The second model is even more powerful in helping us to understand the emergence of endogenous boom-bust cycles.

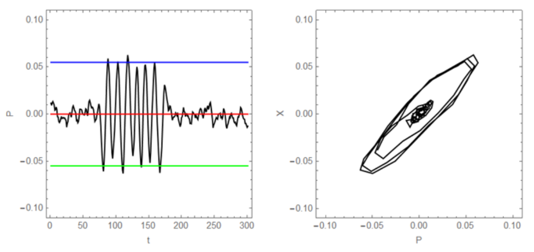
In the case already shown for the first map ($h = 0$) in R_2 without exogenous shocks the trajectory converges to a virtual fixed point, but under the action of some exogenous shock the typical boom-bust dynamics emerges. This is completely endogenous in the more general model just with $h = 0.001$:



Second New Model

d>0, h>0

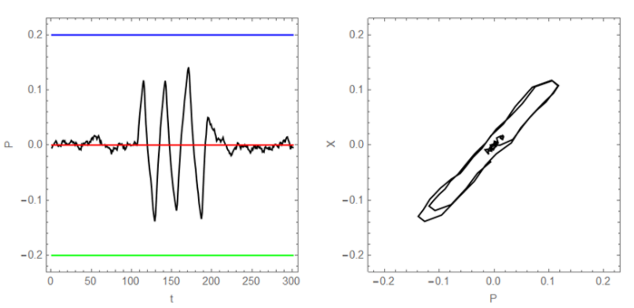
In the full model, where we always have coexistence of attracting cycles with the stable fundamental fixed point, exogenous shocks can bring about irregular switching between both regimes, leading to periods where stock markets do well, i.e. stock prices are close to the fundamental value and do not fluctuate strongly, and periods where stock markets do not well, i.e. stock prices display booms and slumps and fluctuate strongly. An example with parameters in region R_1, at $b = 0.45$, $c = 0.2$, $d = 0.011$, $h = 0.01$, and normally distributed shocks at $\sigma = 0.003$ added at every period:



Second New Model

d>0, h>0

Another example with parameters in region R_2, at $b = 0.05$, $c = 0.1$, $d = 0.02$, $h = 0.01$, and normally distributed shocks with $\sigma = 0.003$ added at every period:

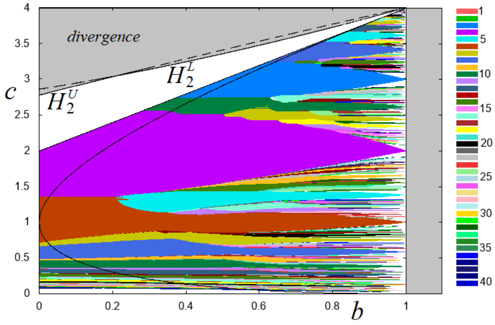


Hence, the full model produces a more interesting and realistic picture of actual stock market dynamics.

d>0, h>0

Robustness

As already remarked, the dynamics evidenced in our model are quite robust. We have used the same value d for the demand both when sentiment traders buy and sell. But also considering two different values, the dynamic behaviors are similar, as evidenced by the 2D bifurcation diagram in which with $h = 0.01$ we have fixed $d_L = 0.02$ and $d_U = 0.03$



2-cycle

$$P_0 = \frac{d_U(1+b) - v d_L}{c\,(c - 2(b+1))},$$

$$X_0 = \frac{-d_L(b+1) + v d_U}{c\,(c - 2(b+1))}$$

homoclinic bifurcation curves

$$H_2^U : \frac{y_0 - x_0/\lambda_2 - h}{1 - 1/\lambda_2} = \frac{\lambda_1 y_0 - x_0 - bh}{c - 1 + \lambda_1}$$

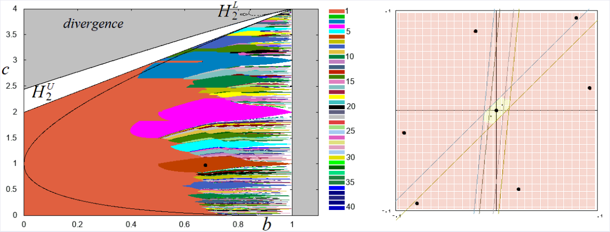
$$H_2^L : \frac{y_0 - x_0/\lambda_2 + h}{1 - 1/\lambda_2} = \frac{\lambda_1 y_0 - x_0 + d_L + bh}{c - 1 + \lambda_1}$$

Second New Model

d>0, h>0

Robustness

We have considered the condition $d > h$ since it seems quite reasonable. The numerical 2D bifurcation diagram in which we have fixed $d = 0.005 < h = 0.01$ evidences some similarity and some differences:



There are parameter regions in which the system has only the attracting fundamental fixed point, but increasing the role of chartists (increasing b) the coexistence with attracting cycles occurs. In the figure at the point $(b, c) = (0.7, 0.9)$ we show the coexistence with a 6-cycle.

Second New Model

d>0, h>0 - concluding remarks

Both models ($d > 0$, $h = 0$) and the full model ($d > 0$, $h > 0$) are quite powerful in understanding the dynamics of the stock market.

In particular, the map with two discontinuities leads to new kind of dynamic behaviours, that have not been seen in other models: the existence of two different kinds of families of cycles issuing from the same point of $b = 1$ (at the center bifurcation), related to symbolic sequences with two symbols and with three symbols).

In the full model, in the whole stability box we have coexistence between the fundamental fixed point and other cycles, the virtual nonfundamental steady states are attracting ghosts, leading to the co-existence of attracting cycles.

d>0, h>0 - concluding remarks

The full model is still under study. In particular the center bifurcation and the two families of periodicity regions issuing from from $(b, c) = (1, c_{m/n})$. Moreover, in a left neighborhood of the line $b = 1$ we can assume that the two different families of periodicity regions persist and are organized according to the Farey summation rule applied to the rotation numbers, but this structure changes as b decreases.

The asymmetric case, with d_L different from d_U, is future work, as well as the system when the condition $d < h$ holds.

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