Discrete dynamic models in social sciences: strategic interaction, rationality, evolution

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International Online GSDUAB Seminar

November 14, *2022*

Time in economics is often discrete, *event driven dynamics*

Two famous exemplary cases

- Cobweb model for price dynamics (Nicholas Kaldor, 1934)
- Duopoly model (Augustine Cournot, 1838)

OUTLINE OF THE LECTURE

- Cobweb with fading memory and maps with vanishing denominator

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- Evolutionary competition between different behavioural rules and replicator dynamics
- Some future extensions

Cobweb model

Consider a good sold in the market at a unit price $p(t)$.

- Demand: $Q^d(t) = D(p_t)$, usually decreasing (hence invertible).
- Supply function $Q^s(t) = S(p_t^e)$ (increasing)
- Economic equilibrium: $Q^d(t) = Q^s(t) \implies D(p_t) = S(p_t^e)$

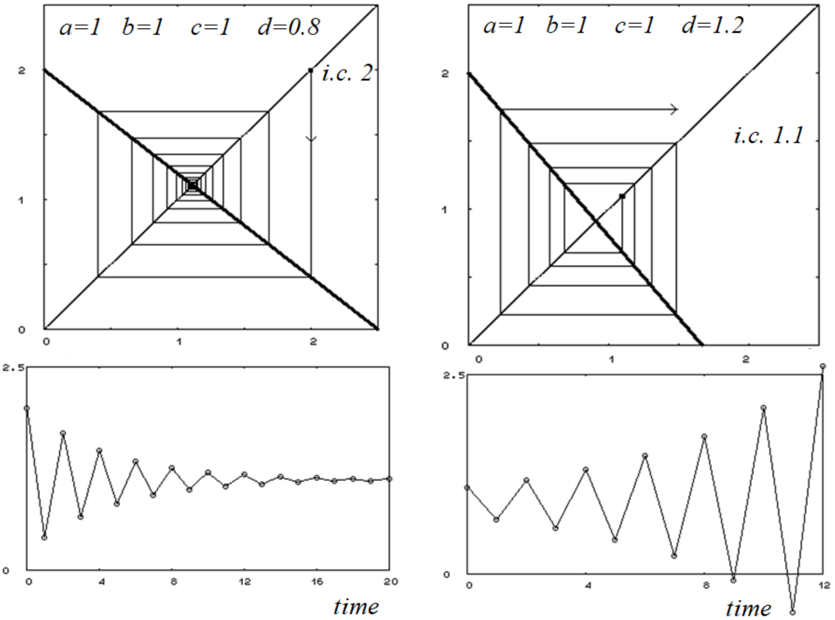
discrete dynamics

$\Delta t = 1$: production lag (maturation period for agricultural products, production time for an industrial process)
Naïve expectations $p_t^e = p_{t-1}$
Matching between demand and supply: $D(p_t) = S(p_{t-1})$

Hence $p_{t+1} = f(p_t) = D^{-1}(S(p_t))$

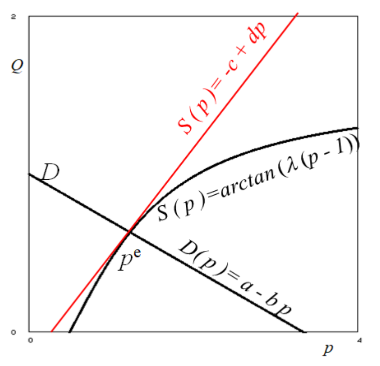
Example: linear demand and linear supply:
$D(p) = a - bp$; $S(p) = -c + dp$

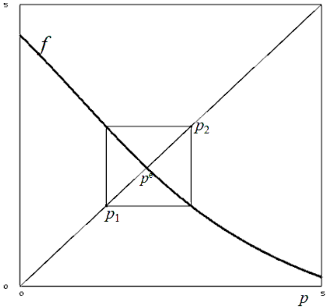
$$p_{t+1} = f(p_t) = -\frac{d}{b}p_t + \frac{a+c}{b}$$

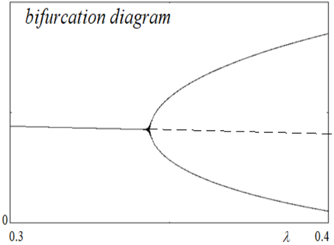

a=1 b=1 c=1 d=0.8
i.c. 2
a=1 b=1 c=1 d=1.2
i.c. 1.1
time
time

Nonlinear supply with saturation

$S(p) = \arctan(\lambda(p-1))$







$D(p_{t+1}) = S(p_t)$ gives:
$p_{t+1} = f(p_t) = \frac{1}{b}\left[a - \arctan(\lambda(p_t - 1))\right]$
nonlinear decreasing map.

Chiarella C. (1988) "The cobweb model. Its instability and the onset of chaos", *Economic Modelling*.

Hommes C. (1991) "Adaptive learning and roads to chaos. The case of the cobweb", *Economic Letters*.

Adaptive expectations

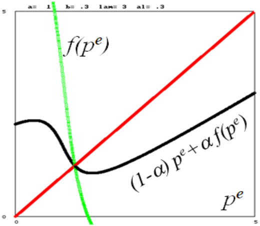
$p^e_{t+1} = p^e_t + \alpha(p_t - p^e_t) = (1-\alpha)p^e_t + \alpha p_t$,
$0 \leq \alpha \leq 1$.
For $\alpha = 1$ reduces to naïve $p^e_{t+1} = p_t$.

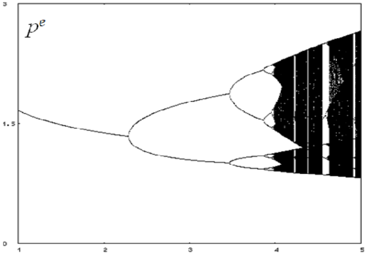
From $p_t = f(p^e_t)$ the law of motion

$$p^e_{t+1} = (1-\alpha)\, p^e_t + \alpha f(p^e_t)$$

in the space of expected prices.
Then $p_t = f(p^e_t)$ from beliefs to realizations

For the model
$p_t = f(p^e_t) = \frac{1}{b}\left[a - \arctan\left(\lambda(p^e_t - 1)\right)\right]$
we get a bimodal map.





Statistical learning

Dimitri (1988), "A short remark on learning of Rational Expectations", *Economic Notes*.

Holmes, Manning (1988) "Memory and market stability: The case of the Cobweb", *Economic Letters*

Cobweb model $p_{t+1} = f(p_{t+1}^{(e)})$
with $p_{t+1}^{(e)}$ average of past prices

$$p_{t+1}^{(e)} = \sum_{k=1}^{t} a_{tk} p_k, \text{ with } a_{tk} \geq 0, \text{ and } \sum_{k=1}^{t} a_{tk} = 1$$

numerically show that "the evolution of the model is very much dependent upon the starting position" and "intermediate run dynamics can be rather complex and of considerable interest".

Fading memory

- Bischi and Gardini (1997) *Int. Jou. of Bifurcation and Chaos*.
- Bischi and Naimzada (1997) *Economic notes*, 1997

Weights distributed as the terms of a geometric sequence of ratio $\rho \in [0, 1]$

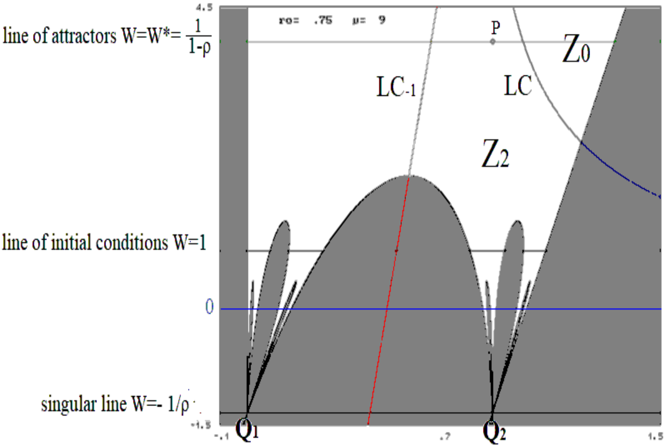
$$a_{tk} = \frac{\rho^{t-k}}{W_t}, \quad \text{with} \quad W_t = \sum_{k=1}^{t} \rho^{t-k} = \begin{cases} \frac{1-\rho^t}{1-\rho} & \text{if } 0 \le \rho < 1 \\ t & \text{if } \rho = 1 \end{cases}.$$

Taking $z_t = p_{t+1}^{(e)} = \sum_{k=1}^{t} \frac{\rho^{t-k}}{W_t} p_k$ and W_t (partial sum of geometric series) as dynamical variables

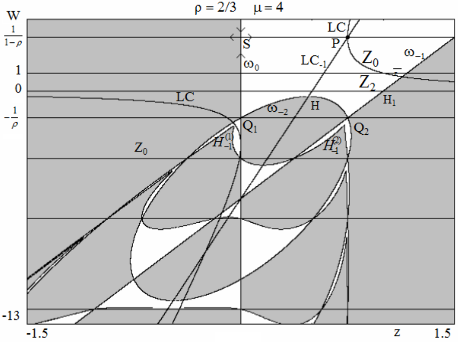
$$T : \begin{cases} z_{t+1} = & \frac{\rho W_t}{1+\rho W_t} z_t + \frac{1}{1+\rho W_t} f(z_t) \\ W_{t+1} = & 1 + \rho W_t \end{cases}$$

2-dim equivalent map, with i.c.$(z_1, W_1) = (p_1, 1)$ and attractors in the limiting invariant line $W = W^* = \frac{1}{1-\rho}$

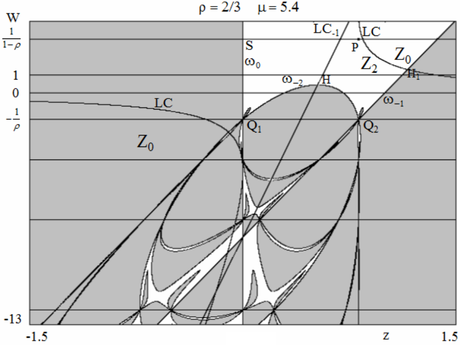
Quadratic map $f(z) = \mu z(1-z)$ (like in the example of Dimitri)



The two preimages of the line $W = 0$ are the points $Q_1 = \left(0, -\frac{1}{\rho}\right)$ and $Q_2 = \left(\frac{\mu-1}{\mu}, -\frac{1}{\rho}\right)$ where the first component of the map assumes the form $\frac{0}{0}$.



ρ = 2/3 μ = 4
W
1/(1−ρ)
1
0
−1/ρ
-13
-1.5
z
1.5
LC
P
S
ω0
LC-1
Z0
ω−1
Z2
H1
H
ω−2
Q1
Q2
Z0



ρ = 2/3 μ = 5.4
W
1/(1−ρ)
1
0
−1/ρ
-13
-1.5
z
1.5
LC-1
LC
S
P
ω0
Z0
Z2
H1
ω−2
H
LC
ω−1
Q1
Q2
Z0

Maps with vanishing denominator and focal points

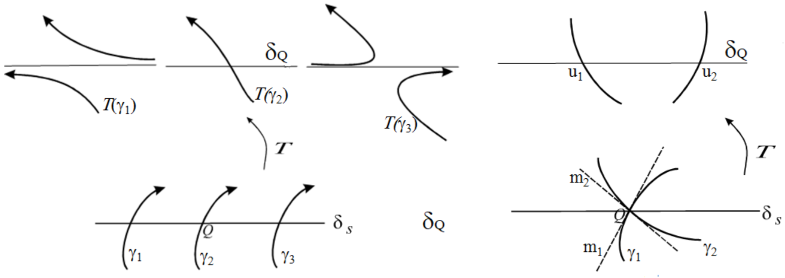
- Bischi, L. Gardini (1997) "Basin fractalization due to focal points in a class of triangular maps", *Int. Jou. of Bifurcation and Chaos*.
- Bischi, Gardini, Mira (1999) "Maps with denominator. Part I: some generic properties", *Int. Jou. of Bifurcation & Chaos*.
- Bischi, Gardini, Mira. (2003) "Plane maps with denominator. Part II: noninvertible maps with simple focal points", *Int. Jou. of Bifurcation and Chaos*.
- Bischi, Gardini, Mira (2005) "Plane Maps with Denominator. Part III: Non simple focal points and related bifurcations", *Int. Jou. of Bifurcation and Chaos*.

$$T : \begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases} \quad \text{where} \quad F(x,y) = \frac{N_1(x,y)}{D_1(x,y)} \text{ and/or } G(x,y) = \frac{N_2(x,y)}{D_2(x,y)}$$

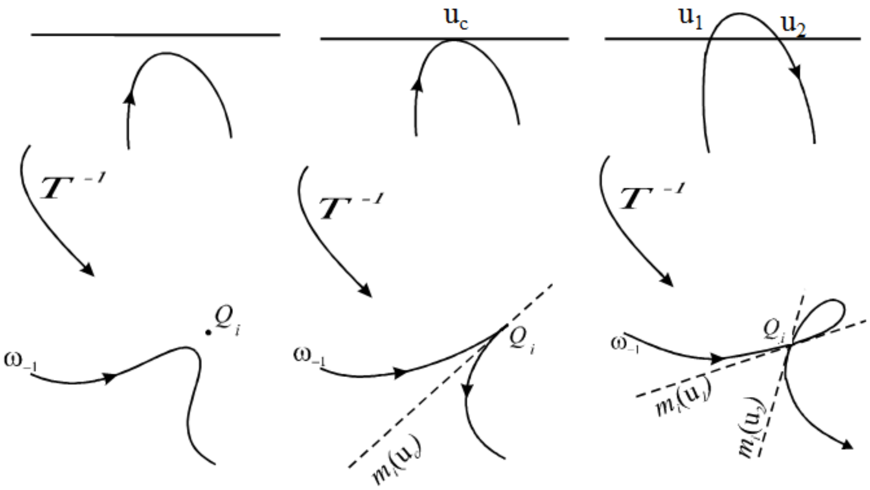
Definitions

In δ_s (Singular set) one denominator vanishes. $Q \in \delta_s$ focal point if at least one component of the map T becomes 0/0 in Q and there exist smooth arcs $\gamma(t)$, with $\gamma(0){=}Q$, such that $\lim_{\tau \to 0} T(\gamma(\tau))$ is finite. The set of all such finite values is the prefocal set δ_Q

One-to-one correspondence between slope of γ through Q_i and point where $T(\gamma)$ crosses δ_Q



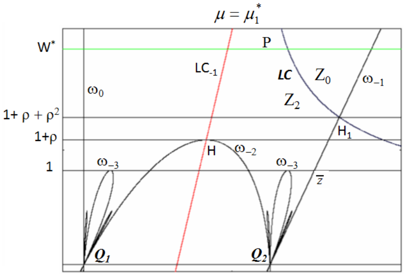
Roughly speaking, a *prefocal curve* is a set of points for which at least one inverse exists which maps (or "focalizes") the whole set into a *focal point*.

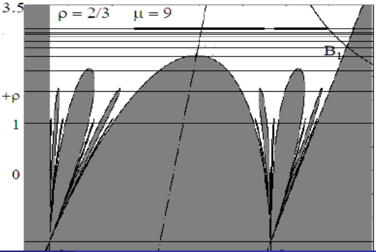


Sequence of bifurcations

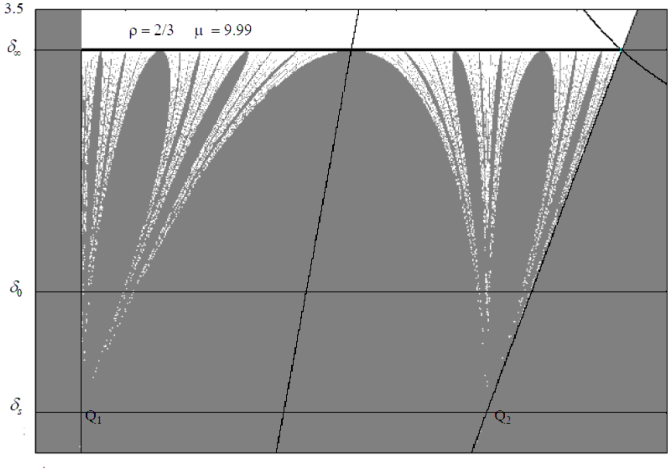
As μ is increased, at $\mu_0^* = 2 + 2\sqrt{1+\rho}$ the vertex H of the parabola ω_{-2} is on the line $W = W_1 = 1 + \rho$ and, as a consequence, the curve ω_{-2} becomes tangent to the line of initial conditions $W = W_0 = 1$

At $\mu = \mu_1^*$ H is on $W = W_2 = 1 + \rho + \rho^2$. At this value of μ two lobes of $\mathcal{B}(\infty)$, bounded by ω_{-3}, reach the line of initial conditions and two new holes are created



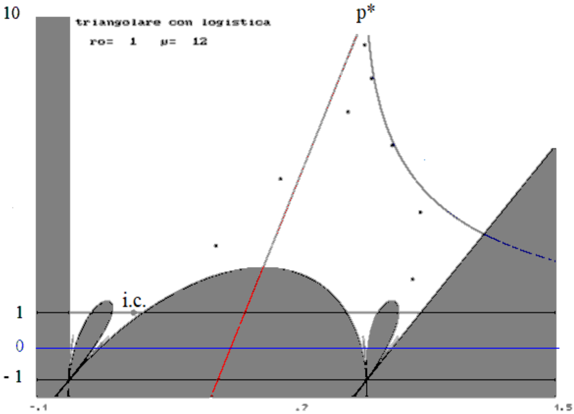


When $\mu = \mu^*_\infty = \lim_{n\to\infty} \mu^*_n = \frac{4-\rho}{1-\rho}$ the vertex H, together with all of its infinite preimages on the top of the lobes, reach the line of the ω-limit sets $W = W^*$. The basin along $W = 1$ is a Cantor set.



Bray, M. (1983) "Convergence to rational expectations equilibrium" in Friedman and Phelps (eds), *Individual forecasting and aggregate outcomes*, Cambridge University Press.

Uniform average, limiting case $\rho = 1$, $W^* = \frac{1}{1-\rho} \to \infty$



Classical Cournot Oligopoly Model with rational players

Market with N firms $i = 1, ..., N$

Inverse demand $p = f(Q)$, with $Q = \sum_{i=1}^{N} q_i$

Cost functions $C_i(q_i)$, $i = 1, ..., N$

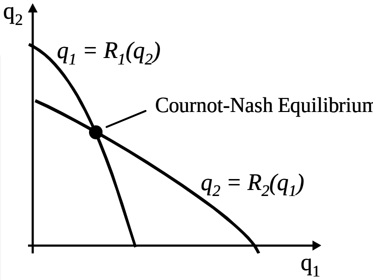
Max. expected profit $\pi_i = pq_i - C_i(q_i)$:

Intersection $q_i = R_i(q_{-i})$ (Cournot-Nash equilibrium) computed and reached in one shot

$$q_i(t+1) = \arg\max_{q_i}[f_i^e\left(q_i + q_{-i}^e\left(t+1\right)\right)q_i - C_i\left(q_i\right)].$$

Rationality, info set, computational ability

- Demand function $f_i^e = f(Q)$, $\forall i$;
- Its own cost function $C_i(q_i)$
- Able to solve the max. problem
- Perfect Foresight $q_{-i}^e(t+1) = q_{-i}(t+1)$;



Dynamic Cournot linear model (1838)

BR: Best Reply dynamics (with naive expectations)

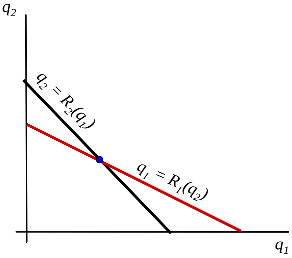
$q_i(t+1) = R_i\left(q^e_{-i}(t+1)\right)$ with $q^e_{-i}(t+1) = q_{-i}(t)$

discrete dynamical system: $q_i(t+1) = R_i\left(q_{-i}(t)\right)$

Linear demand p(t)=a-bQ, linear costs

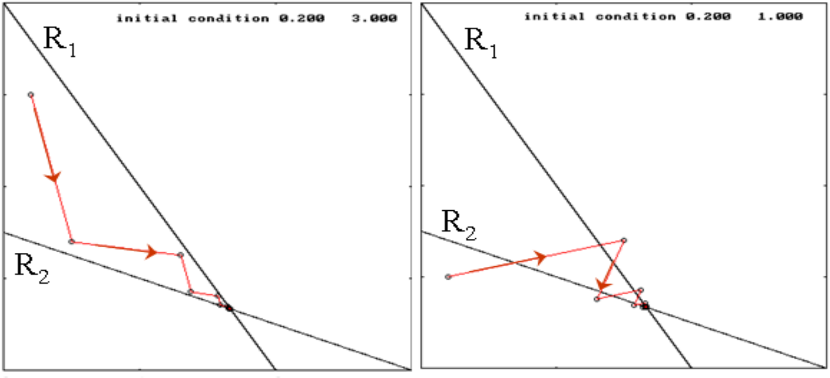
- $\pi_i(t) = (a - b(q_1 + q_2))q_i(t) - c_i q_i$
- FOC: $a - 2bq_i - bq_j - c_i = 0$
- reaction functions with naive expectations:

$$q_1(t+1) = R_1(q_2(t)) = -\frac{1}{2}q_2(t) + \frac{a-c_1}{2b}$$
$$q_2(t+1) = R_2(q_1(t)) = -\frac{1}{2}q_1(t) + \frac{a-c_2}{2b}$$



Unique NE, always stable

Cournot tâtonnement towards Cournot-Nash Equilibrium



Introducing nonlinearities

"Models of duopoly have always held a fascination for mathematically inclined economists"

Shubik, 1981, in *Handbook of Mathematical Economics*

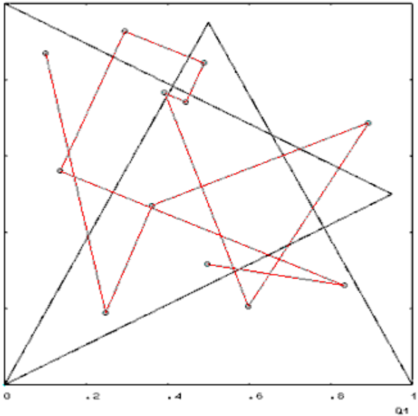
"... and also for economically inclined mathematicians"

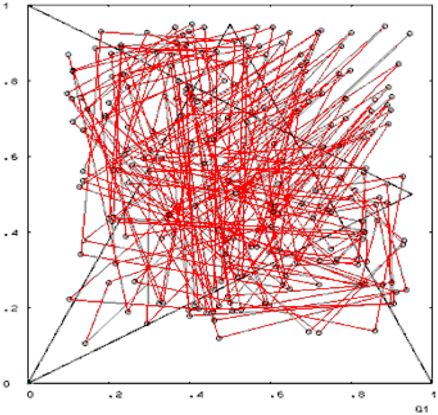
see e.g. Bischi, Chiarella, Kopel, Szidarovszky *Nonlinear oligopolies: Stability and Bifurcations*, Springer 2010

Unimodal reaction functions

Rand, D. (1978) "Exotic Phenomena in games and duopoly models", *Journal of Mathematical Economics*.

Chaotic dynamics, i.e. bounded oscillations with sensitive dependence on initial conditions



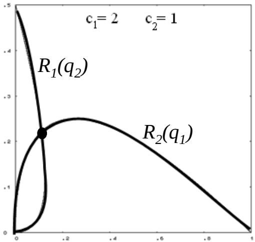


Economically funded unimodal reaction functions

Isoelastic demand $p = \frac{1}{Q}$ *(Puu, CS&F 1991)*

- $\pi_i(t) = \frac{q_i}{q_1+q_2} - c_i q_i$
- FOC: $\frac{q_j}{(q_1+q_2)^2} - c_i = 0$
- Reaction functions

$$q_1 = R_1(q_2) = \sqrt{q_2 / c_1} - q_2$$
$$q_2 = R_1(q_2) = \sqrt{q_1 / c_2} - q_1$$

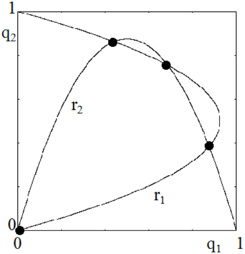


Nonlinear cost (Kopel, CS&F 1996)

- Linear demand: $p = a - b(q_1 + q_2)$
- Cost functions with externalities

$$C_i = d + aq_i - b(1+2\mu)q_i q_j + 2b\mu q_i q_j^2$$

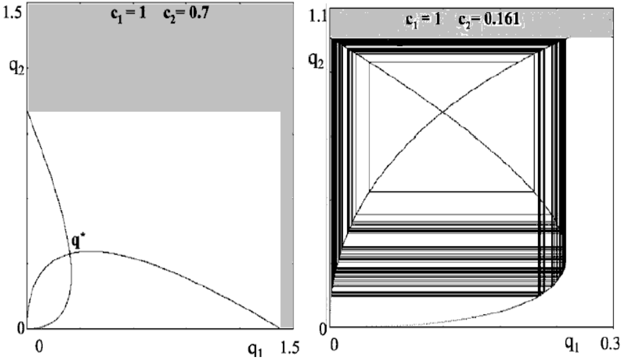
Reaction functions: $q_i = R_i(q_j) = \mu_i q_j (1 - q_j)$



Best reply with naive expectations

Puu, T. 1991. "Chaos in Duopoly pricing", *Chaos, Solitons & Fractals*

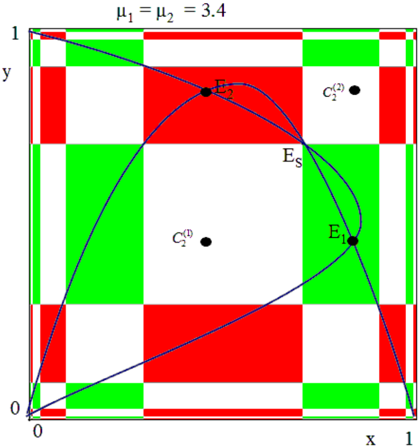
$$\begin{cases} q_1(t+1) = R_1(q_2(t)) = \sqrt{q_2(t)/c_1} - q_2(t) \\ q_2(t+1) = R_2(q_1(t)) = \sqrt{q_1(t)/c_2} - q_1(t) \end{cases}$$

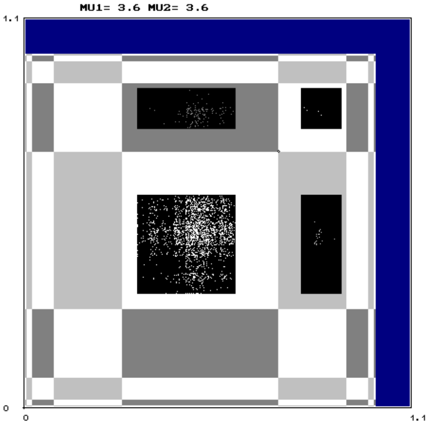


Bischi, Mammana, Gardini (2000). "Multistability and cyclic attractors in duopoly games". *Chaos, Solitons and Fractals*.

"Kopel map" $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T : \begin{cases} x' = r_1(y) = \mu_1 y(1-y) \\ y' = r_2(x) = \mu_2 x(1-x) \end{cases}$$





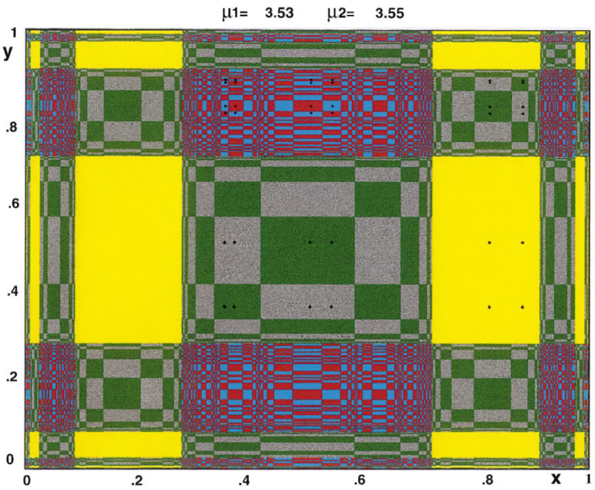


Fig. 2. The black points represent the periodic points of the five coexisting attracting cycles of the map (5) with $\mu_1 = 3.53$ and $\mu_2 = 3.55$. Each basin of attraction, represented by a different color, is formed by disjoint rectangles, given by the immediate basin (containing the periodic points) and all its preimages.

Bounded rationality: partial information

$$q_i(t+1) = \arg\max_{q_i(t+1)} \pi_i^e(t+1)$$

expected profit at time $t+1$ on the basis of information available at time t

1. Expectations on competitor's choices:

$$q_i(t+1) = \arg\max_{q_i} f\left(q_i + Q_i^e(t+1)\right) q_i - C_i\left(q_i, q_{-i}^e(t+1)\right)$$

2. Subjective expected demand function

$$q_i(t+1) = \arg\max_{q_i} f^e\left(q_i + Q_i^e(t+1)\right) q_i - C_i\left(q_i, q_{-i}^e(t+1)\right)$$

Bischi, Chiarella, Kopel, Szidarovszky, (2010) *Nonlinear Oligopolies: Stability and Bifurcations*, Springer.

Adaptive adjustment towards Best Reply

Inertia in adopting the computed output (anchoring attitude)

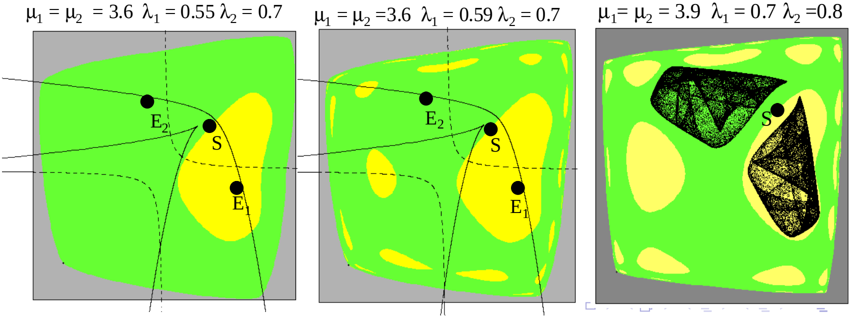
$q_i(t+1) = (1-\lambda_i)\, q_i(t) + \lambda_i R_i\,(q_{-i}(t)), \quad 0 \le \lambda_i \le 1$

$\lambda_i \in [0,1]$ represents the attitude of firm i to adopt the best reply

$(1-\lambda_i)$ is the anchoring, a measure of inertia.

- It reduces to best reply for $\lambda_i = 1$, complete inertia as $\lambda_i \to 0$.
- It has the same (Nash) equilibria as the best reply model

Example: Kopel model $R_i(q_j) = \mu_i q_j\,(1-q_j)$



Adaptive expectations

Bischi, Kopel (2001). "Equilibrium Selection in a Nonlinear Duopoly Game with Adaptive Expectations", *Journal of Economic Behavior and Organization*.

$$q_1(t+1) = R_1\left(q_2^e(t+1)\right)$$
$$q_2(t+1) = R_2\left(q_1^e(t+1)\right)$$

with adaptive expectations

$$q_1^e(t+1) = q_1^e(t) + \alpha_1\left(q_1(t) - q_1^e(t)\right) = (1-\alpha_1)q_1^e(t) + \alpha_1 R_1\left(q_2^e(t)\right)$$
$$q_2^e(t+1) = q_2^e(t) + \alpha_2\left(q_2(t) - q_2^e(t)\right) = (1-\alpha_1)q_1^e(t) + \alpha_2 R_1\left(q_2^e(t)\right)$$

$\alpha_i \in [0,1]$, adaptive adjustment in the beliefs space.

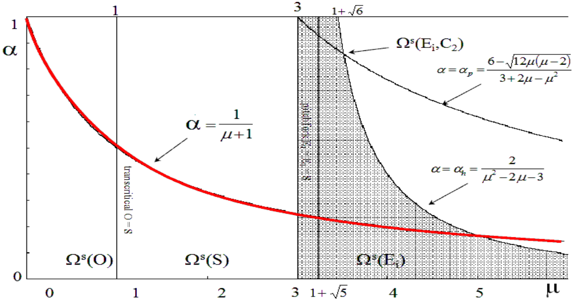
The real outputs at each step: mapping from beliefs to realizations

$$\begin{cases} q_1(t) = R_1\left(q_2^e(t)\right) \\ q_2(t) = R_2\left(q_1^e(t)\right) \end{cases}$$

iterated map $T : \begin{cases} x' = (1-\alpha_1)x + \alpha_1\mu_1 y(1-y) \\ y' = (1-\alpha_2)y + \alpha_2\mu_2 x(1-x) \end{cases}$

Theorem

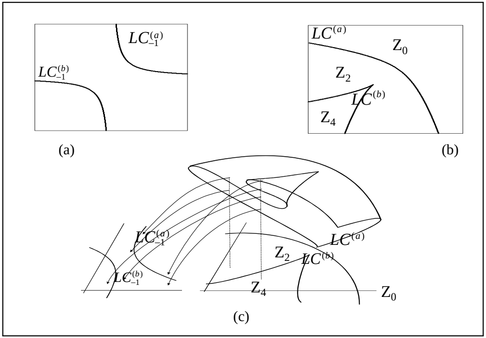
(Global bifurcation of the basins with homogeneous players). *Let $\alpha_1 = \alpha_2 = \alpha$ and $\mu_1 = \mu_2 = \mu$. If $\alpha(\mu+1) < 1$ then the two basins are simply connected sets; if $\alpha(\mu+1) > 1$ they are formed by infinitely many disjoint components.*



Jacobian matrix: $DT(x,y) = \begin{bmatrix} 1-\alpha_1 & \alpha_1\mu_1(1-2y) \\ \alpha_2\mu_2(1-2x) & 1-\alpha_2 \end{bmatrix}$

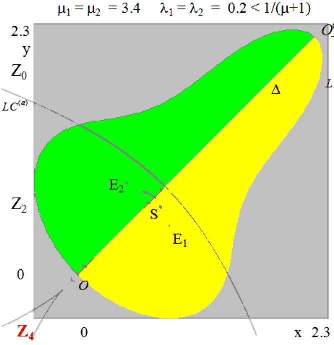
LC_{-1} : $\det DT = 0$, i.e. $\left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right) = \frac{(1-\alpha_1)(1-\alpha_2)}{4\alpha_1\alpha_2\mu_1\mu_2}$

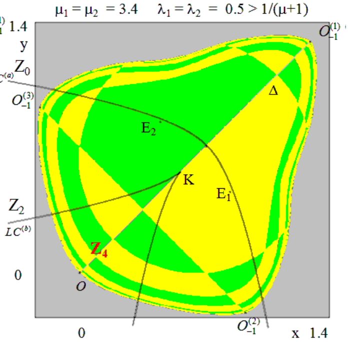
Equilateral hyperbola, union of two branches $LC_{-1} = LC_{-1}^{(a)} \cup LC_{-1}^{(b)}$,
Hence also $LC = T(LC_{-1})$ is formed by two disjoint branches



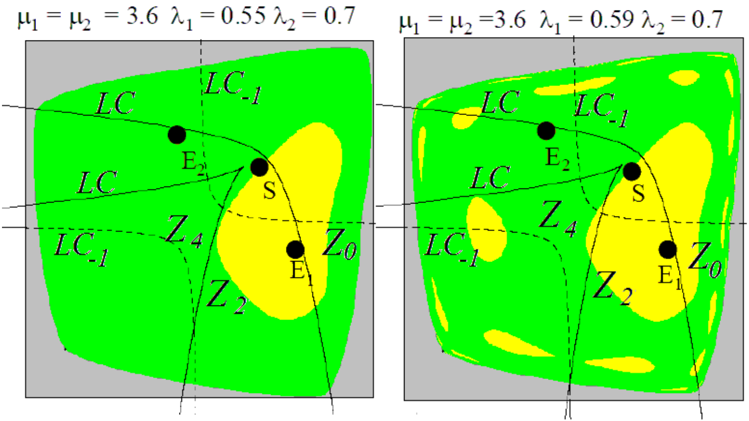
Symmetric case

$K_{-1} = LC_{-1}^{(b)} \cap \Delta = (k_{-1}, k_{-1})$ with $k_{-1} = \frac{\alpha(\mu-1)-1}{2\alpha\mu}$ the eigenvalue $z_{\perp}$ vanishes and the curve $LC^{(b)} = T(LC_{-1}^{(b)})$ has a *cusp point* $K = LC^{(b)} \cap \Delta = (k, k)$ with $k = f(k_{-1}) = \frac{(\alpha(\mu+1)-1)(\alpha\mu+3(1-\alpha))}{4\alpha\mu}$ at $\alpha(\mu+1) = 1$ $K \equiv O$ and the cusp point K.



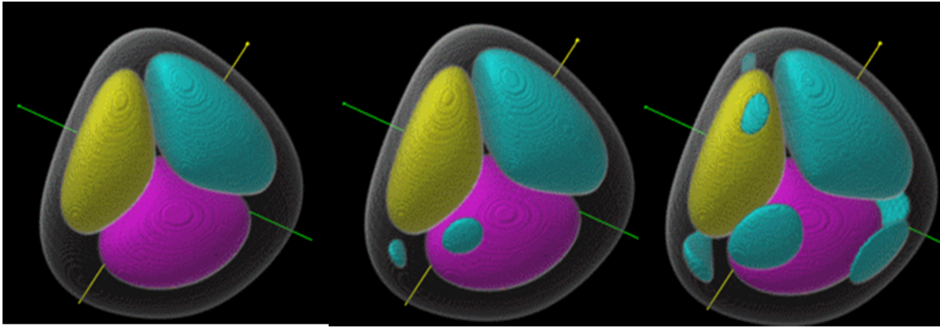


Heterogeneous behaviour: computer aided proof?



Beyond 2D. Case of a triopoly game

Bischi G.I., L. Mroz and H. Hauser (2001). "Studying basin bifurcations in nonlinear triopoly games by using 3D visualization". *NonlinearAnalysis TMA*.



Local Monopolistic Approximation (LMA)

Bischi, Naimzada, Sbragia (2007) "Oligopoly Games with Local Monopolistic Approximation " *Journal of Economic Behavior & Organization*

Firms do not know the demand, at any time get a correct (local) estimate of demand slope by marketing experiments of small quantity variations

$$\frac{\partial f(Q)}{\partial q_i} = \frac{df(Q)}{dQ} \simeq \frac{f(q_i(t) + \Delta q_i + q_{-i}(t)) - f(q_i(t) + q_{-i}(t))}{\Delta q_i}$$

or small price variations: $\frac{\partial f(Q)}{\partial q_i} = \frac{df(Q)}{dQ} = \left[\frac{dQ(p)}{dp}\right]^{-1}$ where $Q = f^{-1}(p)$

Conjectured demand function: Linear and monopolistic approximation

$$p^e(t+1) = p(t) + f'(Q)(q_i(t+1) - q_i(t)), \quad \text{where } p(t) = f(Q(t))$$

FOC becomes $p(t) + 2f'(Q)q_i(t+1) - f'(Q)q_i(t)) - C_i'(q_i(t+1)) = 0$

From $f(Q) + 2f^{'}(Q)q_i(t+1) - f^{'}(Q)q_i(t)) - C_i^{'}(q_i(t+1)) = 0$
A *linear* equation, hence an *explicit* dynamical system, is get with:

LMA with linear cost $C_i = c_i q_i$

$$q_i(t+1) = \frac{1}{2}q_i(t) - \frac{f(Q(t)) - c_i}{2f^{'}(Q(t))} \qquad i = 1, ..., n$$

LMA with quadratic cost $C_i = (c_{i0} + c_i q_i) q_i$

$$q_i(t+1) = \frac{q_i(t)f_i(t) - p(t)}{2[f_i(t) - c_i]} \qquad i = 1, ..., n$$

The steady states are the Cournot-Nash equilibria

Reduced Information set for a form using LMA approach

(i1) No knowledge of demand function, only local estimate of slope;

(i2) No expectations on other firms' future production;

(i3) Solve a quadratic optimization problem, i.e. a linear equation;

No optimization at all: Just following the Profit Gradient

Bischi, Naimzada (2000) "Global Analysis of a Duopoly game with Bounded Rationality", in *Advances in Dyn. Games and applications*

Each firm infers how the market will respond to its production changes by a (correct) estimate of the marginal profit $\frac{\partial \pi_i}{\partial q_i}$.

Gradient (or myopic) adjustment

With this local information a firm increases (decreases) its output if it perceives a positive (negative) marginal profit

$$q_i(t+1) = q_i(t) + v_i q_i(t) \frac{\partial \pi_i(t)}{\partial q_i} \,; \quad i = 1, 2$$

where v_i is a relative speed of adjustment, being $\frac{q_i(t+1) - q_i(t)}{q_i(t)} = v_i \frac{\partial \pi_i}{\partial q_i}$.

Example: linear demand p = a - bQ, linear costs Ci = ciqi

Profit: $\Pi_i(q_1, q_2) = q_i\left[a - b(q_1 + q_2) - c_i\right]$

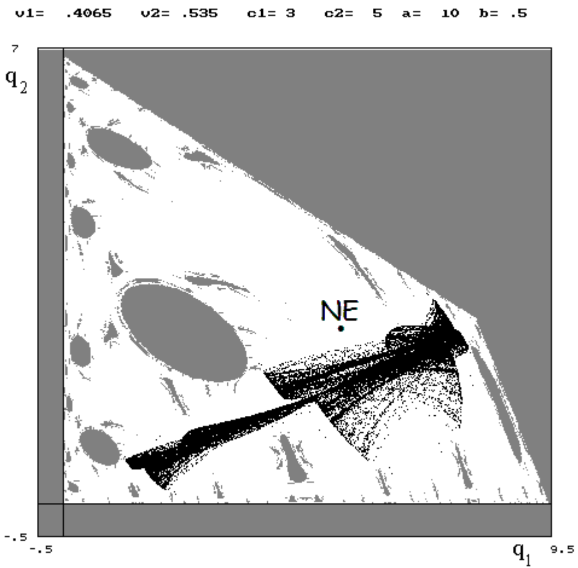
Marginal profit $\frac{\partial \Pi_i}{\partial q_i} = a - c_i - 2bq_i - bq_j \quad , \quad i, j = 1, 2, \; j \neq i.$

Model with profit gradient relative adjustment

$$\begin{cases} q_1(t+1) = & (1 + v_1(a - c_1))q_1(t) - 2bv_1 q_1^2(t) - bv_1 q_1(t) q_2(t) \\ q_2(t+1) = & (1 + v_2(a - c_2))q_2(t) - 2bv_2 q_2^2(t) - bv_2 q_1 q_2(t) \end{cases}$$

Invariant axes with boundary equilibria
$E_0 = (0, 0)$, $E_1 = \left(\frac{a - c_1}{2b}, 0\right)$, $E_2 = \left(0, \frac{a - c_2}{2b}\right)$
Interior (Nash) equilibrium
$E_* = \left(\frac{a + c_2 - 2c_1}{3b}, \frac{a + c_1 - 2c_2}{3b}\right)$

Complex attractors and basins



Case of identical players

The dynamical system is the same if the variables are swapped:
$T \circ P = P \circ T$, $P : (x_1, x_2) \to (x_2, x_1)$ reflection through the diagonal Δ.
This implies that the diagonal is mapped into itself, i.e., $T(\Delta) \subseteq \Delta$:
Identical players starting from identical initial conditions behave identically for each time (*synchronized trajectories)* governed by the map

$$\mathbf{x}(t+1) = f(\mathbf{x}(t)) \quad \text{with} \quad f = T|_{\Delta} : \Delta \to \Delta.$$

"representative agent" whose dynamics summarize the common behavior of the synchronized competitors.

Bischi , Gallegati, Naimzada (1999). "Symmetry-breaking bifurcations and representative firm in dynamic duopoly games". *Annals of Operations Research*

Bischi, Stefanini, Gardini (1998) “Synchronization, intermittency and critical curves in duopoly games”, *Mathematics and Computers in Simulations*.

A trajectory starting out of Δ, i.e. with $x_0 \neq y_0$, is said to synchronize if $|x_1(t) - x_2(t)| \to 0$ as $t \to +\infty$.

Problem

Let A_s be an attractor of the one-dimensional restriction. Is it also an attractor for the two-dimensional map T?

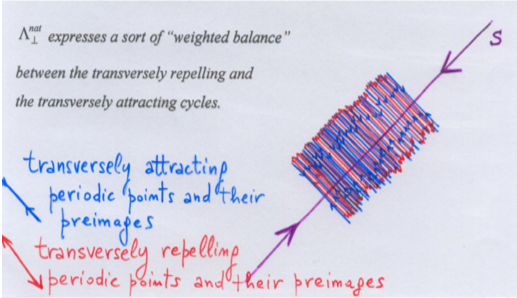
A question of *transverse stability*: stability of A_s with respect to perturbations transverse to Δ

Interesting when dynamics on Δ are chaotic (*chaos synchronization*)
The key property is that a chaotic set A_s includes infinitely many periodic points which are unstable in the direction along Δ.
$DT(x,x) = \{T_{ij}(x)\} : T_{11} = T_{22}$ and $T_{12} = T_{21}$.
Eigenvalues $\lambda_{\parallel}(x) = T_{11}(x) + T_{12}(x)$
and $\lambda_{\perp}(x) = T_{11}(x) - T_{12}(x)$
with related eigenvectors $\mathbf{v}_{\parallel} = (1,1)$, $\mathbf{v}_{\perp} = (1,-1)$
Transverse Lyapunov exponents $\Lambda_{\perp} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N} \ln|\lambda_{\perp}(s_i)|$ where $\{s_i = f^i(s_0), i \geq 0\}$ is a trajectory embedded in A_s.

Spectrum of transverse Lyapunov exponents computed at the infinitely many periodic cycles

$$\Lambda_{\perp}^{\min} \leq ... \leq \Lambda_{\perp}^{nat} \leq ... \leq \Lambda_{\perp}^{\max}$$

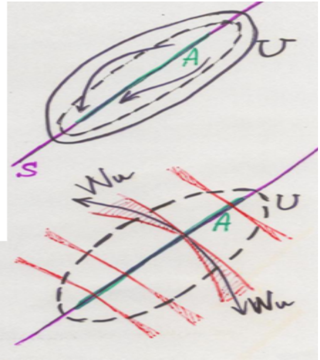
$\Lambda_{\perp}^{nat}$ computed along a generic aperiodic trajectory, is a "weighted balance" between the transversely repelling and attracting cycles.



If $\Lambda_{\perp}^{\max} < 0$ (all cycles embedded in A_s are transversely stable) A_s is asymptotically stable.
If $\Lambda_{\perp}^{\max} > 0$, while $\Lambda_{\perp}^{nat} < 0$, A_s is not Lyapunov stable, but is a *Milnor attractor*

Definition. *A closed invariant set A is said to be a weak attractor in Milnor sense (or simply Milnor attractor) if its stable set* $B(\mathcal{A})$ *has positive Lebesgue measure.*

Note that a topological attractor is also a Milnor attractor, whereas the converse is not true.



If $\mathcal{A} \subset \Delta$ is a chaotic attractor of $T|_{\Delta}$ then it is a non-topological Milnor attractor if

(a) $\Lambda_{\perp}^{\text{max}} > 0$

(b) $\Lambda_{\perp}^{nat} < 0$.

$\Lambda_{\perp}^{max}$ from negative to positive marks a *riddling (or bubbling) bifurcation*. Two possible scenarios according to the fate of locally repelled trajectories:
(**L**) they can be reinjected towards Δ (after some bursts far from Δ before synchronizing, *on-off intermittency*);
(**G**) they may belong to the basin of another attractor (*riddled basins*)

A bridge between critical sets and chaos synchronization

Locally repelled trajectories folded back toward A_s by the action of the non linearities acting far from Δ, described by using the *critical curves* as the reinjection is due to their folding action.

Bischi, Gardini (1998) “Role of invariant and minimal absorbing areas in chaos synchronization”, *Physical Review E*

Bischi,Gardini (2000) Global Properties of Symmetric Competition Models with Riddling and Blowout Phenomena", *Discrete Dynamics in Nature and Society*

Bischi, Cerboni Baiardi (2017) "Bubbling, Riddling, Blowout and Critical Curves", *Journal of Difference Equations and Applications*

An example:Dynamic marketing model

- Bischi, Gardini,Kopel (2000) "Analysis of Global Bifurcations in a Market Share Attraction Model", *Jou. of Economic Dynamics and Control*
- Bischi, Gardini (2000) "Global Properties of Symmetric Competition Models with Riddling and Blowout Phenomena", *Discrete Dynamics in Nature and Society*.

$$
\begin{aligned}
x_i(t+1) &= x_i(t) + \lambda_i x_i(t)\Pi_i(t) = \\
&= x_i(t) + \lambda_i x_i(t)\left(B\frac{a_i x_i^{\beta_i}(t)}{\sum_{j=i}^{n} a_j x_j^{\beta_j}(t)} - x_i(t)\right) \qquad i = 1, .., N
\end{aligned}
$$

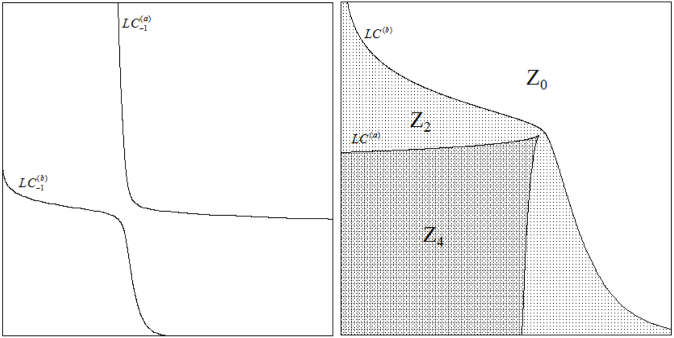
N=2: Symmetric case $\lambda_1 = \lambda_2 = \lambda, \quad a_1 = a_2 = a, \ \ \beta_1 = \beta_2 = \beta$

Restriction of the symmetric map to Δ

$f(x) = (1 + \frac{1}{2}\lambda B)x - \lambda x^2$

Jacobian matrix on the diagonal:

$$DT(x, x; \lambda, B, \beta,) = \begin{bmatrix} 1 - 2\lambda x + \frac{\lambda B(\beta+2)}{4} & -\frac{\lambda B \beta}{4} \\ -\frac{\lambda B \beta}{4} & 1 - 2\lambda x + \frac{\lambda B(\beta+2)}{4} \end{bmatrix}.$$



Bischi, Cerboni Baiardi (2017) "Bubbling, Riddling, Blowout and Critical Curves", *Journal of Difference Equations and Applications*

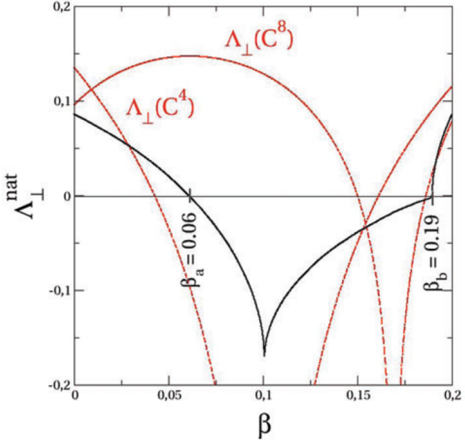


Figure 8. For $B = 10$ and $\lambda = 2(a_1 - 1)/B$, the natural transverse Lyapunov exponent (black line) of the four-band chaotic attractor $\mathcal{A}_s \subset \Delta$ is represented as the normal parameter β varies, as well as the transverse Lyapunov exponents of the cycles of period 4 and 8 embedded in A_s (red lines).

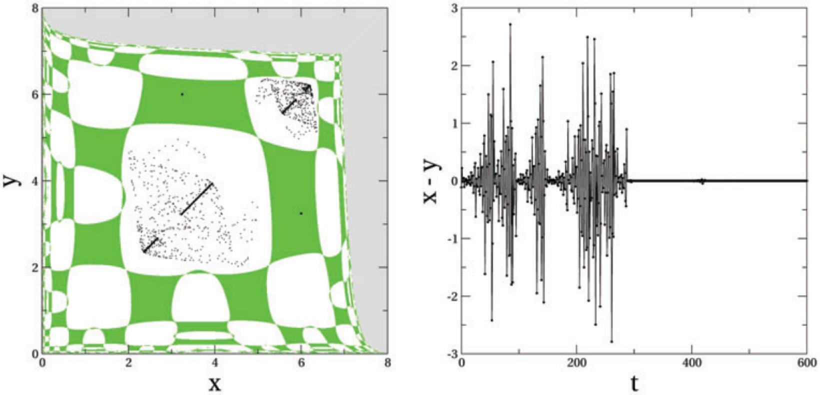


Figure 9. A numerical simulation of the system (14) obtained for $B = 10, \lambda = 2(\overline{\mu}_2 - 1)/B$ and $\beta = 0.09$ at which $\Lambda_{\perp}^{nat} < 0$ while $\Lambda_{\perp}(C^8) > 0$). Left. A trajectory in the phase space (x_t, y_t) whose transient part is out of Δ that synchronizes along the Milnor attractor $\mathcal{A}_s$ in the long run. The white region is the basin of attraction of $\mathcal{A}_s$ whereas the points in the grey region generate interrupted trajectories, involving negative values of the state variables. The further green region is the basin of attraction of a stable period-two cycle. Right. The displacement $x_t - y_t$ vs. time.

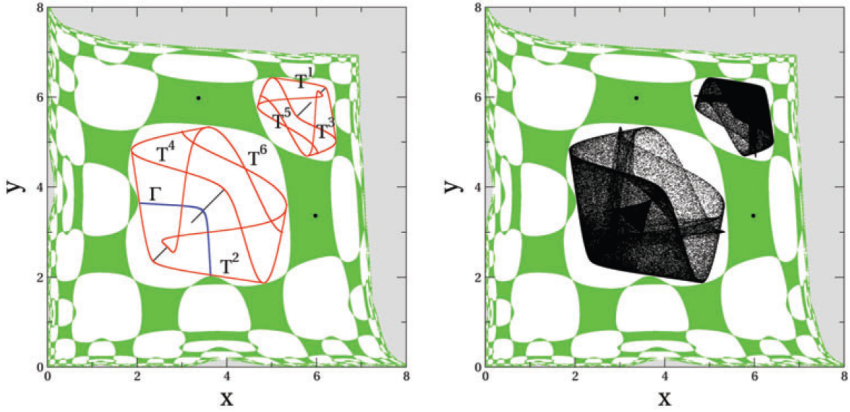
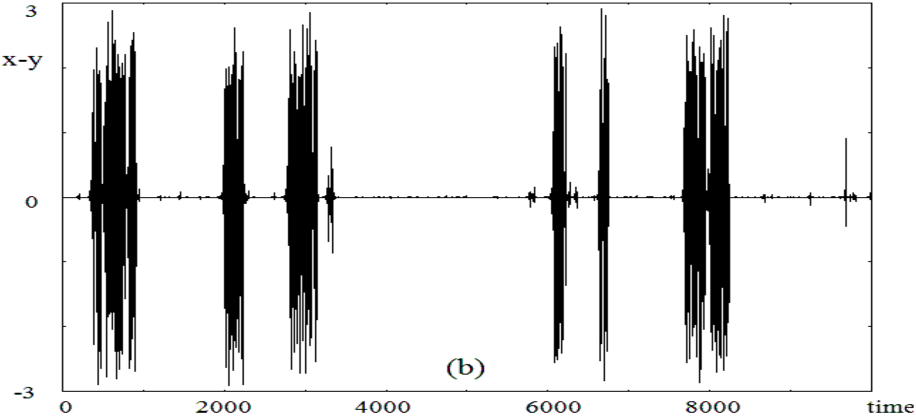


Figure 10. Left. Minimal invariant absorbing area $\mathcal{A}$ obtained by six iteration of the generating arc Γ (blue line) where $T^k = T^k(\Gamma)$, $k = 1, \ldots, 6$ (red lines). Right. The effect of the parameters' mismatch $a_x = 0.514961$ and $a_y = 0.51496$. Other parameters are as in Figure 9.

With parameters' mismatch bursts never stop



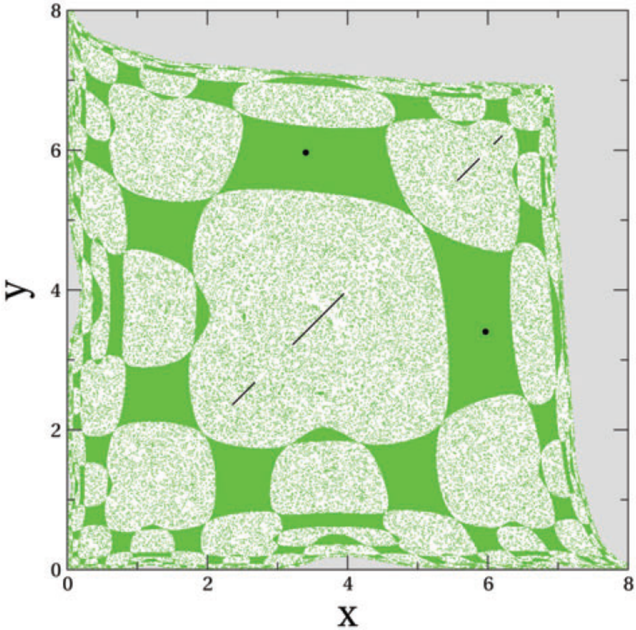
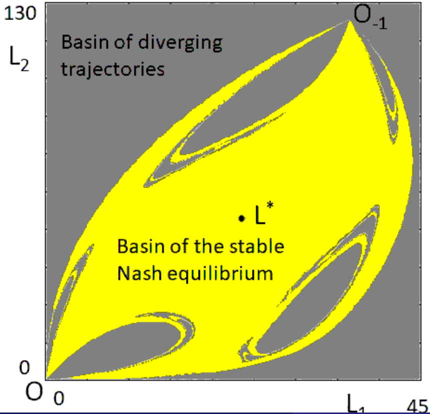


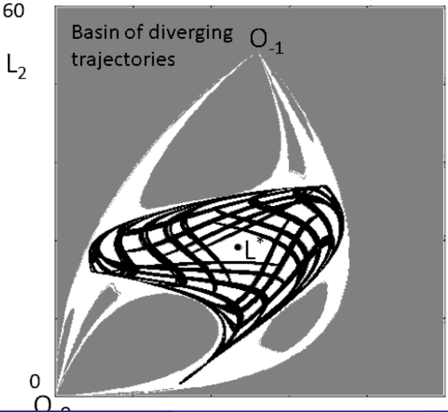
Figure 11. Global riddling at $\beta = 0.0945$.

Games and lobes: isoelastic demand p = 1/Q

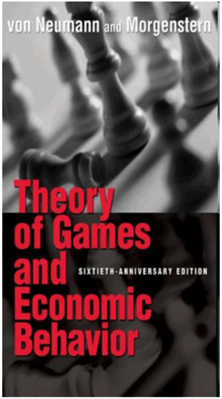
$\pi_i(q_1, q_2) = \frac{q_i}{q_1+q_2} - c_i q_i$ hence $\frac{\partial \pi_i}{\partial q_1} = \frac{q_j}{(q_1+q_2)^2} - c_i$

gradient dynamics
$$\begin{cases} q_1(t+1) = & q_1(t)\left(1 - c_1 v_1 + v_1 \frac{q_2(t)}{(q_1(t)+q_2(t))^2}\right) \\ q_1(t+1) = & q_2(t)\left(1 - c_2 v_2 + v_2 \frac{q_1(t)}{(q_1(t)+q_2(t))^2}\right) \end{cases}$$

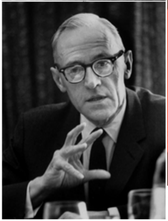


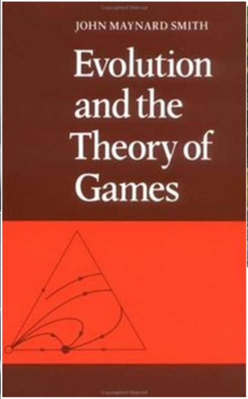


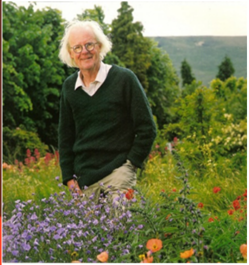
Heterogenous players and evolution of different bevaviors



John von Neumann
Oskar Morgenstern
Princeton, 1947







John Maynard Smith
Cambridge, 1982

Competitions of behavioural rules: replicator dynamics

- Population of N agents, partitioned into k groups according to the strategy (or behavior) adopted $S = \{S_1, ..., S_k\}$.
- $N_i(t)$ agents follow behavior S_i at time t, $\sum_{i=1}^{k} N_i(t) = N$
- $r_i(t) = \frac{N_i(t)}{N(t)}$, $\sum_{i=1}^{k} r_i(t) = 1$

Selection mechanism: The growth of r_i is proportional to payoff obtained π_i compared with average payoff $\bar{\pi} = \sum_{i=1}^{k} r_i \pi_i$.
Monotone transformation of payoffs $u(\pi_i) = \exp(\beta\pi_i)$, $\beta > 0$, and consequently $\bar{u} = \sum_{i=1}^{k} r_i \exp(\beta\pi_i)$.

Exponential replicator dynamics

$$\begin{aligned} q_i(t+1) &= H_i\left(q_1(t), ..., q_n(t), r_1(t), ..., r_n(t)\right) \quad i = 1, ..., N \\ r_i(t+1) &= r_i(t) \frac{e^{\beta\pi_i(t)}}{\sum_{j=1}^{n} r_j(t) e^{\beta\pi_j(t)}} \end{aligned}$$

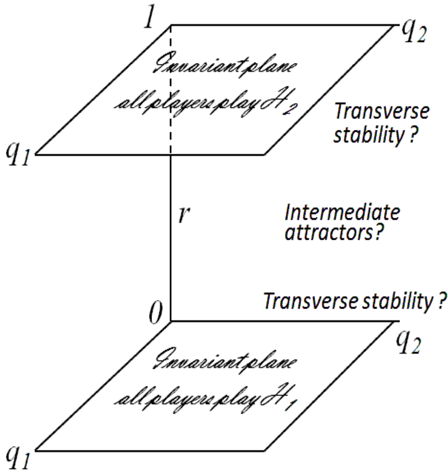
Compare two decision strategies

N firms partitioned into 2 groups according to behavior adopted

$$\begin{cases} q_1(t+1) = H_1(q_1(t), q_2(t), r(t)) \\ q_2(t+1) = H_2(q_1(t), q_2(t), r(t)) \\ r(t+1) = r(t)\frac{e^{\beta\pi_1(t)}}{r(t)e^{\beta\pi_1(t)}+(1-r(t))e^{\beta\pi_2(t)}} \end{cases}$$

$$Q(t) = N\left[r(t)q_1(t) + (1-r(t))q_2(t)\right]$$

with $r(t) = \frac{n_1(t)}{N}$ evolving according to exponential replicator



Example: Best Reply versus LMA

Bischi, Lamantia, Radi (2015). "An evolutionary Cournot model with limited market knowledge". *J. Econ. Behavior & Organization*.

BR with isoelastic demand p=1/Q, linear cost, naive expectations

$$q_1(t+1) = R_1(q_2(t)) = \sqrt{\frac{q_2(t)}{c_1}} - q_2(t)$$
$$q_2(t+1) = R_2(q_1(t)) = \sqrt{\frac{q_1(t)}{c_2}} - q_1(t)$$

LMA with isoelastic demand p=f(Q)=1/Q and linear costs

from $q_i(t+1) = \frac{1}{2}q_i(t) - \frac{f(Q(t))-c_i}{2f'(Q(t))}$ $\quad i = 1, 2$

with $p = f(Q) = \frac{1}{Q}$, $Q = q_1 + q_2$, we get:

$$q_1(t+1) = \frac{1}{2}\left[2q_1(t) + q_2(t) - c_1\left(q_1(t) + q_2(t)\right)^2\right]$$
$$q_2(t+1) = \frac{1}{2}\left[q_1(t) + 2q_2(t) - c_2\left(q_1(t) + q_2(t)\right)^2\right]$$

Evolutionary pressure based on observed profits

$$\pi_{BR} = px - (c_x x + K_x) = \left(\frac{1}{Q} - c_x\right) x - K_x$$

$$\pi_{LMA} = py - (c_y y + K_y) = \left(\frac{1}{Q} - c_y\right) y - K_y$$

$K_x \geq K_y$ information costs of *BR* and *LMA* behaviors..
The fraction $r(t)$ updated according to *exp. replicator dynamics*

$$\boxed{r(t+1) = r(t)\frac{e^{\beta\pi_{BR}(t)}}{r(t)e^{\beta\pi_{BR}(t)}+(1-r(t))e^{\beta\pi_{LMA}(t)}}}$$

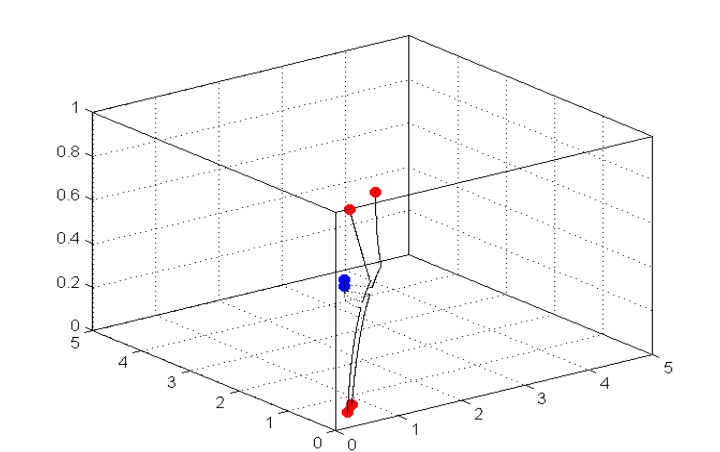
$\beta > 0$ *intensity of choice*:
$\beta = 0$ agents do not switch;
$\beta = \infty$ implies $r(t) \to 1$ if $\pi_{BR}(t) > \pi_{LMA}(t)$ and $r(t) \to 0$ if $\pi_{BR}(t) < \pi_{LMA}(t)$.

- Steady states: $r = 0$; $r = 1$; any $r^* \in (0, 1)$ such that $\pi_{BR} = \pi_{LMA}$.

Coexistence of cyclic attractors and path dependence

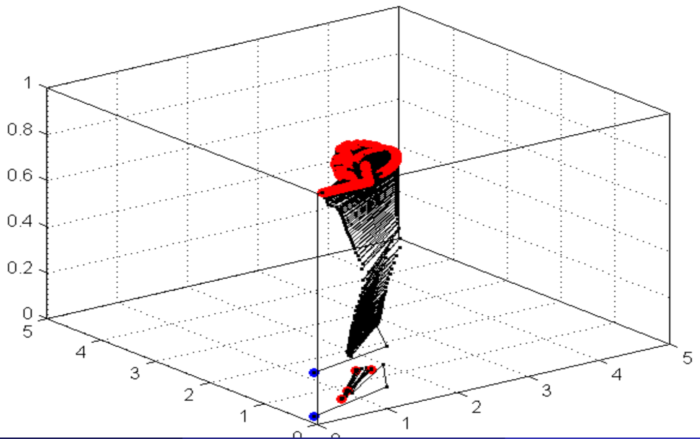
$\lambda = \alpha = 0.3$, $c = 0.1$, $\delta = 0$, $\beta = 1$, $K_x = 0.01$, $K_y = 0$, $N = 15$, two different i.c.

Two periodic attractors in pure strategies (red)



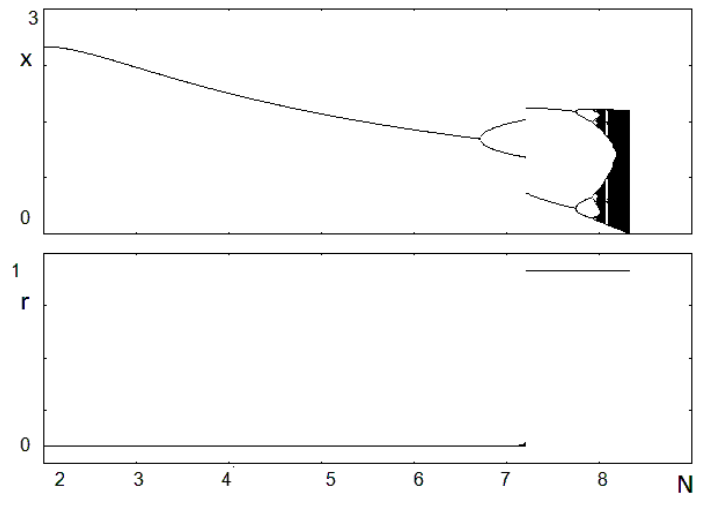
Path dependence: coexistence of chaotic (BR) and periodic (LMA) attractors

$\lambda = 0.6$, $\alpha = 0.7$, $c = 0.1$, $\delta = 0$, $\beta = 1$, $K_x = 0.01$, $K_y = 0$, $N = 8$, two different i.c.



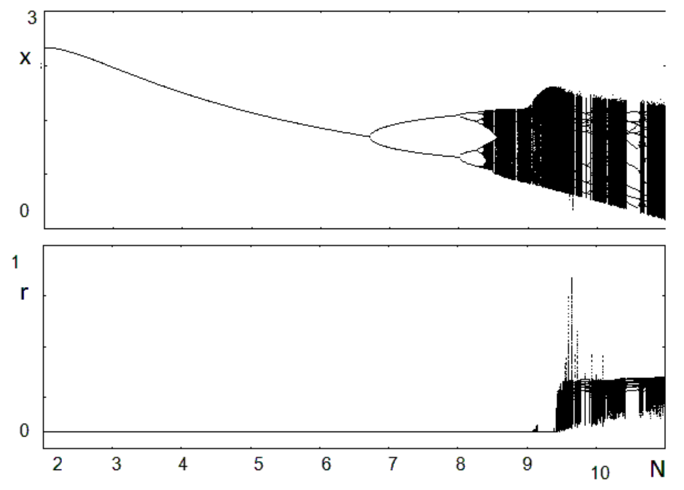
Transverse stability switch

$\lambda = 0.6,\ \alpha = 0.7,\ c = 0.1,\ \delta = 0,\ \beta = 1,\ K_x = 0.01,\ K_y = 0$



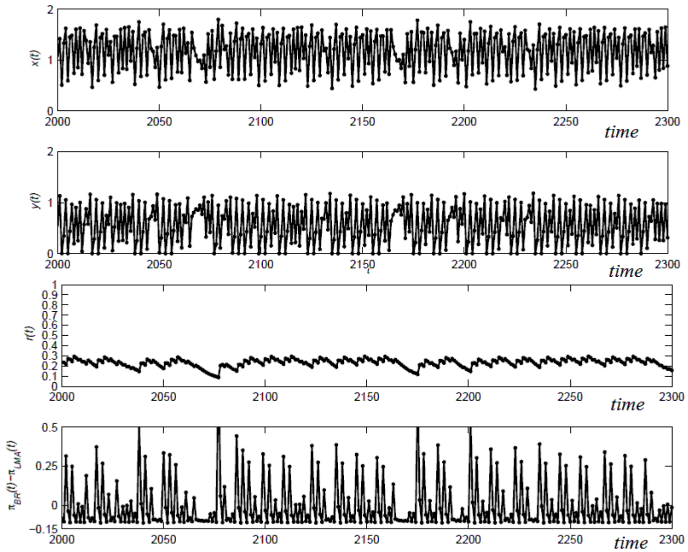
Attractors with intermediate r values

$\lambda = 0.5$, $\alpha = 0.7$, $c = 0.1$, $\delta = 0$, $\beta = 1$, $K_x = 0.1$, $K_y = 0$,i.C.$(x(0), y(0), r(0)) = (0.1, 0.2, 0.5)$



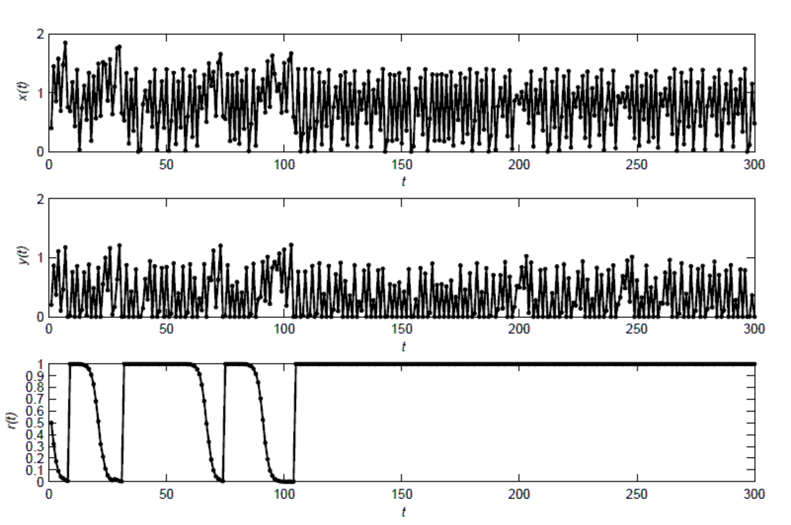
Coexistence chaotic dynamics

Same parameters and $N = 10$



Intermittent dynamics

Same parameters but info cost increased at $K_x = 0.8$



Further extensions

Bischi, Lamantia, Scardamaglia (2020) "On the influence of memory on complex dynamicsof evolutionary oligopoly models", *Nonlinear Dynamics*

Fitness measured as *accumulated payoff* instead of current payoff

$$U_i(t) = (1-\omega)\,\pi_i(t) + \omega U_i(t-1)$$

$\omega \in [0,1]$ *memory parameter* :
for $\omega = 0$, $U_i(t) = \pi_i(t)$
for $\omega = 1$, uniform mean of all the payoffs of the past.

Recursive formula (accumulated payoff)

$$U_i(t) = (1-\omega)\sum_{k=0}^{t-1} \omega^k \pi_i(t-k) + \omega^t U_i(0), \qquad i = 1, 2$$

Model with memory

$$T:\begin{cases} x_1(t+1) = H_1(x_1(t), x_2(t), r(t)) \\ x_2(t+1) = H_2(x_1(t), x_2(t), r(t)) \\ r(t+1) = R(r(t), m(t)) = \frac{r(t)}{r(t)+(1-r(t))e^{-\beta m(t)}} \\ m(t+1) = (1-\omega)(\pi_1(t+1) - \pi_2(t+1)) + \omega m(t) \end{cases}$$

$$m(t) = U_1(t) - U_2(t)$$

- Often memory has a stabilizing effect, but not always.

Other "competitions" between different behaviors

Cerboni Baiardi, Lamantia, Radi (2015) "Evolutionary competition between boundedly rational behavioral rules in oligopoly games", *Chaos, Solitons & Fractals*

Competition between Local Monopolistic Approximation and Gradient dynamics.

Radi (2017) “Walrasian versus Cournot behavior in an oligopoly of boundedly rational firms”, *Journal of Evolutionary Economics*

Competition between Best Reply and a Walrasian rule.

Bischi, Lamantia, Radi (2013) "Multi-species exploitation with evolutionary switching of harvesting strategies", *Nat. Res. Modeling*

Hybrid model: Fish grows in continuous time, fishers switch (according to profit-driven replicator) the harvesting strategy at discrete periods

Radi, Lamantia, Tichý (2021) «Hybrid dynamics of multi-species resource exploitation » *Decisions in Economics and Finance*

Through a discretization of the continuous variables, the problem is reformulated as three-dimensional iterated map.

Some further evolutionary dynamics

On exponential replicator switching function

- Cabrales, Sobel (1992) "On the limit points of discrete selection dynamics", *J. Econ.Theory*.
- Hofbauer, Sigmund (2003) “Evolutionary Game Dynamics”, *Bulletin of The American Mathematical Society*.

On evolutionary dynamics with Logit switching functions

- Brock, Hommes (1997) A rational route to randomness. *Econometrica*.
- Droste, Hommes, Tuinstra (2002) "Endogenous Fluctuations Under Evolutionary Pressure in Cournot Competition", *Games and Economic Behavior*.

Other imitation switching mechanisms

- Bischi, Dawid, Kopel (2003) «Spillover Effects and the Evolution of Firm Clusters» *Jou. Econ. Behavior & Organization*