Limit Cycles in Two-dimensional predator-prey systems

Talk at **International Online GSDUAB Seminar**

4 April 2022

Robert Kooij

TUDelft

Joint work with ***André Zegeling***
Guangxi Normal University, Guilin, China

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About me

TUDelft

1988 - 1993

1994 - 1996

kpn
Royal Dutch Telecom

1997 - 2003

Applied → TNO

I'm Back

SUTD SINGAPORE UNIVERSITY OF TECHNOLOGY AND DESIGN

2018 - 2019

Theoretical → TUDelft

2005 - now

Robustness of Complex Networks

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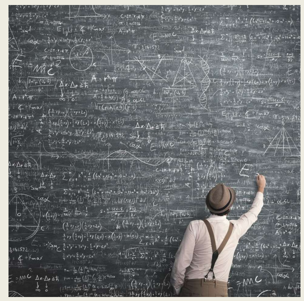
Overview

1. Trip down memory lane
2. Classical results for predator-prey systems
3. Some high level new results
4. One detailed new result
5. Wrap-up



5 Open Problems

Trip down memory lane

1988 – 1993 PhD student of

- **Prof. John W. Reyn**
  - Quadratic systems

$$\begin{cases} \frac{dx}{dt} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ \frac{dy}{dt} = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \end{cases}$$

  - Limit cycles in polynomial systems

Trip down memory lane

Mathematics Genealogy Project

John William Reyn

MathSciNet

Ph.D. Technische Universiteit Delft 1961

Dissertation: *Differential-Geometric Considerations on the Hodograph Transformation for Irrotational Conical Flow*

Name	School	Year
Bakker, Pieter	Technische Universiteit Delft	1988
van Horssen, Wim	Technische Universiteit Delft	1988
de Jager, Paul	Technische Universiteit Delft	1989
van der Beek, Clemens	Technische Universiteit Delft	1989
Zegeling, André	Technische Universiteit Delft	1991
Kooij, Robert	Technische Universiteit Delft	1993
Blom, Caspar	Technische Universiteit Delft	1994
de Winkel, Marco	Technische Universiteit Delft	1996
Haaker, Timber	Technische Universiteit Delft	1996
Huang, Xianhua	Technische Universiteit Delft	1996
Boon, Roland	Technische Universiteit Delft	1997
Timochouk, Leonid	Technische Universiteit Delft	1997
Boertjens, Gerdineke	Technische Universiteit Delft	2000

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Trip down memory lane

Strong relationship with UAB

Trip down memory lane

Stefan Banach International Mathematical Center
Warszawa, Mokotowska 25

SINGULARITIES IN DIFFERENTIAL EQUATIONS AND PFAFF SYSTEMS
October 2 – October 18, 1995

MONDAY, Oct.9

9.30 YE YANGIAN
Qualitative and bifurcation theory of quadratic differential systems

10.30 R. KOOIJ
Limit cycle[illegible] in two-[illegible]ensional predator-prey systems

11.20 COFFEE & TEA BR[illegible]

11.50 V. GROMEK
Algebraic non-integrability of Painlevé equations and the first integrals

15.00 V. ROMANOVSKII
On cyclicity of center or focus for a cubic system

TUESDAY, Oct. 10

9.30 J. LLIBRE
On the non-existence, existence and uniqueness of limit cycles

10.30 YE YANGIAN Dulac function
~~New~~ method for the proof of the existence of limit cycles around a weak focus of order 2 in quadratic differential systems

11.20 COFFEE & TEA BREAK

11.50 V. KOSTOV
On the Deligue-Simpson problem

15.00 A. CIMA
Injectivity and global attractors

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Trip down memory lane

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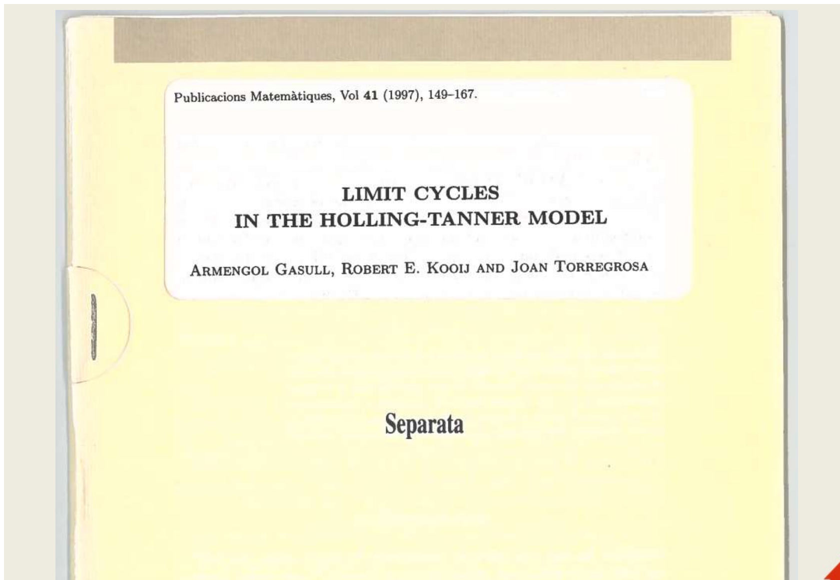
Trip down memory lane



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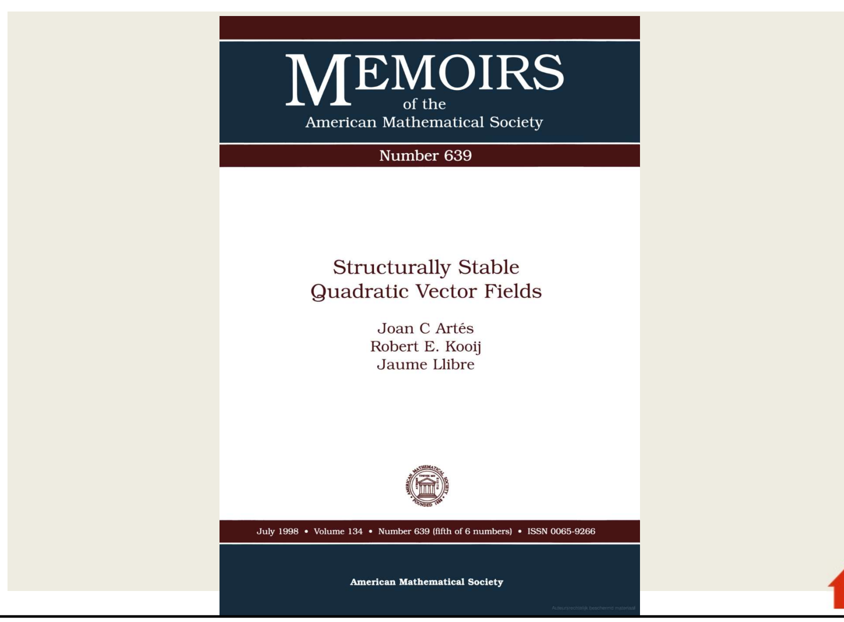
Trip down memory lane



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Trip down memory lane

Google Scholar

Tuning an IP-based Network Transporting Telephony and Videophony J Janssen, DD Vleeschauwer, MJC Buchli, RE Kooij 9th COST 263	3	2001
Cubic Liénard equations with quadratic damping having two antisaddles F Dumortier, RE Kooij, C Li Qualitative theory of dynamical systems 1 (2), 163-209	11	2000
Coexistence of centers and limit cycles in polynomial systems RE Kooij, A Zegeling The Rocky Mountain Journal of Mathematics, 621-640	7	2000
Periodic orbits in planar systems modelling neural activity RE Kooij, F Giannakopoulos Quarterly of Applied Mathematics 58 (3), 437-457	6	2000
End-to-end delay models for interactive services on a large-scale IP network M Mandjes, K van der Wal, R Kooij, H Bastiaansen Proceedings of the 7th workshop on performance modelling and evaluation of …	31	1999

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Trip down memory lane

Google Scholar

Analyzing information availability in ICN under link failures E Rietberg, L D'Acunto, R Kooij, H van den Berg 2018 International Conference on Information Networking (ICOIN), 199-204	3	2018
Quadratic systems with a symmetrical solution A Zegeling, R Kooij Electronic Journal of Qualitative Theory of Differential Equations 2018 (32 …		2018
Multi-criteria robustness analysis of metro networks X Wang, Y Koç, S Derrible, SN Ahmad, WJA Pino, RE Kooij Physica A: Statistical Mechanics and its Applications 474, 19-31	70	2017
The reliability of a gas distribution network: A case study W Pino, D Worm, R van der Linden, R Kooij 2016 International Conference on System Reliability and Science (ICSRS), 122-129	4	2016
Modeling region-based interconnection for interdependent networks X Wang, RE Kooij, P Van Mieghem Physical Review E 94 (4), 042315	14	2016

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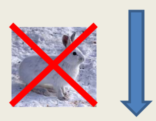
Overview

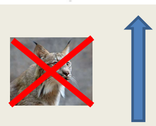
Classical results for predator-prey systems

Classical results for predator-prey systems

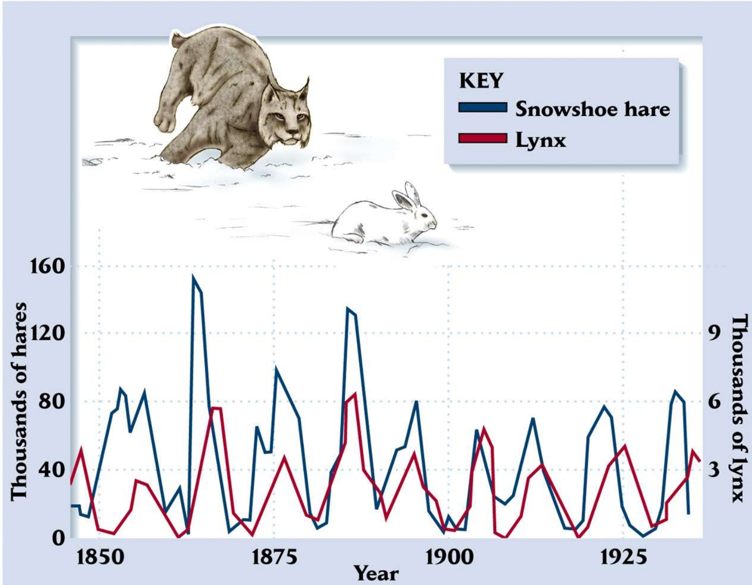
Predator 掠夺者



Prey 猎物



Classical results for predator-prey systems



Data from Hudson's Bay Company (fur trading)

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Classical results for predator-prey systems

- Lotka-Volterra model (1925)

1. Prey grow exponentially in absence predators.
2. Predators will go extinct in absence prey.
3. Predators can consume unbounded quantities of prey.
4. Both populations are moving randomly through a homogeneous environment

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Classical results for predator-prey systems

System of differential equations

$$\begin{cases} \frac{dx}{dt} = \alpha x - \beta xy \\ \frac{dy}{dt} = -\delta y + \gamma \beta xy \end{cases}$$

x(t) = density of the prey population
y(t) = density of the predator population

α = growth rate prey
δ = death rate predator

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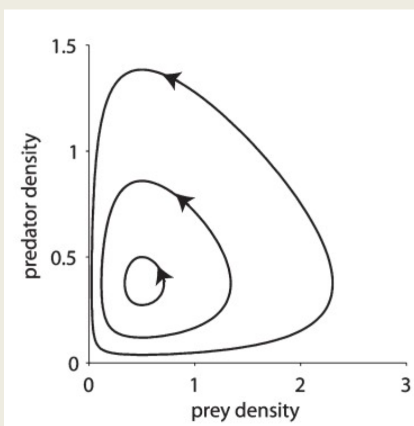
Classical results for predator-prey systems

- Phase plane analysis



- Oscillations of varying amplitude
- Predators never go extinct

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Classical results for predator-prey systems

- **Unrealistic assumption 1**
  - Absence of predators i.e. y = 0

$$\frac{dx}{dt} = \alpha x$$

exponential growth of prey

  - Logistic equation

$$\frac{dx}{dt} = \alpha x(1 - \frac{x}{K})$$

x(t) → *K* for *t* → ∞

K = carrying capacity

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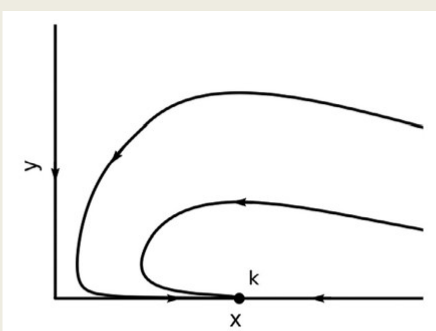
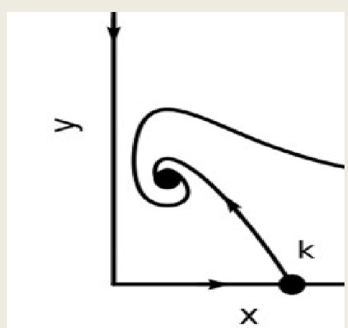
Classical results for predator-prey systems

- Logistic growth rate prey



large δ
extinction predator



small δ
co-existence

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Classical results for predator-prey systems

- **Unrealistic assumption 2**

Predators can consume unbounded quantities of prey

$$\frac{dx}{dt} = \alpha x - \beta x y$$

$$\frac{dy}{dt} = -\delta y + \gamma \beta x y$$

consumption rate of prey per predator as function of prey density = **Functional Response = *p(x)***

- βx: unbounded increase
- saturation predator
- *p(x)* is bounded

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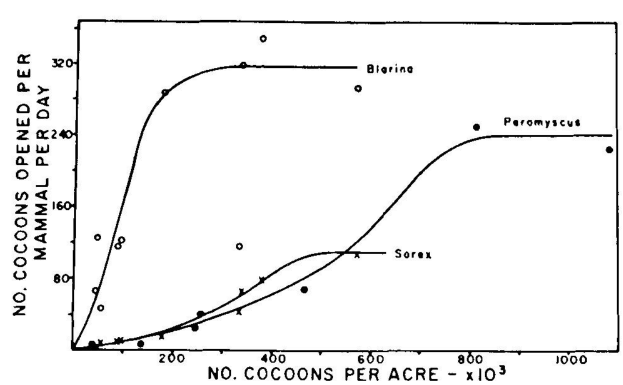
Classical results for predator-prey systems

Experimental results by Holling (1959)
Functional response
(predator: *small mammals*, prey: *small flies*)

THE CANADIAN ENTOMOLOGIST May 1959



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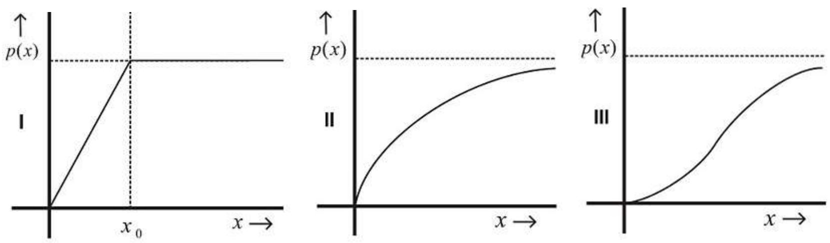
Classical results for predator-prey systems

Choices for $p(x)$



Holling I

$$p(x) = x \quad 0 \le x \le 1$$
$$p(x) = 1 \quad x > 1$$

Holling II

$$p(x) = \frac{x}{c + x}$$

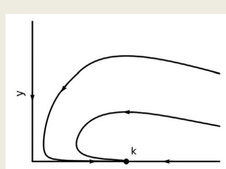
Holling III

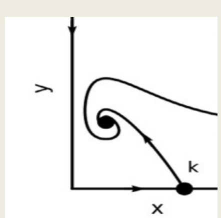
$$p(x) = \frac{x^2}{c + x^2}$$

analytical

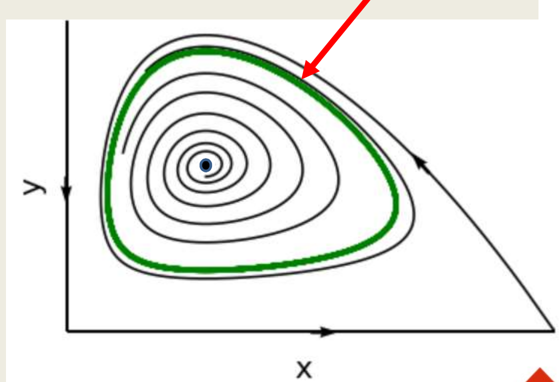
Classical results for predator-prey systems

- Logistic growth rate prey + Holling II or III





Limit cycle (极限环)



Classical results for predator-prey systems

- Determine number of limit cycles
  - In general unsolved (Hilbert 16th problem)
  - Non-existence (Dulac functions)
  - Uniqueness (Liénard equation)
  - Upper bounds (perturbation methods)

$$\begin{cases} \frac{dx}{dt} = ax(1-\frac{x}{K}) - p(x)y \\ \frac{dy}{dt} = -\delta y + \gamma p(x)xy \end{cases}$$

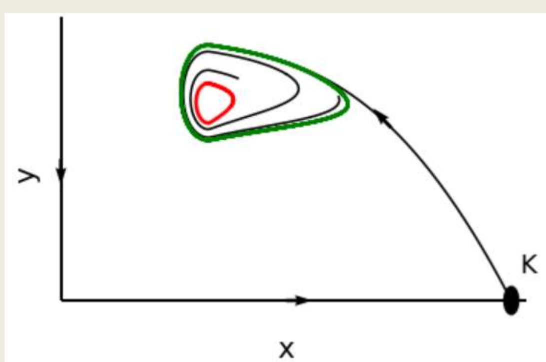
p(x) Holling II or III

At most one limit cycle [Cheng, 1981]

Classical results for predator-prey systems

p(x) Holling I

Example with *at least* two limit cycles [Liu, 1988]



Intermezzo

Holling II

$$p(x) = \frac{Ax}{1 + AHx}$$

A: attack rate
H: prey handling rate

disc equation

The Canadian Entomologist

Vol. XCI — Ottawa, Canada, July 1959 — No. 7

Some Characteristics of Simple Types of Predation and Parasitism[1]

By C. S. Holling

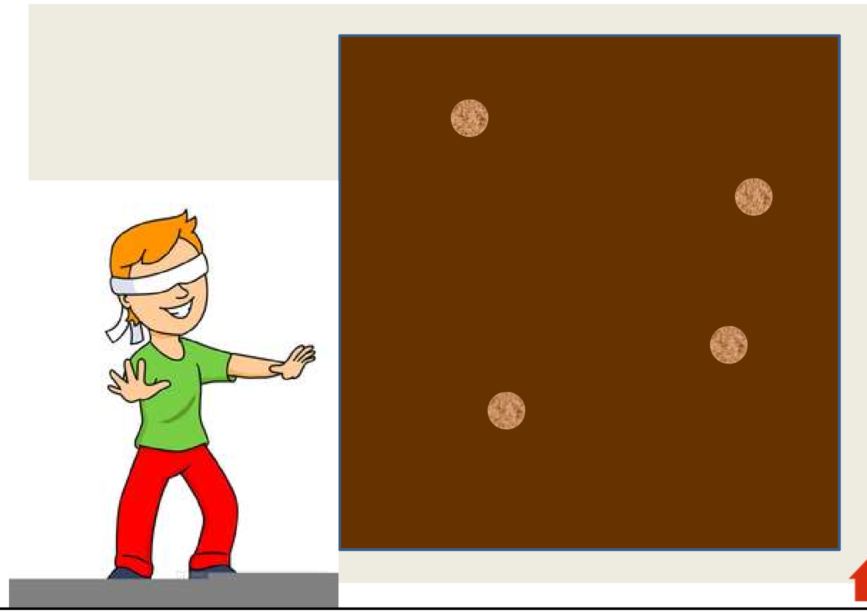
Forest Insect Laboratory, Sault Ste. Marie, Ontario

Intermezzo

Artificial Predator-Prey Situations

- “prey”
  - sandpaper discs 4 cm in diameter
  - Thumb-tacked to a 9 square feet table
- “predator”
  - Blindfolded person, in front of table
  - Searching discs one minute by tapping with finger
  - Disc found: remove and put aside; continue search

Intermezzo



Intermezzo

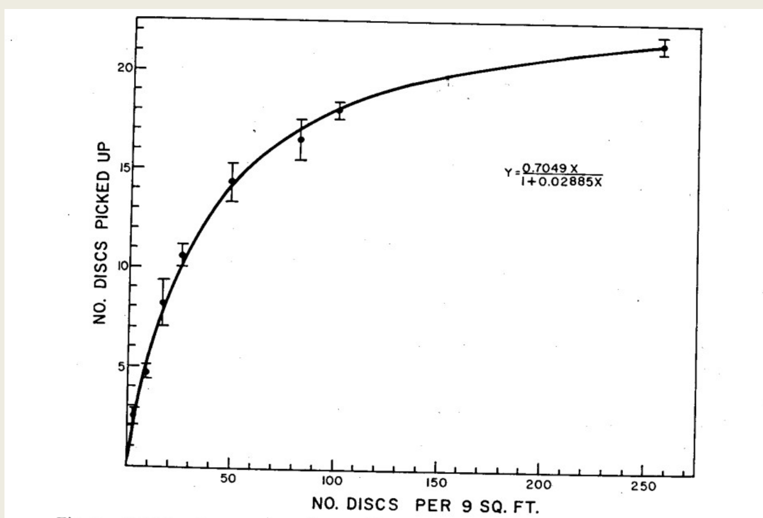


Fig. 1. Functional response of a subject searching for sandpaper discs by touch. (Averages ± 2 S.E. of 8 replicates.)

Intermezzo

z: number of discs removed
T_s: time available for searching
a: rate of searching multiplied by the probability of finding a disc
x: density of discs

$$z = aT_s x$$

T: total time of one experiment
b: time to pick up one disc

$$T_s = T - bz$$

$$z = a(T - bz)x \Rightarrow z = \frac{Tax}{1 + abx}$$

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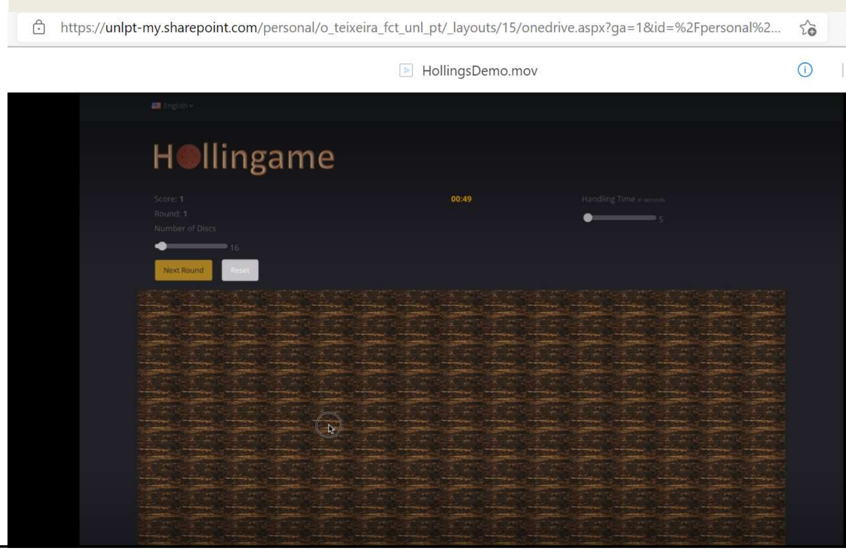
Open Problem 1

- Repeat the sandpaper experiment through a webinterface



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Intermezzo

The Canadian Entomologist

Vol. XCI — Ottawa, Canada, July 1959 — No. 7

Some Characteristics of Simple Types of Predation and Parasitism[1]

By C. S. Holling

Forest Insect Laboratory, Sault Ste. Marie, Ontario

Acknowledgments

I wish to acknowledge the considerable assistance rendered by Dr. R. M. Belyea and Mr. A. W. Ghent through discussion and criticism of the manuscript. I must also thank Miss Patricia Baic, whose 'predatory' behaviour provided data for the major portion of this paper.

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Open Problem 2

- Can we find the whereabouts of Miss Patricia Baic?

Google

MyHeritage

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Overview

Some high level new results

$$(*)\quad \begin{cases} \dfrac{dx}{dt} = \dfrac{\phi}{k}x(k-x) - p(x)y, \\ \dfrac{dy}{dt} = -\delta y + p(x)y, \end{cases}$$

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Some high level new results

1. Impact of non-analytical Functional Response (other than Holling I)
2. Conditions for “at most two limit cycles” for Holling I?
3. Examples with more than 2 limit cycles

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Some high level new results

Chaos, Solitons and Fractals 123 (2019) 163–172

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

ELSEVIER

Frontiers

Predator-prey models with non-analytical functional response

Robert E. Kooij[a,b,*], André Zegeling[c]

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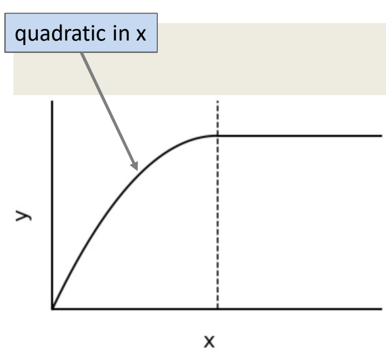
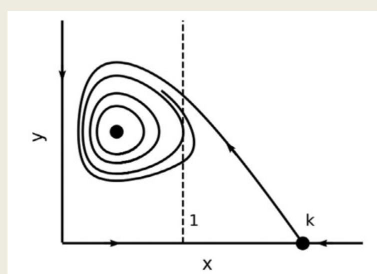
Some high level new results





Holling II - like
At most one limit cycle

Period annulus case
No limit cycles

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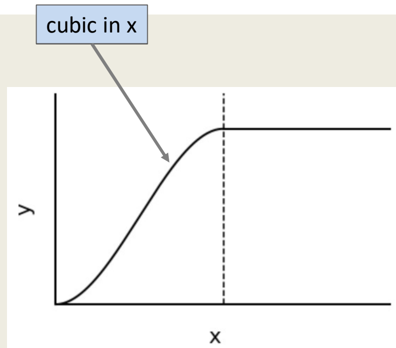
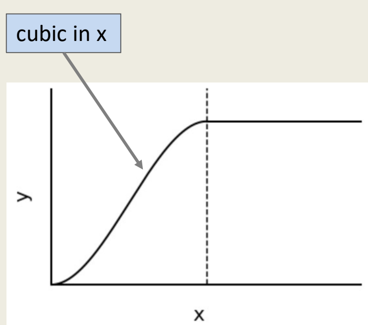
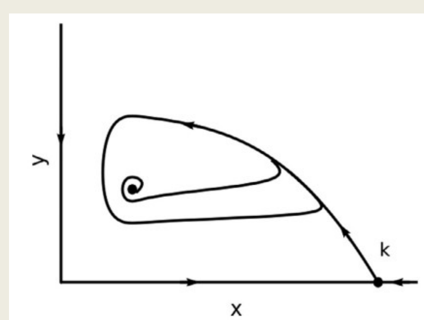
Some high level new results



Holling III - like
at least two limit cycles

Open Problem 3





- Uniqueness of limit cycle when equilibrium is unstable?

Some high level new results

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Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 269 (2020) 5434–5462

Journal of Differential Equations
www.elsevier.com/locate/jde

Singular perturbations of the Holling I predator-prey system with a focus

André Zegeling[a,*], Robert E. Kooij[b]

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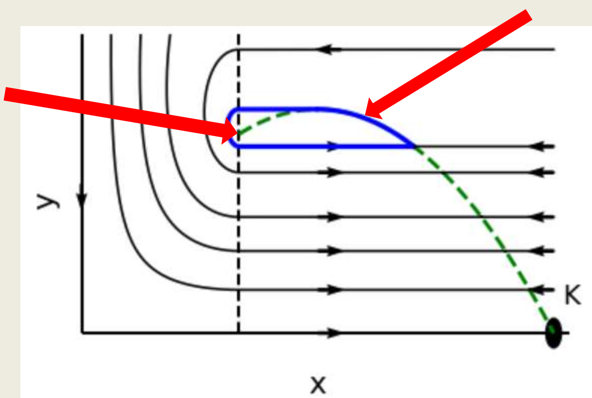
Some high level new results

$\delta = 1$; *singular system*



$\delta = 1 - \varepsilon \qquad 0 < \varepsilon << 1$

Exactly 2 limit cycles!

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Some high level new results

Qualitative Theory of Dynamical Systems (2021) 20:65
https://doi.org/10.1007/s12346-021-00501-w

Check for updates

Several Bifurcation Mechanisms for Limit Cycles in a Predator–Prey System

André Zegeling[1] · Robert E. Kooij[2]

Received: 7 December 2020 / Accepted: 24 June 2021

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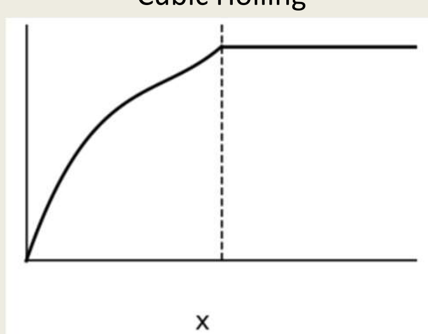
Some high level new results

$$p(x) = x\big(1 + (a_0 + a_1 x)(x - 1)\big); \qquad 0 \le x \le 1$$

$$p(x) = 1; \qquad x > 1$$

"Cubic Holling"



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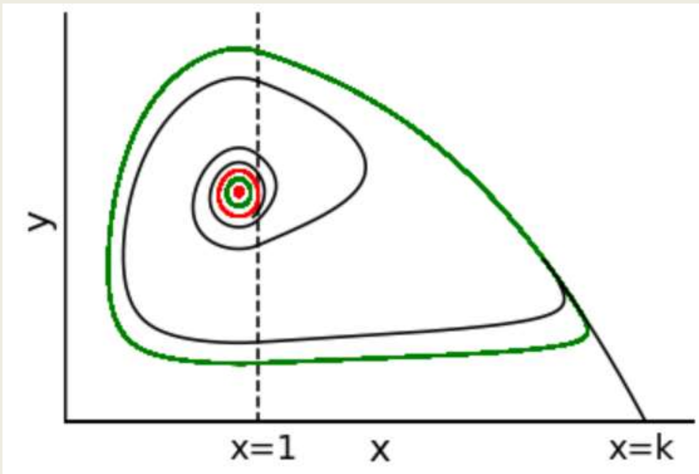
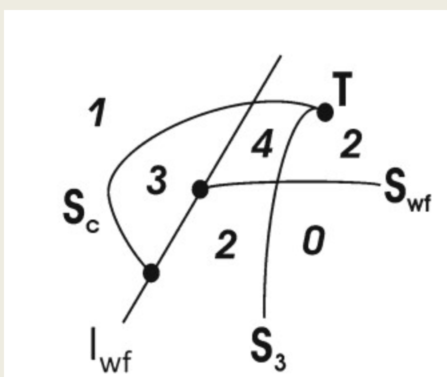
Some high level new results



At least 4 limit cycles!

Open Problem 4

- Validate the bifurcation diagrams numerically



Overview

One detailed new result

One detailed new result

International Journal of Bifurcation and Chaos, Vol. 31, No. 10 (2021) 2150154 (11 pages)

DOI: 10.1142/S0218127421501546

Co-existence of a Period Annulus and a Limit Cycle in a Class of Predator–Prey Models with Group Defense

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André Zegeling
Guilin University of Aerospace Technology,
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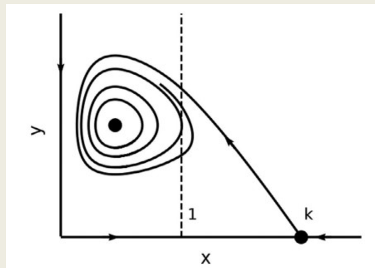
zegela1@yahoo.com

Received November 24, 2020; Revised March 12, 2021

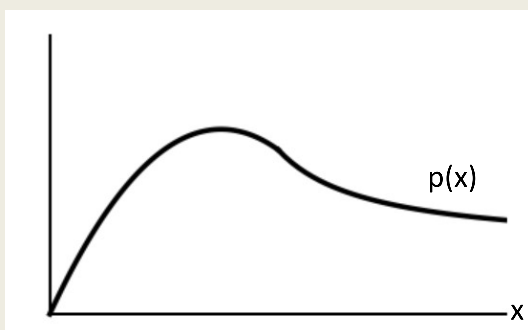
One detailed new result

- Previous result: period annulus



- So far: p'(x) ≥ 0
- Now allow: group defense

One detailed new result



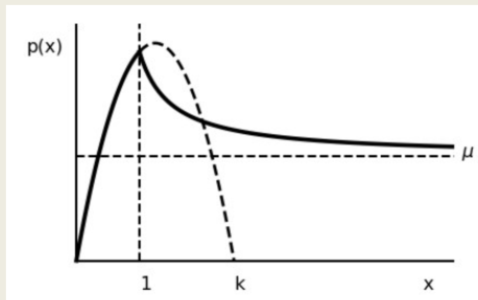
(*)

$$\frac{dx}{dt} = \frac{\phi}{k}x(k-x) - p(x)y,$$

$$\frac{dy}{dt} = -\delta y + p(x)y,$$

One detailed new result

- Strip I: $0 \leq x \leq 1$: $p(x) = x(k - x)$
- Strip II: $x \geq 1$: $p(x) = \xi(x)$



- $\delta < \mu$ → a single equilibrium in first quadrant

One detailed new result

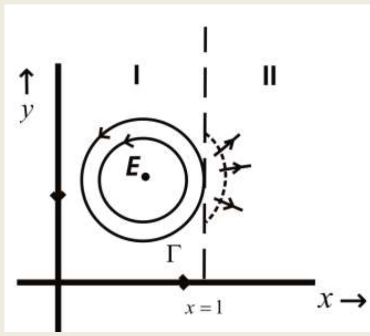
- Strip I

$$\frac{dx}{dt} = \frac{\phi}{k}x(k-x) - x(k-x)y,$$
$$\frac{dy}{dt} = -\delta y + x(k-x)y.$$

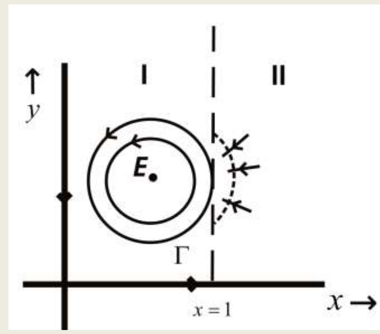
- Integrating factor $\frac{1}{x(k-x)y}$
- Period annulus in Strip I
- Use closed orbits extended to Strip II

One detailed new result

- Define S = ξ'(1) – k + 2



S < 0



S ≥ 0

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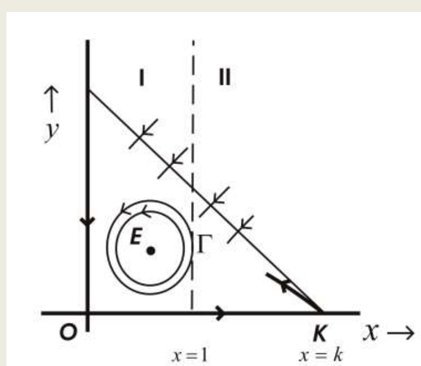
One detailed new result

- System (*) is bounded



- S < 0: period annulus unstable on outside
  - **Existence stable limit cycle!**

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One detailed new result

- Can we proof uniqueness of the limit cycle?
- Transform system (*) to Liénard system

$$\frac{dx}{dt} = F(x) - \Psi(y), \quad \frac{dy}{dt} = g(x)$$

- Show that *F(x)*, *f(x)=F'(x)*, *Ψ(y)* and *g(x)* satisfy certain conditions (Zhang Zhifen Theorem)
  - But: ZZF assumes isolated zero(s) of *f(x)*
  - Here: *f(x) = 0* on an interval

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One detailed new result

Proposition 1. *Consider the generalized Liénard system (26) and let $F(x), g(x)$ be continuous, piecewise differentiable functions on the open interval (r_1, r_2), and let $\Psi(y)$ be a continuously differentiable function on $\mathbb{R}$ such that*

(i) *there exists $x_g \in (r_1, r_2)$ such that $(x - x_g)g(x) > 0$ for $x \neq x_g$,*
(ii) *$\Psi(y)$ is monotonically increasing, and there exists a y_g such that $\Psi(y_g) = 0$,*
(iii) *$F(x) \equiv 0$ for $r_1 \leq x \leq x_0$, where $x_0 > x_g$,*
(iv) *there exists an $x_F > x_0$ such that $F(x_F) = 0$,*
(v) *$F(x) > 0$ for $x_0 < x < x_F$,*
(vi) *$f(x) < 0$ for $x_F \leq x \leq r_2$ where the function $f(x)$ is defined by $\frac{dF(x)}{dx}$,*

then in the strip $r_1 < x < r_2$ system (26) has at most one limit cycle, which is stable and hyperbolic if it exists.

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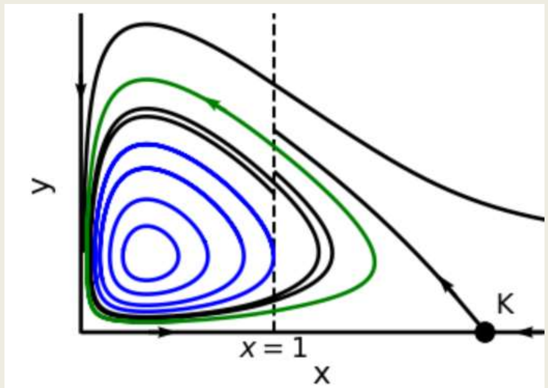
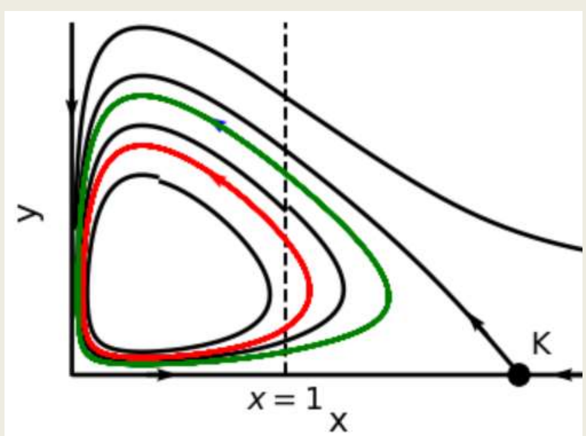
One detailed new result

- **Theorem**: For S < 0 system (*) has a unique limit cycle
  - stable and hyperbolic
  - period annulus in its interior



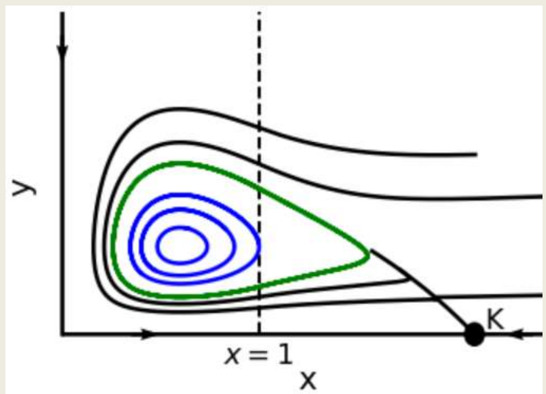
One detailed new result

- System (*) is structurally unstable



Open Problem 5

- The case with two equilibria in first quadrant



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Wrap-up

- Two-dimensional predator-prey systems
- Overview classical results
- New non-analytical functional responses
- Richer dynamics
  - 4 limit cycles
  - Co-existence period annulus and limit cycle
  - …
- 5 Open Problems

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Thanks for your attention!

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