Continuous Limit in Dynamics with Choice

Sanja Gonzalez Živanović
Lev Kapitanski*

Department of Mathematics and Computer Science,
Barry University, Miami Shores, Fl
*Department of Mathematics, University of Miami, Coral Gables, Fl

ICDEA, Barselona, Spain

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Variable Time Step Dynamics With Choice

- We study systems that switch their modes of operation (regimes) at discrete moments of time.
- The intervals between switching – dwell times – may vary.
- The number of modes may be finite or infinite.
- One-step evolution map: $S_j^\tau : x \to S_j^\tau(x)$,
  - τ – dwell time
  - j – regime used during this time interval
- Dynamics: $x_{n+1} = S_{j_n}^{\tau_n}(x_n)$
- We study all possible regime switchings and dwell time sequences (deterministic picture)

Example: Switched System

$$\dot{x} = f_{w(n)}(x) \quad \text{on the interval } [t_n, t_{n+1}),$$

- $t_0 < t_1 < \dots$ are the switching times,
- N different regimes labeled by elements of set $\mathcal{J} = \{1, 2, \dots, N\}$
- Each $f_{w(n)}$ is taken from a finite set of functions $\{f_0, \dots, f_{N-1}\}$,
- $w : \mathbb{N}_0 \to \{1, 2, \dots, N\}$ is a regime switching function,
- Time intervals between switching (dwell times) $h(n) = t_{n+1} - t_n$
- x_0 – initial state; after n steps the state will be x_n

$$x_{n+1} = S_{w(n)}^{h(n)}(x_n)$$

Example: Difference Equations

- discrete switched system:

$$x_{n+1} = x_n + h(n)\, f_{w(n)}(x_n)\,,$$

is already in the form

$$x_{n+1} = S_{w(n)}^{h(n)}(x_n)$$

We develop a language to describe the dynamics of systems of this sort.

We call it Variable Time Step Dynamics with Choice.

Encoding regime switching and dwell time sequences

Suppose there are 2 regimes (labeled 0 and 1) and 2 possible dwell times, *a* and *b*.

The strategy of regime switching is given by an infinite sequence (word, string)

$$w = 1011010010000111 0... \quad w(0) = 1, w(1) = 0, w(2) = 1, \ldots$$

The strategy of dwell times switching is given by an infinite sequence

$$h = aabbabaabb... \quad h(0) = a, h(1) = a, h(2) = b, \ldots$$

The corresponding trajectory of x is

$$x \to S_1^a(x) \to S_0^a(S_1^a(x)) \to S_1^b(S_0^a(S_1^a(x))) \to \ldots$$

Mathematical Setting

- X – the state space
- $\mathcal{J}$ – the set encoding different regimes
- $\mathcal{I}$ – the set of allowed dwell times
  $\mathcal{I} \subset (0, +\infty)$, could be an interval, or a finite set,...
- $\Sigma_{\mathcal{J}}$ – the space of one-sided infinite strings with symbols in $\mathcal{J}$
  $\Sigma_{\mathcal{I}}$ – the space of one-sided infinite strings with symbols in $\mathcal{I}$
- $\sigma : \Sigma_{\mathcal{J}} \to \Sigma_{\mathcal{J}}$ – the shift operator: If $w = w(0)w(1)w(2)\cdots \in \Sigma_{\mathcal{J}}$, then $\sigma(w) = w(1)w(2)\ldots$
- $\sigma : \Sigma_{\mathcal{I}} \to \Sigma_{\mathcal{I}}$ – the shift operator: If $h = h(0)h(1)h(2)\cdots \in \Sigma_{\mathcal{I}}$, then $\sigma(h) = h(1)h(2)\ldots$
- the maps $S_j^\tau : X \to X, \quad j \in \mathcal{J}, \tau \in \mathcal{I}$

Mathematical Setting

The right way to describe:

Variable time step dynamics with choice is a discrete time dynamics on $\mathfrak{X} = X \times \Sigma_{\mathcal{J}} \times \Sigma_{\mathcal{I}}$ generated by iterations of the map

$$\mathfrak{S} : \ (x, w, h) \mapsto \left(S_{w(0)}^{h(0)}(x), \, \sigma(w), \, \sigma(h) \right)$$

after n steps:

$$\mathfrak{S}^n(x, w, h) = \left(S_{w[n]}^{h[n]}(x), \, \sigma^n(w), \, \sigma^n(h) \right)$$

$$S_{w[n]}^{h[n]}(x) = S_{w(n-1)}^{h(n-1)} \circ \cdots \circ S_{w(1)}^{h(1)} \circ S_{w(0)}^{h(0)}(x)$$

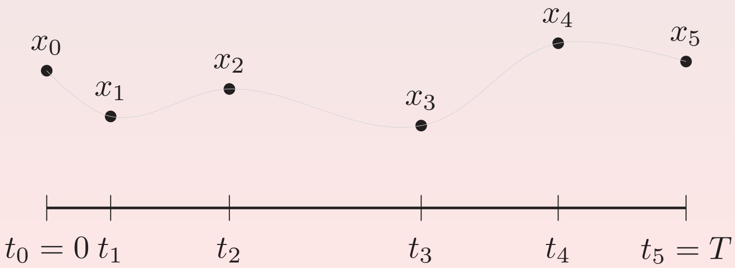
Reachable states at time T

Given an interval $[0, T]$, consider different partitions of $[0, T]$,

$$0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = T, \qquad h(i) = t_{i+1} - t_i$$

and different orders of switching modes, $w(0)\, w(1) \ldots w(m-1)$, to obtain all possible

$$x_m = S_{w[m]}^{h[m]}(x_0) = S_{w(m-1)}^{h(m-1)} \circ \cdots \circ S_{w(0)}^{h(0)}(x_0)\,, \quad \sum_j h(j) = T$$

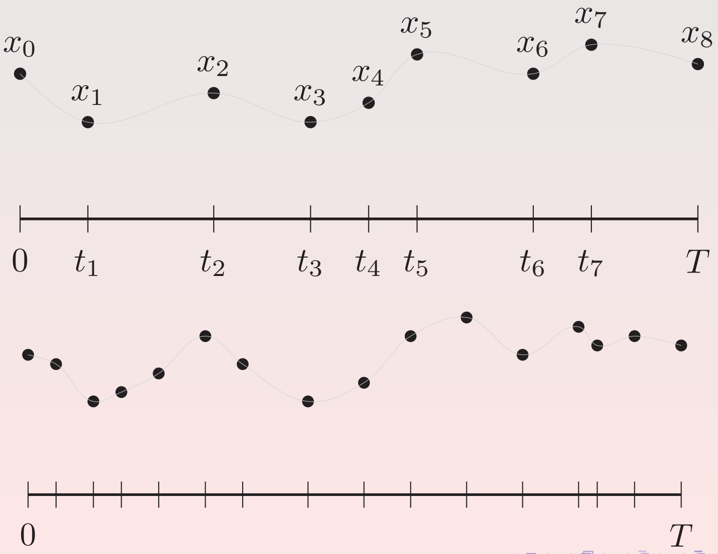


Mathematical Setting

In this talk:

- $X = \mathbb{R}^d$,
- $\mathcal{J}$ is compact metric space,
- $\mathcal{I} = (0, \epsilon]$,
- Each $S_j^\tau : X \to X$, is continuous, bounded for all $j \in \mathcal{J}$, $\tau \in (0, \epsilon]$

Continuous Limit in Dynamics with Choice



Continuous Limit in Dynamics with Choice

Make the partition finer and finer...

In the limit: all the points reachable at time T are all the limits of the form $\lim_{k\to\infty} S^{h_k[n_k]}_{w_k[n_k]}(x_0)$

- Define the reachable sets: For every $x \in X$, set $\mathcal{F}_T(x) = x$ if $T = 0$, and, if $T > 0$,

$$\mathcal{F}_T(x) = \Big\{ y \in X \mid y = \lim_{k\to\infty} S^{h_k[n_k]}_{w_k[n_k]}(x) \text{ for some sequence } (w_k, h_k) \subset \Sigma_{\mathcal{J}} \times \Sigma_{(0,\epsilon_k]}, \text{ with } \epsilon_k \searrow +0 \; n_k \nearrow +\infty, \text{ and } \sum_{i=0}^{n_k-1} h_k(i) = T \text{ for all } k \geq 0 \Big\}$$

Assumptions

- On any bounded set $A \subset X$, the maps S_j^τ are uniformly continuous with respect to j and τ.
- For any bounded B,

$$d_H(B, S_{w(m-1)}^{h(m-1)} \circ \cdots \circ S_{w(0)}^{h(0)}(B))$$

is small if the total time $\sum_{i=0}^{m-1} h(i)$ is small, independetely of the choice of w and h.

Properties

- $\mathcal{F}_T(x)$ is a nonempty, compact set.
- $\mathcal{F}_T(x)$ is continuous with respect to T,
- $\mathcal{F}_T(B)$ is compact.

Properties

For any $A, B \subset X$ bounded,

- If $A \subset B$, $\mathcal{F}_T(A) \subset \mathcal{F}_T(B)$
- $\mathcal{F}_T(A \cup B) = \mathcal{F}_T(A) \cup \mathcal{F}_T(B)$
- $\overline{\bigcup_{x \in B} \mathcal{F}_T(x)} \subset \overline{\bigcup_{x \in \bar{B}} \mathcal{F}_T(x)} \subset \mathcal{F}_T(B) = \mathcal{F}_T(\bar{B})$
- For any $T_1, \ T_2 > 0$ we have $\mathcal{F}_{T_1}\left(\mathcal{F}_{T_2}(B)\right) \subset \mathcal{F}_{T_1+T_2}(B)$.

Additional Assumption

For every $T > 0$, there exists a modulus of continuity function ω^T such that for any sequence $(w, h) \subset \Sigma_{\mathcal{J}} \times \Sigma_{(0,\epsilon]}$, if $\sum_{i=0}^{n-1} h(i) = T$, then

$$\sup_{x,y:d(x,y)\leq\delta} d(S_{w[n]}^{h[n]}(x), S_{w[n]}^{h[n]}(y)) \leq \omega^T(\delta)$$

For a fixed δ, the function $\omega^T(\delta)$ is nondecreasing in T.

Properties

- $\mathcal{F}$ has the semi-group property, i.e.,

$$\mathcal{F}_{T_2}(\mathcal{F}_{T_1}(x)) = \mathcal{F}_{T_1+T_2}(x).$$

- $\overline{\bigcup_{x\in B}\mathcal{F}_T(x)} = \bigcup_{x\in \bar{B}}\mathcal{F}_T(x) = \mathcal{F}_T(B) = \mathcal{F}_T(\bar{B})$
- For every $T > 0$, $\mathcal{F}_T$ is continuous in the sense that if $x_n \to x$, then

$$d_H(\mathcal{F}_T(x_n), \mathcal{F}_T(x)) \to 0$$

- The triple $(\mathbb{R}^d, \mathcal{F}, \mathbb{R}_+)$ is a multivalued semi-dynamical system.

Special Case

- $\dot{x}(t) = f_i(x(t)), \quad x(0) = x_0, \; i = 1, \dots, N$
- $f_1, \dots, f_N : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous maps
- Denote by $S_i^\tau(x_0)$ the solution of $\dot{x}(t) = f_i(x(t)), \quad x(0) = x_0$ at time τ. We have

$$S_i^\tau(x_0) = x_0 + \int_0^\tau f_i(x(s))\, ds.$$

Special Case

- It follows from general theory of ODEs that the maps S_i^T are continuous and bounded.
- The maps S_i^T satisfy all the Assumptions previously stated.
- The set $\mathcal{F}_T(x_0)$ is non-empty and has all the properties stated before.

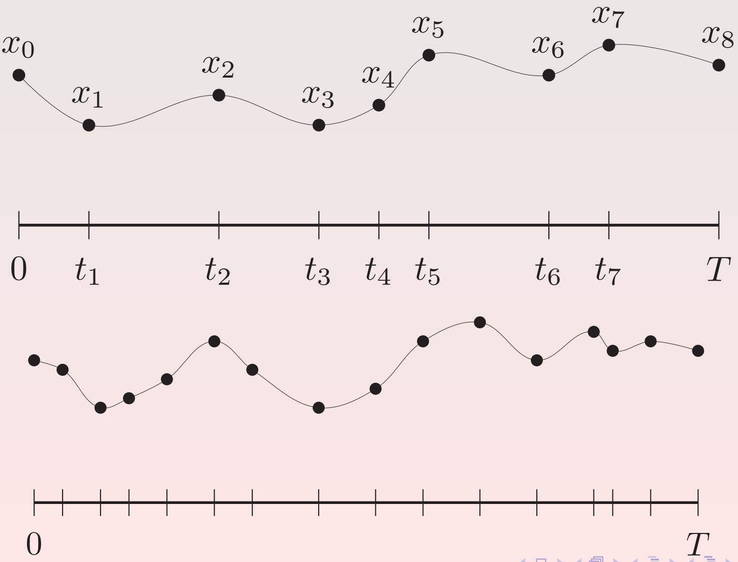
Special Case

- For any $(w,h) \in \Sigma_{\mathcal{J}} \times \Sigma_{(0,\epsilon]}$, iterates are defined as

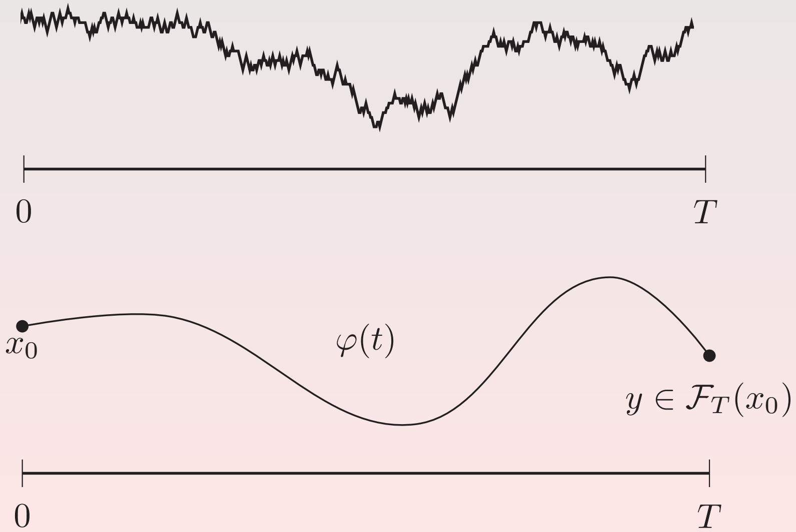
$$S_{w[n]}^{h[n]}(x_0) = \gamma(h[n]) = x_0 + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f_{w(i)}(\gamma(s))ds, \qquad t_{i+1} - t_i = h(i)$$

$$\gamma(t) = x_i + \int_{t_i}^{t} f_{w(i)}(\gamma(s))ds, \qquad t_i \le t \le t_{i+1}$$

γ curves



Limiting curve



Special Case

- Every sequence $(\gamma_k) \subset C([0, T])$ has a convergent subsequence.
- $\mathcal{CL}_{[0,T]}(x_0) = \{\varphi \in C([0, T]) \mid \varphi = \lim_{k\to\infty} \gamma_k\}$
  – variable time step continuous limit dynamics
- By construction, $\mathcal{F}_t(x_0) = \bigcup_{\varphi(\cdot)} \varphi(t)$

Differential Inclusions (DIs)

- Relaxed DI:

$$\dot{x}(t) \in \overline{co}\,\{\, f_1(x(t)),\ f_2(x(t)),\ \ldots, f_N(x(t))\,\}, \quad x(0) = x_0$$

- $\dot{x} \in F(t, x)$
- Solution set of DI is

$$\mathcal{DI}_{[0,T]}(x_0) = \Big\{ x(t) \,|\, x(t) \,\text{is a.c.}, \dot{x}(t) \in \overline{co}\{f_1(x(t)), \ldots, f_N(x(t)),\ \text{for a.e. } t \in [0, T],\ x(0) = x_0 \Big\}$$

Main Theorem

Main Theorem

1) $\mathcal{DI}_{[0,T]}(x_0) = \mathcal{CL}_{[0,T]}(x_0)$,

2) In particular, reachable set of the differential inclusion at time t is $\mathcal{F}_t(x_0)$, i.e., $\mathcal{DI}_t(x_0) = \mathcal{F}_t(x_0)$.

Sketch of the Proof

Proof: $\mathcal{CL}_{[0,T]}(x_0) \subset \mathcal{DI}_{[0,T]}(x_0)$

- is standard in the theory of DIs.

Proof: $\mathcal{CL}_{[0,T]}(x_0) \supset \mathcal{DI}_{[0,T]}(x_0)$

a) Every solution of DI, $x(\cdot) \in \mathcal{DI}_{[0,T]}(x_0)$, is a solution of the so-called control system

$$\dot{x}(t) = \sum_{i=1}^{N} \alpha_i(t)\, f_i(x(t)), \qquad x(0) = x_0$$

where $\alpha_i(\cdot)$ are some measurable functions with $\sum_{i=1}^{N} \alpha_i(t) = 1$ for a.e. $t \in [0, T]$.

Sketch of The Proof

Proof: $\mathcal{CL}_{[0,T]}(x_0) \supset \mathcal{DI}_{[0,T]}(x_0)$

b) First we show that every solution of the control system can be approximated by solutions of the system

$$\dot{x}(t) = \sum_{i=1}^{N} b_i(t)\, f_i(x(t)), \quad x(0) = x_0$$

where each $b_i(\cdot)$ is a step function, and $\sum_{i=1}^{N} b_i(t) = 1$.

Denote the solution of this system by $S^t_{\sum_{i=1}^{N} b_i f_i}(x_0)$.

Sketch of The Proof - Splitting Method

Proof: $\mathcal{CL}_{[0,T]}(x_0) \supset \mathcal{DI}_{[0,T]}(x_0)$

c) The key step:

$$S^T_{\sum_{i=1}^N b_i f_i}(x_0) = \lim_{M\to\infty} \left(S_N^{b_N \frac{T}{M}} \circ S_{N-1}^{b_{N-1} \frac{T}{M}} \circ \cdots \circ S_1^{b_1 \frac{T}{M}} \right)^M (x_0)$$

Recall the VTSDWC:

$$S_{w[n]}^{h[n]}(x_0) = \gamma(h[n]) = x_0 + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f_{w(i)}(\gamma(s))ds, \qquad t_{i+1} - t_i = h(i)$$

$$\gamma(t) = x_i + \int_{t_i}^{t} f_{w(i)}(\gamma(s))ds, \qquad t_i \le t \le t_{i+1}$$

In our case, for each M we have periodic sequences w (the same for all M) and h_M,
$w = 123\cdots N123\cdots N123\cdots$ and
$h_M(0) = b_1 T/M$, $h_M(1) = b_2 T/M$, etc. For example,

$$S_{w[N+2]}^{h_M[N+2]}(x_0) = S_2^{b_2 \frac{T}{M}} \circ S_1^{b_1 \frac{T}{M}} \circ S_N^{b_N \frac{T}{M}} \circ S_{N-1}^{b_{N-1} \frac{T}{M}} \circ \cdots \circ S_1^{b_1 \frac{T}{M}}(x_0)$$

We prove that $\gamma_M(t) \underset{M\to\infty}{\to} S^t_{\sum_{i=1}^N b_i f_i}(x_0)$ for every $t \in [0, T]$.

Motivation: Splitting Method

- The classical Lie product formula

$$e^{t(A+B)} = \lim_{M\to\infty} \left(e^{\frac{t}{M}A} e^{\frac{t}{M}B} \right)^M$$

where A and B are square matrices.

- This formula shows how to approximate the evolution function of the equation

$$\dot{x} = (A+B)x$$

by the evolution functions of the equations $\dot{x} = Ax$ and $\dot{x} = Bx$.

Conclusion

- We give a general framework to study systems that switch regimes at discrete moments of time.
- We study the aggregate dynamics for all possible regime switchings and time switchings, rather then approximating, optimizing, averaging.
- We explain what a continuous limit in VTSDWC is (when the dwell times go to zero).
- In the special case of a switched system, the continuous limit in VTSDWC coincides with the solutions of the corresponding differential inclusion.