Functional envelopes of dynamical systems – old and new results

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Based on:

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Functional envelopes of dynamical systems – old and new results

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1.-4. by [AKS], 5. by [AKS]+[A], 6. by [AKS]+[M], 7. by [DSS].

1. Definition

(X, f) ...... dyn. system (X– compact metric, $f : X \to X$ cont.)

$S(X)$ .... all cont. maps $X \to X$; with compact-open topology
($S_U(X)$ ... unif. metric, $S_H(X)$ ... Hausdorff metric)
topol. semigroup with respect to the comp. of maps

$F_f : S(X) \to S(X)$
$F_f(\varphi) = f \circ \varphi$ uniformly cont. (for each of the two metrics)

$(S(X), F_f)$ ...... **functional envelope of** (X, f)
trajectory of φ: $\boxed{\varphi, f \circ \varphi, f^2 \circ \varphi, \dots}$

$(S_U(X), F_f)$ and $(S_H(X), F_f)$ are topol. conjugate, but in general not compact
$\Rightarrow$ the same topological properties,
but not necessarily the same metric properties

2. Motivation

1) **Functional difference equations** (Sharkovsky et al.)

$$x(t+1) = f(x(t)), \quad t \geq 0, \qquad f : [a,b] \to [a,b] \text{ continuous}$$

Every $\varphi : [0,1) \to [a,b]$ gives a solution $x : [0,\infty) \to [a,b]$:

$$\begin{aligned} x(t) &= \varphi(t), \qquad t \in [0,1) \\ x(t+1) &= f(\varphi(t)) \\ x(t+2) &= f^2(\varphi(t)) \\ &\dots \end{aligned}$$

we see here $\boxed{\varphi, f \circ \varphi, f^2 \circ \varphi, \dots}$

x continuous $\iff$ φ continuous and $\varphi(1^-) = f(\varphi(0))$

In such a case we can view the boxed maps as continuous maps $[0,1] \to [a,b]$, rather than $[0,1) \to [a,b]$.

Finally, if $[a,b] = [0,1] =: I$, the boxed sequence is the $\boxed{\text{trajectory of } \varphi \text{ in } (S(I), F_f)}$ (i.e. in the fc. envelope of (I, f)).

2. Motivation

2) **Semigroup theory**

S - topological semigroup ...... **density index** $D(S)$ = least n such that S contains a dense subsemigroup with n generators (∞ if no such finite n exists).

$$D(S(X)) = \begin{cases} 2, & \text{if } X = I^k \text{ (Schreier, Ulam, Sierpinski ...} \\ & \qquad\qquad \text{... Cook, Ingram, Subbiah (35 years story))} \\ 2, & \text{if } X = \text{Cantor set} \\ \infty, & \text{if } X = \mathbb{S}^k. \end{cases}$$

$D(S(X)) = 2$ ..... $\exists\, \varphi, f$ such that the family of maps

$$\boxed{\varphi}, f, \ \varphi^2, \boxed{f \circ \varphi}, \varphi \circ f, f^2, \ \varphi^3, f \circ \varphi^2, \varphi \circ f \circ \varphi, \boxed{f^2 \circ \varphi}, \ldots$$

is dense in $S(X)$. *Can the smaller family of boxed maps be dense in* $S(X)$? (i.e., can the orbit of φ in the fc. envelope $(S(X), F_f)$ be dense?)

2. Motivation

3) **Dynamical systems theory**

2^X = closed subsets of the cpct. space X, with Hausdorff metric
Quasi-factor of (X, f) = (closed, here) any subsystem of $(2^X, f)$.
No distinction between maps and their graphs $\Rightarrow$

$$\boxed{(S_H(X), F_f) = \text{a quasi-factor of } (X \times X, \mathrm{id} \times f)}\,.$$

$R_X := \{\mathrm{range}(\varphi) : \varphi \in S(X)\}$ with Hausdorff metric. Then

$$\boxed{(R_X, f) = \text{a quasi-factor of } (X, f)}\,.$$

Moreover, $\boxed{(R_X, f) \text{ is a } \textbf{factor} \text{ of } (S(X), F_f)}$
[$f(\mathrm{range}(\varphi)) = \mathrm{range}(f \circ \varphi)$ and so $\varphi \mapsto \mathrm{range}(\varphi)$
is a homomorphism of $(S(X), F_f)$ onto (R_X, f)].

$\Rightarrow$ connection between properties of $(S(X), F_f)$ and (R_X, f).

2. Motivation

$$\begin{array}{ccccccc}
S_H(X) & \xrightarrow{F_f} & S_H(X) & \xleftarrow{quasi-f.} & X \times X & \xrightarrow{\mathrm{id} \times f} & X \times X \\
{\scriptstyle range}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle (a,b)\mapsto b} \\
R_X & \xrightarrow{f} & R_X & \xleftarrow{quasi-f.} & X & \xrightarrow{f} & X \\
 & \text{commutes} & & & & \text{commutes} &
\end{array}$$

3. Some of the results on properties related to the simplicity of a system

Fact. $(S(X), F_f)$ contains an isomorphic copy of (X, f) (the copy is made of constant maps). Hence the name 'functional envelope'.

Corollary. All properties which are **hereditary down** (i.e. are inherited by subsystems) carry over from $(S(X), F_f)$ to (X, f) (if the property is metric, then regardless of whether S_U or S_H).

Examples: isometry, equicontinuity, uniform rigidity, distality, asymptoticity, proximality.

Direction from f to F_f:

(X, f)	isom.	equi.	u.rig.	dist.	asymp.	prox.
$(S_U(X), F_f)$	+	+	+	+	–	–
$(S_H(X), F_f)$	+	+	+	–	–	–

(X, f) distal .... $(S_H(X), F_f)$ may contain asymptotic pairs
(X, f) asymptotic ... $(S_U(X), F_f)$ and $(S_H(X), F_f)$ may contain distal pairs

4. Some of the results on orbit closures, ω-limit sets and range properties

Definition. Let P be a property a map from $S(X)$ may or may not have. It is said to be a **range property** if

$$\mathrm{range}\,\theta = \mathrm{range}\,\varphi \Longrightarrow (\varphi \text{ has } P \Leftrightarrow \theta \text{ has } P)$$

and it is said to be a **range down property** if

$$\mathrm{range}\,\theta \subseteq \mathrm{range}\,\varphi \Longrightarrow (\varphi \text{ has } P \Rightarrow \theta \text{ has } P).$$

Obviously, a range down property is a range property.

4. Some of the results on orbit closures, ω-limit sets and range properties

Some of many results for the illustration:

Theorem. The following are range down properties:
(i) the compactness of an orbit closure,
(ii) having a nonempty ω-limit set,
(iii) recurrence,
(iv) the simultaneous compactness and minimality of an orbit closure (the minimality of an orbit closure is only a range prop.)

5. Some of the results on dense orbits

$D(S(X)) > 2 \Rightarrow$ no dense orbits in $(S(X), F_f)$
$D(S(X)) = 2 \Rightarrow$?

Answer:

– *dense orbits in functional envelopes may exist (Example: Fc. envelope of the full shift on $A^{\mathbb{N}}$ contains dense orbits. ($A = \{0,1\} \Rightarrow A^{\mathbb{N}} =$Cantor, $A = [0,1] \Rightarrow A^{\mathbb{N}} =$ Hilbert cube)*

– *for many X, even if $D(S(X)) = 2$, there are no dense orbits in the functional envelope $(S(X), F_f)$ regardless of the choice of f:*

Theorem. Let X be a nondegenerate compact metric space satisfying (at least) one of the following conditions:
(a) X admits a stably non-injective continuous selfmap,
(b) X contains no homeo. copy of X with empty interior in X.
Then there are no dense orbits in the functional envelope $(S(X), F_f)$.

– covers all manifolds etc.

5. Some of the results on dense orbits

In particular, we see: If K is a Cantor set, then $(S(K), F_f)$ may contain dense orbits (i.e. may be topologically transitive).

Theorem (Akin 2007, personal communication): If K is a Cantor set and (K, f) is weakly mixing, then $(S(K), F_f)$ is also weakly mixing.

6. Some of the results on topological entropy

F_f is uniformly continuous on $S_U(X)$ and $S_H(X)$ and so one can study the topological entropy of fc. envelopes.

$$d_U \geq d_H \quad \Longrightarrow \quad \mathrm{ent}_{\,U}(F) \geq \mathrm{ent}_{\,H}(F) \geq \mathrm{ent}(f)$$

Examples and theorem:

- $\mathrm{ent}(f) = 0$ (even an asymptotic countable system or a nondecreasing interval map), $\mathrm{ent}_U(F_f) = +\infty$
  So:
  $\mathrm{ent}(f) = 0 \not\Rightarrow \mathrm{ent}_U(F_f) = 0$ (even on the interval)
- $\mathrm{ent}(f) = 0$ (even an asymptotic countable system), $\mathrm{ent}_H(F_f) = +\infty$
  However:
  Theorem (Matviichuk 2011): If f is a tree map, then
  $$\mathrm{ent}(f) = 0 \Rightarrow \mathrm{ent}_H(F_f) = 0$$
  $$\mathrm{ent}(f) > 0 \Rightarrow \mathrm{ent}_H(F_f) = +\infty$$

7. Some of the results on expansivity

homeo $f : X \to X$... *expansive* if $\exists \varepsilon > 0\ \forall x, y \in X, x \neq y$
$\exists n \in \mathbb{Z} : d(f^n(x), f^n(y)) > \varepsilon$
... *continuum-wise expansive* or *c-w expansive* if
$\exists \varepsilon > 0\ \forall K$- a subcontinuum of X
$\exists n \in \mathbb{Z} : \operatorname{diam} f^n(K) > \varepsilon$

map $f : X \to X$ ... *positively expansive* (*pos. c-w expansive*) if
... $\exists n \geq 0$ ...

$$\begin{array}{ccccc}
(S_H(X), F_f) \text{ exp.} & \Longrightarrow & (X, f) \text{ exp.} & \Longleftrightarrow & (S_U(X), F_f) \text{ exp.} \\
\Downarrow & & \Downarrow & & \\
(S_H(X), F_f) \text{ c-w exp.} & \Longrightarrow & (X, f) \text{ c-w exp.} & \Longleftrightarrow & (S_U(X), F_f) \text{ c-w exp.}
\end{array}$$

7. Some of the results on expansivity

Theorem

Let X be a compact metric space.

1. *If X contains an infinite, zero dimensional subspace Z such that Z is open in X, then $(S_H(X), F_f)$ is never exp./pos. exp.*
2. *If X contains an arc, then $(S_H(X), F_f)$ is never c-w exp./pos. c-w exp. (hence, never exp./pos. exp.).*