Averaging theorems for dynamic equations on time scales

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Historical background

- Lagrange and celestial mechanics (18th century)
- Reduction of

$$x'(t) = F(t, x(t), \varepsilon), \quad x(t_0) = x_0$$

(where F is T-periodic in the first variable) to the standard form

$$x'(t) = \varepsilon f(t, x(t)) + O(\varepsilon^2), \quad x(t_0) = x_0$$

- Expand $f(t, x)$ into Fourier series with respect to t and neglect all time-dependent terms, keeping only

$$f^0(x) = \frac{1}{T}\int_0^T f(t, x)\, dt$$

- Averaged equation: $y'(t) = \varepsilon f^0(y(t)), \quad y(t_0) = x_0$
- 20th century: proofs of asymptotic validity, nonperiodic averaging

Classical averaging theorems

Solutions of the initial-value problem

$$x'(t) = \varepsilon f(t, x(t)) + \varepsilon^2 g(t, x(t), \varepsilon), \quad x(t_0) = x_0,$$

can be approximated by solutions of the averaged equation

$$y'(t) = \varepsilon f^0(y(t)), \quad y(t_0) = x_0,$$

where

$$f^0(y) = \frac{1}{T}\int_{t_0}^{t_0+T} f(t, y)\, dt$$

if f is a T-periodic function in the first variable and

$$f^0(y) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} f(t, y)\, dt$$

otherwise.

Quality of the approximation

Periodic case:
Given a $d > 0$, there is an $\varepsilon_0 > 0$ and a $c > 0$ such that

$$\|x(t) - y(t)\| \leq c\varepsilon$$

for every $\varepsilon \in (0, \varepsilon_0]$ and $t \in [t_0, t_0 + d/\varepsilon]$.

Nonperiodic case:
Given a $d > 0$ and a $\delta > 0$, there is an $\varepsilon_0 > 0$ such that

$$\|x(t) - y(t)\| \leq \delta$$

for every $\varepsilon \in (0, \varepsilon_0]$ and $t \in [t_0, t_0 + d/\varepsilon]$.

Averaging theorems for other types of equations

- Ordinary differential equations with impulses
- Retarded functional differential equations
- Dynamic equations on time scales
- Generalized ordinary differenital equations

Generalized ordinary differential equations

- Interval $I \subseteq \mathbb{R}$
- $F : \mathbb{R}^n \times I \to \mathbb{R}^n$

A function $x : I \to \mathbb{R}^n$ is called a solution of the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DF(x,t), \quad x(a) = x_0$$

whenever

$$x(s) = x_0 + \int_a^s DF(x(\tau),t)$$

for every $s \in I$, where the integral on the right-hand side is the Kurzweil integral.

Kurzweil integration

A function $F : [a,b] \times [a,b] \to \mathbb{R}^n$ is called Kurzweil integrable over $[a,b]$ if there exists a vector $I \in \mathbb{R}^n$ such that given an $\varepsilon > 0$, there is a function $\delta : [a,b] \to \mathbb{R}^+$ such that

$$\left\| \sum_{j=1}^{k} (F(\tau_j,\alpha_j) - F(\tau_j,\alpha_{j-1})) - I \right\| < \varepsilon$$

for every partition with division points

$$a = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{k-1} \leq \alpha_k = b$$

and tags $\tau_j \in [\alpha_{j-1},\alpha_j]$ such that

$$[\alpha_{j-1},\alpha_j] \subset (\tau_j - \delta(t_j), \tau_j + \delta(\tau_j)), \quad j \in \{1,\ldots,k\}.$$

Notation: $I = \int_a^b DF(\tau,t)$.
$F(\tau,t) = f(\tau)t \Rightarrow$ Henstock-Kurzweil integral $\int_a^b f(s)\,ds$
$F(\tau,t) = f(\tau)g(t) \Rightarrow$ Kurzweil-Stieltjes integral $\int_a^b f(s)\,dg(s)$

Classical ODEs vs. GODEs

An ordinary differential equation

$$x'(t) = f(x(t),t), \quad x(t_0) = x_0$$

is equivalent to the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DF(x,t), \quad x(t_0) = x_0,$$

where $F(x,t) = \int_{t_0}^t f(x,s)\,ds$.

Periodic averaging for GODEs

$B \subset \mathbb{R}^n$, $\Omega = B \times [0,\infty)$, $\varepsilon_0 > 0$, $L > 0$, $F : \Omega \to \mathbb{R}^n$, $G : \Omega \times (0,\varepsilon_0] \to \mathbb{R}^n$.
Assume there exists a $T > 0$ and a function $M : B \to \mathbb{R}^n$ such that $F(x,t+T) - F(x,t) = M(x)$ for every $x \in B$ and $t \in [0,\infty)$. Let

$$F_0(x) = \frac{F(x,T)}{T}, \quad x \in B.$$

Then, under certain assumption on F, G, and M, the solutions of

$$\frac{dx}{d\tau} = D\left[\varepsilon F(x,t) + \varepsilon^2 G(x,t,\varepsilon)\right], \quad x(0) = x_0,$$

can be approximated by solutions of

$$y'(t) = \varepsilon F_0(y(t)), \quad y(0) = x_0,$$

i.e. there exists a constant $K > 0$ such that

$$\|x(t) - y(t)\| \le K\varepsilon, \quad \varepsilon \in (0,\varepsilon_0],\ t \in [0, L/\varepsilon].$$

Extension of time scale functions

Given a real number $t \le \sup \mathbb{T}$, let

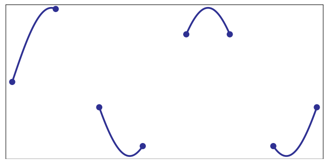
$$t^* = \inf\{s \in \mathbb{T};\, s \ge t\}.$$

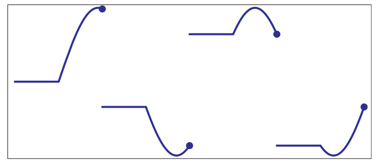
Further, let

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ (-\infty, \infty) & \text{otherwise.} \end{cases}$$

Given a function $x : \mathbb{T} \to \mathbb{R}^n$, define $x^* : \mathbb{T}^* \to \mathbb{R}^n$ by

$$x^*(t) = x(t^*), \quad t \in \mathbb{T}^*.$$





Dynamic equations and GODEs

Let $X \subset \mathbb{R}^n$ and assume that $f : X \times \mathbb{T} \to \mathbb{R}^n$ satisfies certain conditions. If $x : \mathbb{T} \to X$ is a solution of

$$x^{\Delta}(t) = f(x(t),t), \quad x(t_0) = x_0, \tag{1}$$

then $x^* : \mathbb{T}^* \to X$ is a solution of

$$\frac{dx}{d\tau} = DF(x,t), \quad x(t_0) = x_0, \tag{2}$$

where $F(x,t) = \int_{t_0}^{t} f(x,s^*)\, dg(s)$ and $g(s) = s^*$.

Conversely, every solution $y : \mathbb{T}^* \to X$ of (2) has the form $y = x^*$, where $x : \mathbb{T} \to X$ is a solution of (1).

Periodic averaging on time scales

Let $\mathbb{T}$ be a T-periodic time scale ($t \in \mathbb{T}$ implies $t + T \in \mathbb{T}$ and $\mu(t) = \mu(t+T)$) and f a T-periodic function in t. Consider the initial-value problems

$$\begin{aligned} x^{\Delta}(t) &= \varepsilon f(t,x(t)) + \varepsilon^2 g(t,x(t),\varepsilon), \quad x(t_0) = x_0, \\ y'(t) &= \varepsilon f^0(y(t)), \quad y(t_0) = x_0, \end{aligned}$$

where $f^0(y) = \frac{1}{T}\int_{t_0}^{t_0+T} f(t,y)\,\Delta t$.

Then (under certain assumptions on f and g), given a $d > 0$, there is a $c > 0$ such that

$$\|x(t) - y(t)\| \le c\varepsilon$$

for every $\varepsilon \in (0,\varepsilon_0]$ and $t \in [t_0, t_0 + d/\varepsilon]_{\mathbb{T}}$.

Periodic averaging on time scales (2nd version)

Let $\mathbb{T}$ be a T-periodic time scale ($t \in \mathbb{T}$ implies $t + T \in \mathbb{T}$ and $\mu(t) = \mu(t + T)$) and f a T-periodic function in t. Consider the initial-value problems

$$\begin{aligned} x^\Delta(t) &= \varepsilon f(t, x(t)) + \varepsilon^2 g(t, x(t), \varepsilon), \quad x(t_0) = x_0, \\ y^\Delta(t) &= \varepsilon f^0(y(t)), \quad y(t_0) = x_0, \end{aligned}$$

where $f^0(y) = \frac{1}{T}\int_{t_0}^{t_0+T} f(t, y)\,\Delta t$.

Then (under certain assumptions on f and g), given a $d > 0$, there is a $c > 0$ such that

$$\|x(t) - y(t)\| \le c\varepsilon$$

for every $\varepsilon \in (0, \varepsilon_0]$ and $t \in [t_0, t_0 + d/\varepsilon]_{\mathbb{T}}$.

Application: Existence of periodic solutions

Let $\mathbb{T}$ be a T-periodic time scale, $t_0 \in \mathbb{T}$, $p_0 \in \mathbb{R}^n$, $r > 0$, $\varepsilon_0 > 0$. Consider functions $f : [t_0, \infty)_{\mathbb{T}} \times B_r(p_0) \to \mathbb{R}^n$ and $g : [t_0, \infty)_{\mathbb{T}} \times B_r(p_0) \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$, which are T-periodic in the first argument and satisfy certain additional conditions.

If $f^0(p_0) = 0$ and the matrix $\frac{\partial f^0}{\partial x}(p_0)$ is invertible, then there exist numbers $\varepsilon_1 \in (0, \varepsilon_0)$, $C > 0$ and a continuous function $p : [-\varepsilon_1, \varepsilon_1] \to B_r(p_0)$ such that $p(0) = p_0$ and for every $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$, the initial-value problem

$$x^\Delta(t) = \varepsilon f(t, x(t)) + \varepsilon^2 g(t, x(t), \varepsilon), \quad x(t_0) = p(\varepsilon)$$

has a unique solution $x : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^n$, which is T-periodic and satisfies

$$\|x(t) - p_0\| \le C|\varepsilon|, \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

Example

Consider the time scale $\mathbb{T} = \mathbb{Z}$ and the difference equation

$$\Delta x(t) = \varepsilon(1 - x(t) + (-1)^t), \quad t \in \{0, 1, 2, \ldots\},$$

whose right-hand side is 2-periodic in t.
The corresponding averaged equation is $\Delta y(t) = \varepsilon f^0(y(t))$, where $f^0(x) = 1 - x$.
Equilibrium solution: $y(t) = p_0 = 1$, $\frac{\partial f^0}{\partial x}(p_0) = -1$
The previous theorem guarantees that the original difference equation has a 2-periodic solution near p_0 whenever $|\varepsilon|$ is sufficiently small.
Can be found analytically:

$$x(t) = 1 + (-1)^t\varepsilon/(\varepsilon - 2)$$

For $\varepsilon \in [-1, 1]$, we have $|x(t) - 1| \le |\varepsilon|$ for every $t \in \{0, 1, 2, \ldots\}$.

Nonperiodic averaging on time scales

Let $\mathbb{T}$ be a time scale with $\sup \mathbb{T} = \infty$ and $\lim_{t\to\infty} \mu(t)/t = 0$, $c > 0$, and $B_c = \{x \in \mathbb{R}^n; \|x\| < c\}$. Consider $f : B_c \times [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^n$ and the initial-value problems

$$\begin{aligned} x^\Delta(t) &= \varepsilon f(t, x(t)), \quad x(t_0) = x_0, \\ y'(t) &= \varepsilon f^0(y(t)), \quad y(t_0) = x_0, \end{aligned}$$

where

$$f^0(y) = \lim_{T\to\infty,\, T\in\mathbb{T}} \frac{1}{T}\int_{t_0}^{t_0+T} f(y, s)\,\Delta s.$$

Then (under certain assumptions on f), given a $d > 0$ and a $\delta > 0$, there is an $\varepsilon_0 > 0$ such that

$$\|x(t) - y(t)\| \le \delta$$

for every $\varepsilon \in (0, \varepsilon_0]$ and $t \in [t_0, t_0 + d/\varepsilon]_{\mathbb{T}}$.

Open questions

- The condition $\lim_{t\to\infty} \mu(t)/t = 0$ guarantees that the assumptions of the GODE averaging theorem are satisfied. Is it possible to weaken or relax the condition on μ?
- Does there exist a nonperiodic averaging theorem where the averaged equation is a dynamic equation defined on the same time scale as the original equation?

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