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A QRT-system of two order-one homographic difference equations : conjugation to rotations, periods of solutions, sensitiveness to initial conditions

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I. A geometric definition for an homographic system

The “homographic” system in ${\mathbb{R}_*^+}^2$ is

$$u_{n+1}u_n = 1 + \frac{d}{v_n}, \quad v_{n+1}v_n = 1 + \frac{d}{u_{n+1}}, \quad \text{for } d > 0.$$

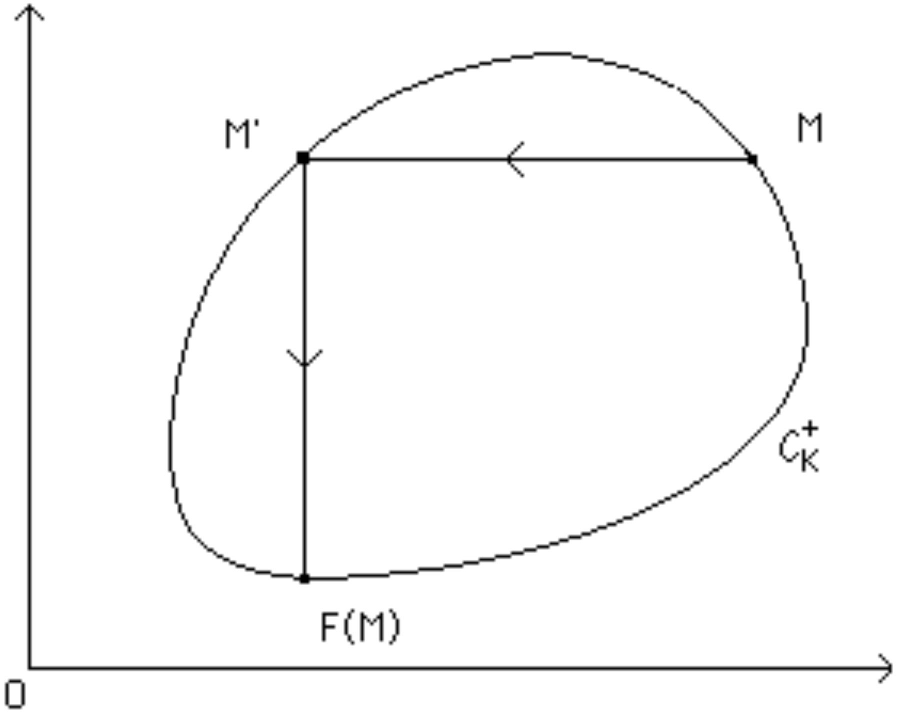
The associated dynamical system is (U, F), where $U = {\mathbb{R}_*^+}^2$, $F(x, y) = (X, Y)$, where

$$Xx = 1 + \frac{d}{y}, \quad Yy = 1 + \frac{d}{X},$$

so that if $M_n := (u_n, v_n)$ then $F(M_n) = M_{n+1}$.

As every QRT-map, there is a geometric construction of F. Let $\mathcal{C}_K$ be the family of cubic curves in the plane, $xy(x+y) + (x+y) + d - Kxy = 0$, with $d > 0$, $K \in \mathbb{R}$. It is of degree 2 in x and in y.

Map $F : U \to U$ is defined by the geometric method :



The cubic curves $\mathcal{C}_K$ are *invariant* and the quantity defined in U by

$$G(x,y) := x + y + \frac{1}{x} + \frac{1}{y} + \frac{d}{xy}$$

is *invariant* under the action of F : $G \circ F = G$. The curve $\mathcal{C}_K^+$ is the K-level set of G in U.

II. Critical point of G and fixed point of F

The first result concerns the sequences (u_n, v_n).

Theorem 1. *The map F has exactly one fixed point $L = (\ell, \ell)$ where ℓ is the positive solution of the equation* $t^3 - t - d = 0$.

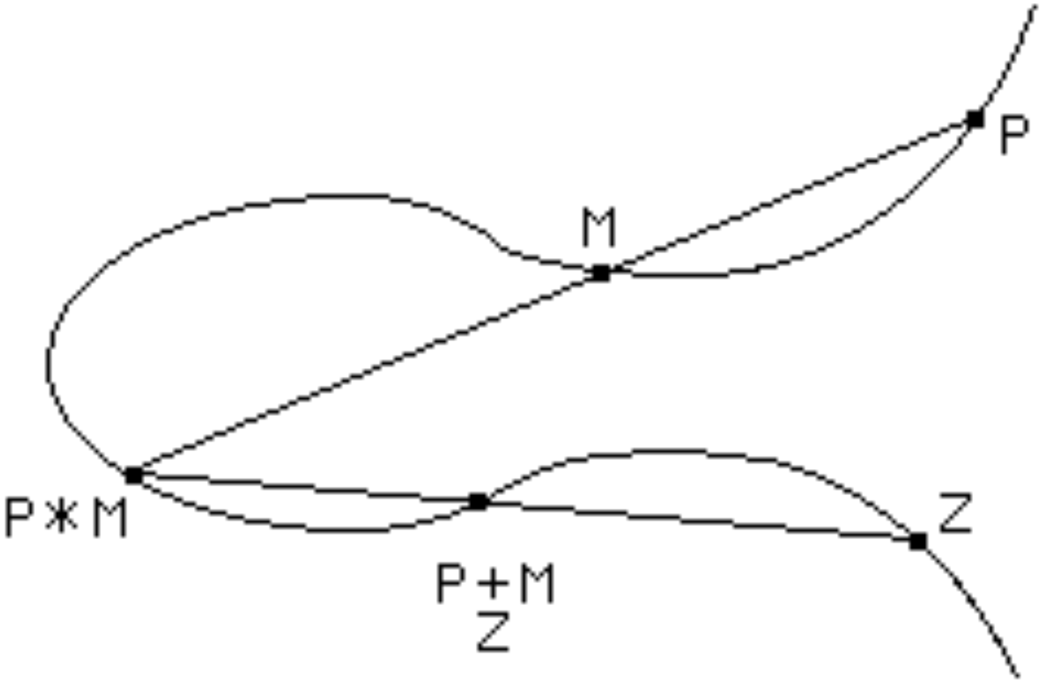
$G \to +\infty$ at the infinite point of U, and L is its unique critical point, where G attains its strict minimum K_m. The solutions of the homographic system are permanent ; if $(u_0, v_0) \neq L$, then the solution diverges. The equilibrium L is localy stable. Moreover, for $K > K_m$ the positive component $\mathcal{C}_K^+$ of the cubic $\mathcal{C}_K$ is diffeomorphic to the circle $\mathbb{T}$and surrounds the point L.

III. The dynamical system in terms of the group law on the cubic

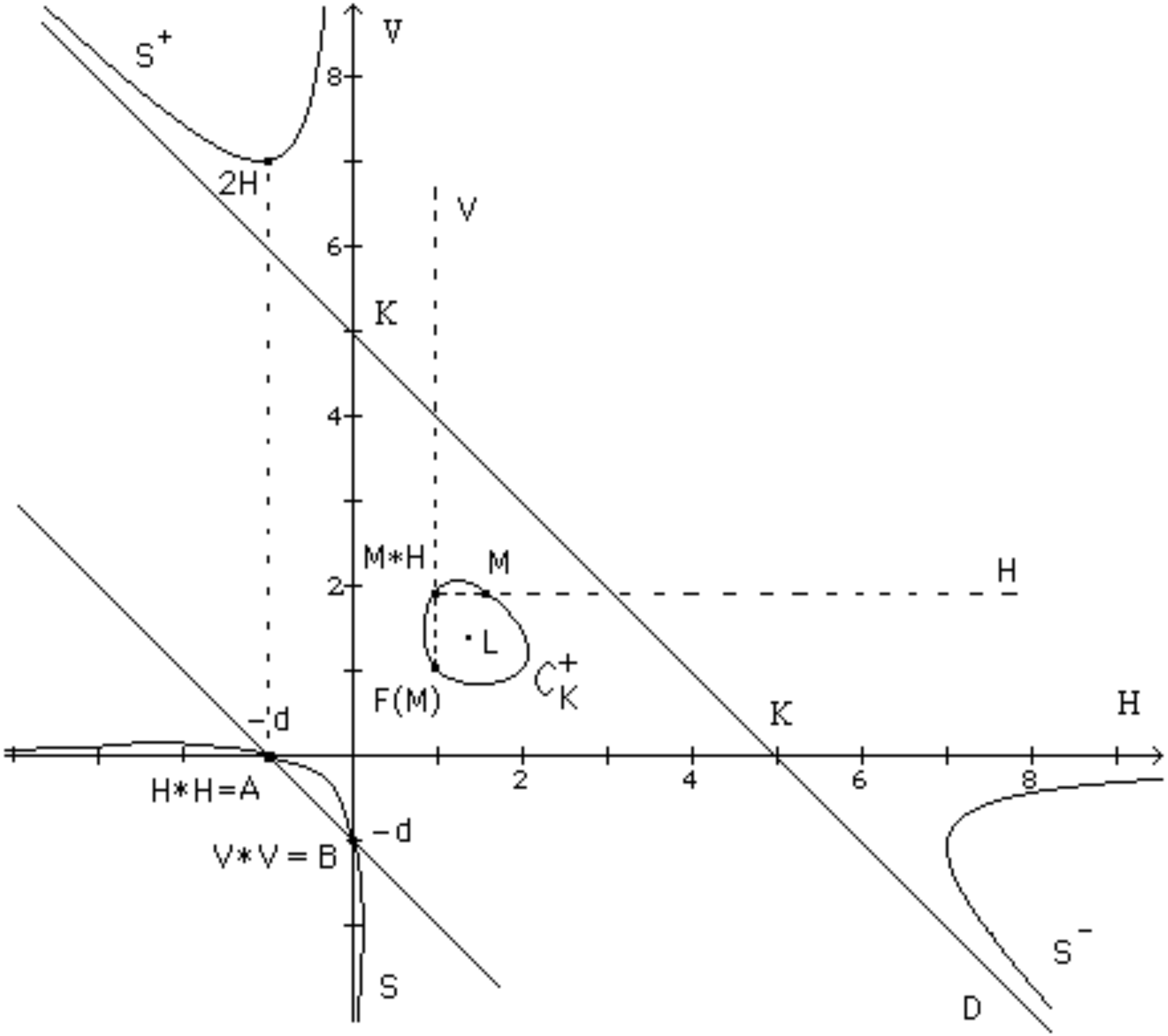
We denote $\overline{\mathcal{C}_K}$ the extension of $\mathcal{C}_K$ in $\mathbb{P}^2(\mathbb{R})$, and $\widetilde{\mathcal{C}_K}$ its extension in $\mathbb{P}^2(\mathbb{C})$. We have natural extensions of F as $\overline{F}$ and $\widetilde{F}$ to these spaces. Now we can extend also the geometric definition of F to $\overline{F}$ and $\widetilde{F}$, by intersection of the cubics with horizontal and vertical lines in $\mathbb{P}^2(\mathbb{R})$ and $\mathbb{P}^2(\mathbb{C})$.

The crucial property of $\widetilde{\mathcal{C}_K}$ is that it is regular and so elliptic.

Let us recall that on an elliptic cubic curve we have abelian group laws : the tangent-chord group laws. We choose a zero element Z on the cubic and define $P \underset{Z}{+} M$ as $(P * M) * Z$, where $A * B$ denotes the third point of the cubic on the line (AB).



Now, the cubic $\widetilde{\mathcal{C}_K}$ has three points at infinity, H, V and D. One sees that the restriction of the map $\widetilde{F}$ to $\widetilde{\mathcal{C}_K}$ is nothing but the map $M \mapsto M \underset{V}{+} H$.



Case d = 1, K = 5

Proposition 1. *If* $M_0 = (u_0, v_0) \in \mathcal{C}_K^+ \subset U$, *then*

$$(u_n, v_n) = M_n = F^n(M_0) = M_0 \underset{V}{+} nH.$$

So (u_n, v_n) *is* k*-periodic iff* $kH = V$ *in the group law, that is iff* H *has for order a divisor of* k.

If a point $M_0 \in U$ *is* k*-periodic, then all points of the curve* $\mathcal{C}_K$ *containing* M_0 *are* k*-periodic.*

Example of calculations with the group law : the “homographic” difference equations have no non-constant 4-periodic solution, that is $4H \neq V$.

First it is easy to see the opposite of a point X of $\overline{\mathcal{C}_K}$ for the group law $\underset{V}{+}$:

$$\underset{V}{-} X = X * B, \quad \text{where} \quad B = V * V = (0, -d, 1).$$

IV. Conjugation of $F_{|\mathcal{C}_K^+}$ to a rotation on the circle via Weierstrass' function $\wp$

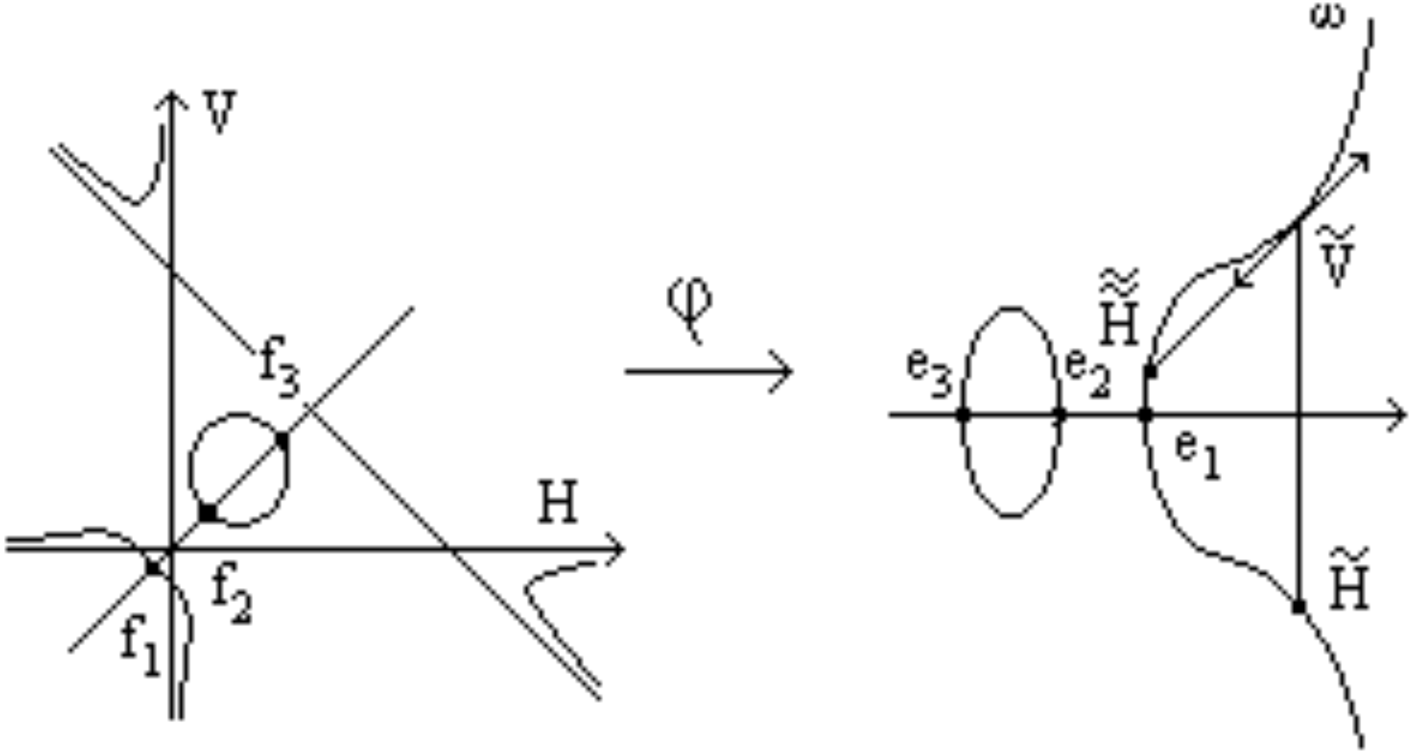
We will transform $\widetilde{\mathcal{C}_K}$ in a standard cubic in normal form. We start with the linear projective transformations $\mathcal{T}_1$:

$$2X = x + y, \quad 2Y = y - x, \quad T = x + y - Kt.$$

Then we make a triple affinity $\mathcal{T}_2$ and a translation $\mathcal{T}_3$ on x. We obtain a new cubic Γ_K in normal form with coefficients depending on K and d

$$Y^2T = 4X^3 - g_2XT^2 - g_3T^3.$$

We put $\phi := \mathcal{T}_3 \circ \mathcal{T}_2 \circ \mathcal{T}_1$, it is a linear projective real transformation of $\widetilde{\mathcal{C}_K}$ onto Γ_K.



We put $\phi(H) := \widetilde{H}$ and $\phi(V) := \widetilde{V}$.

By the linear projective map ϕ, the addition of H on $\widetilde{\mathcal{C}_K}$ for the chord-tangent law $\underset{V}{+}$ with zero element V (that is the map $\widetilde{F}$) is conjugated to the addition of $\widetilde{H}$ on Γ_K for the chord-tangent law $\underset{\widetilde{V}}{+}$ with zero element $\widetilde{V}$.

If ω is the infinite point on Γ_K at the vertical direction, the standard group chord-tangent law $\underset{\omega}{+}$ on Γ_K with ω as zero element is isomorphic to the standard group law on $\mathbb{T}^2$, via the parametrization of Γ_K by the Weierstrass' function $\wp$.

So we will make a supplementary isomorphism on Γ_K. We define a group isomorphism ψ of $(\Gamma_K, \widetilde{V})$ onto (Γ_K, ω) by

$$\psi : \Gamma_K \to \Gamma_K : M \mapsto M \underset{\widetilde{V}}{+} \omega = (M * \omega) * \widetilde{V}.$$

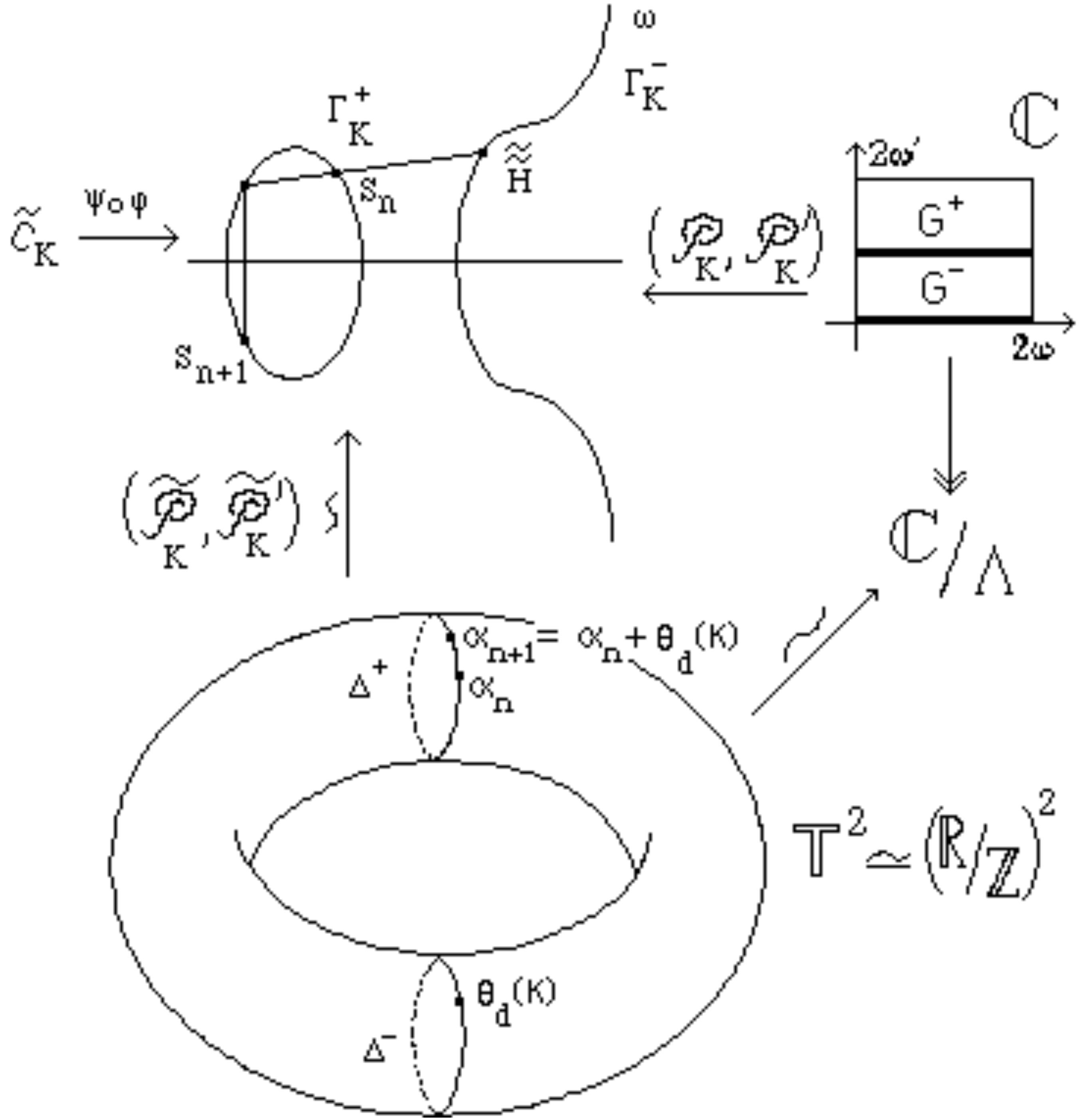
The fact that ψ transforms the addition $\underset{\widetilde{V}}{+}$ in the addition $\underset{\omega}{+}$ is not an obvious fact in general. But in our particular case there is an elementary computer-assisted proof (for example with Maple).

Now, the map $\widetilde{F}$ is conjugated by $\psi \circ \phi$ to the addition of $\widetilde{\widetilde{H}} = \psi(\widetilde{H})$ on (Γ_K, ω)

Γ_K is parametrized by $X = \wp_K(z)$, $Y = \wp'_K(z)$ for $z \in \mathbb{C}$ or in $[0, 2\omega(K)] \times [0, 2\omega'(K)]$, because $\wp_K$ is doubly periodic with the group of periods

$$\Lambda = \{2n\omega(K) + 2im\omega'(K) | (n, m) \in \mathbb{Z}^2\}.$$

We know that Γ_K^+ is parametrized for $z \in G^+ := [0, 2\omega(K)] \times \{i\omega(K)\}$ and that $\Gamma_K^- := \Gamma_K \setminus \Gamma_K^+$ is parametrized for $z \in G^- := [0, 2\omega(K)] \times \{0\}$.


$\widetilde{C}_K$
$\psi \circ \varphi$
Γ_K^+
ω
Γ_K^-
S_n
S_{n+1}
$\widetilde{\widetilde{H}}$
$(\wp_K, \wp'_K)$
$\mathbb{C}$
$2\omega'$
G^+
G^-
2ω
$\mathbb{C}/\Lambda$
$(\widetilde{\wp}_K, \widetilde{\wp}'_K)$ $\wr$
$\alpha_{n+1} = \alpha_n + \theta_d(K)$
α_n
Δ^+
$\sim$
$\theta_d(K)$
Δ^-
$\mathbb{T}^2 \simeq (\mathbb{R}/\mathbb{Z})^2$

So we get the

Theorem 2. *For $d > 0$ and $K \in]K_m, +\infty[$ the restriction of the map F to $\mathcal{C}_K^+$ is conjugated to the rotation on the circle $\mathbb{T}$ with angle $2\pi\theta_d(K) \in]0, \pi[$ given by the following formula :*

$$2\theta_d(K) = \frac{\int_0^{\sqrt{\frac{e_1 - e_3}{\nu}}} \frac{\mathrm{d}u}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}}{\int_0^{+\infty} \frac{\mathrm{d}u}{\sqrt{(1+u^2)(1+\varepsilon u^2)}}},$$

where $X(K)$ is the abscisse of $\widetilde{\widetilde{H}}$ and where one has $\nu := X(K) - e_1 > 0$ and $\varepsilon := \dfrac{e_1 - e_2}{e_1 - e_3}$ (functions of K and d).

IV. The possible periods of periodic solutions of the homographic system

If the rotation number $\theta_d(K)$ is rational, equal to $\frac{p}{q}$ irreducible, then the points $M_0 \in \mathcal{C}_K^+$ are periodic with minimal period q. But if $\theta_d(K)$ is irrational, then the points $M_0 \in \mathcal{C}_K^+$ have a dense orbit in the curve $\mathcal{C}_K^+$. The questions are :

* How are distributed these two types of points in U, for a given d ?

* What are the possible periods of periodic solutions $M_n = (u_n, v_n)$ for a given d ?

Theorem 3. *Let d be positive.*

(1) *It exists a partition of $U \setminus \{L\}$ in two dense sets A_d and B_d, each of them union of invariant curves $\mathcal{C}_K^+$, such that every point in A_d is periodic and every point in B_d has a dense orbit in the positive part of the cubic which passes through it.*

(2) *It exists an integer $N(d)$ such that every integer $q \geq N(d)$ is the minimal period of some solution of the homographic system.*

Proposition 2. *One has* $\lim_{K\to+\infty} \theta_d(K) = \frac{3}{7}$, *and*

$$\theta_m(d) := \lim_{K\to K_m} \theta_d(K) = \frac{1}{\pi}\cos^{-1}\left(\frac{\ell^2-1}{2\ell^2}\right).$$

So we have the inclusion $\mathrm{Im}(\theta_d) \supset < \frac{3}{7}, \theta_m(d) >$, *where* $< a, b > :=]\min(a,b), \max(a,b)[$. *The function* $d \mapsto \theta_m(d)$ *is continuous on* $]0, +\infty[$ *and decreasing from* $\dfrac{1}{2}$ *to* $\dfrac{1}{3}$, $\theta_m(d) = 3/7$ *iff* $d = d_0 := \frac{2\sin(\pi/14)}{[1-2\sin(\pi/14)]^{3/2}} \approx 1.076$. *The map* θ_{d_0} *is non constant and not one-to-one. For each* d *in some open interval* I *containing* d_0 *the map* θ_d *is not one-to-one and not constant.*

Now we make d vary and ask about possible periods for some K and some d.

Theorem 4. *Every integer, except 2, 3, 4, 6, 10, is the minimal period of some solution* (u_n, v_n) *for some* $d > 0$.

The long proof uses three principal ingredients :

* the inclusion $\bigcup_{d>0} \mathrm{Im}(\theta_d) \supset]1/3, 1/2[\setminus \{3/7\}$,

* the prime number theorem,

* the proof that for $K > K_m$ the geometric form of the cubic $\overline{\mathcal{C}_K}$ is those of the previous third figure.

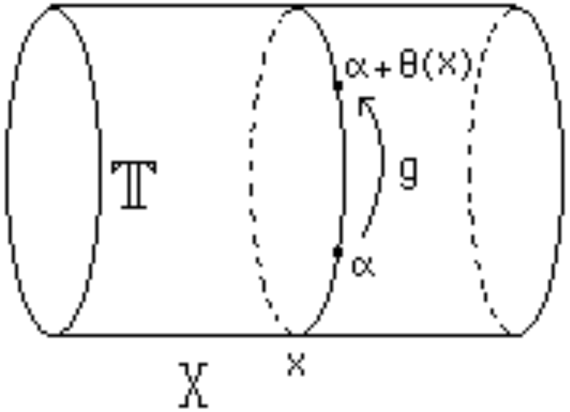
The goal is to find prime numbers $p \in]q/3, q/2[$ wich do not divide q, for q sufficiently large, then to use a computer for q not too large for seeing if some ratio p/q works, and then to study the particular cases by studying the geometrical equation $qH = V$.

V. Chaotic behaviour of the dynamical system (U, F)

Theorem 5. *For every compact set $\mathcal{K} \subset U$ with $L \notin \mathcal{K}$ it exists a number $\delta(\mathcal{K}) > 0$ such that for every point $M \in \mathcal{K}$ and every neighborhood W of M it exists $M' \in W$ such that $\mathrm{dist}(F^n(M), F^n(M') \geq \delta(\mathcal{K})$ for infinitely many integers n.*

This uses the the continuity of the map $K \mapsto \wp_K \ :]K_m, +\infty[\to \mathcal{C}\big(G^+, U\big)$ for the uniform norm (with tedious calculations), and the following probably known result.

Proposition 3. *Let X be a metric space. Let be also $\theta : X \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ a continuous map such that for every non-empty open set U, the set $\theta(U)$ contains a non-empty open set. Define the map $g : X \times \mathbb{T} \to X \times \mathbb{T} : (x, \alpha) \to (x, \alpha + \theta(x))$.*



Then the dynamical system $(X \times \mathbb{T}, g)$ has δ-sensitiveness to initial conditions for every $\delta \in]0, 1/2[$.

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