Translation arcs and Lyapunov stability in two dimensions

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$$x_{n+1} = h(x_n), \quad x_n \in \mathbb{R}^2$$

$h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ continuous + one-to-one

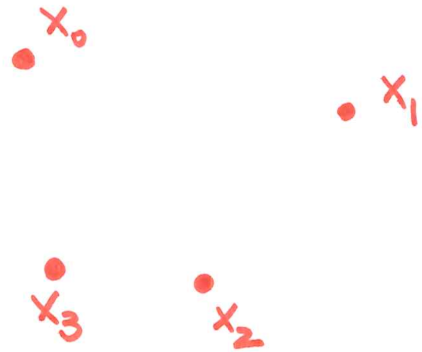
+ orientation-preserving

$$h \in \mathcal{E}_+$$

$$\left[\; h(\mathbb{R}^2) = \mathbb{R}^2 : \quad h \in \mathcal{H}_+ \;\right]$$

$$x_{n+1} = h(x_n), \quad x_n \in \mathbb{R}^2$$

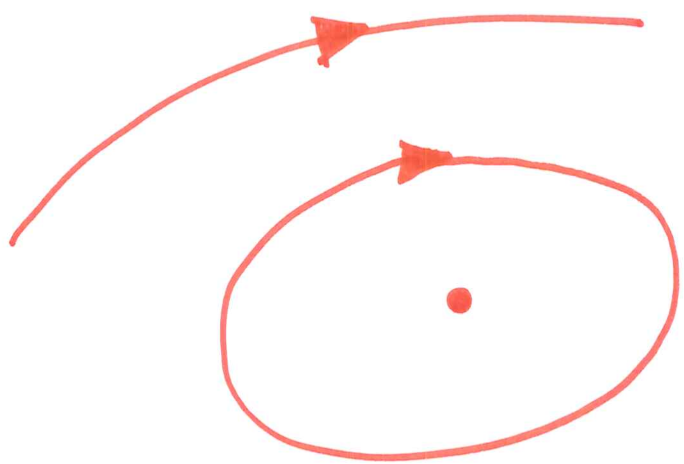
$$h \in \mathcal{E}_+$$



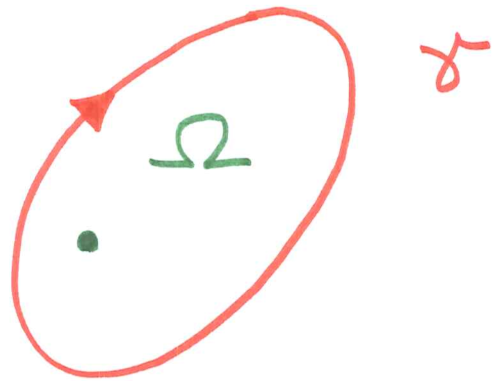
$$\dot{x} = X(x), \quad x \in \mathbb{R}^2$$

$X : \mathbb{R}^2 \to \mathbb{R}^2$ continuous

+ uniqueness i.v.p.



The index of a closed orbit



γ closed orbit of $\dot{x} = X(x)$

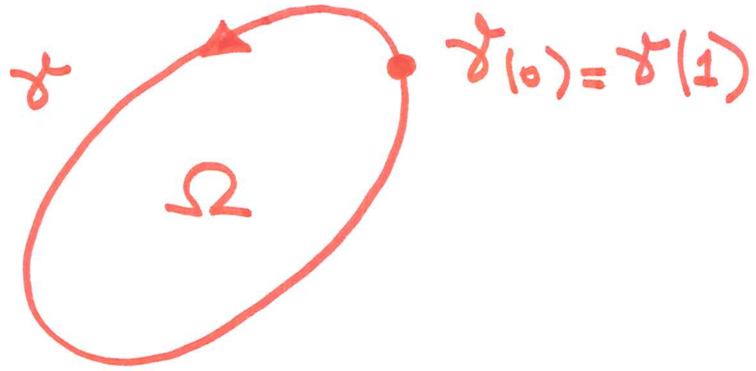
$\Longrightarrow$

The index of X around γ $= 1$

[index $\neq 0 \Rightarrow \exists$ equilibrium on Ω]

Index / rotation number / Brouwer degree ...

Topological degree



$f: \overline{\Omega} \to \mathbb{R}^2$ continuous

$f(x) \neq 0 \quad \forall x \in \gamma = \partial\Omega$

$\gamma = \gamma(t)$, $t \in [0,1]$, parameterization

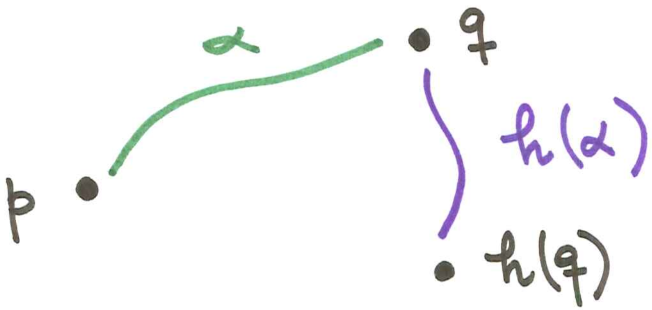
$$f(\gamma(t)) = r(t)\left(\cos\theta(t), \sin\theta(t)\right)$$

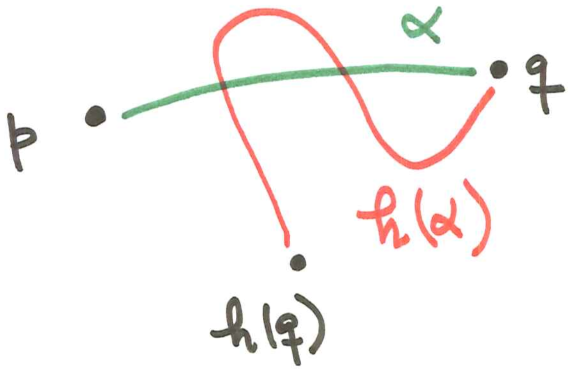
$$d[f, \Omega, 0] = \frac{1}{2\pi}\left(\theta(1) - \theta(0)\right)$$

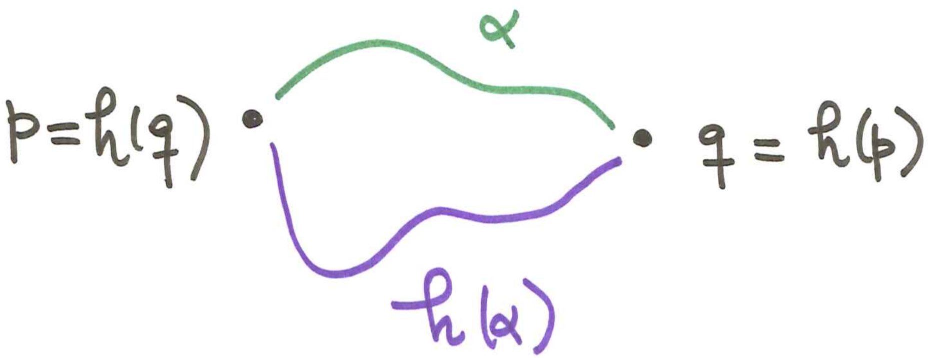
Translation arcs

$$x_{n+1} = h(x_n)$$

$$\alpha = \widehat{pq}\,, \quad h(p) = q\,, \quad h(\alpha \setminus \{p\}) \cap (\alpha \setminus \{p\}) = \phi$$

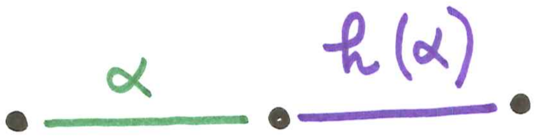


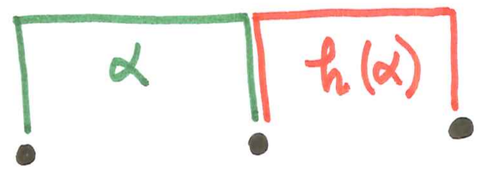




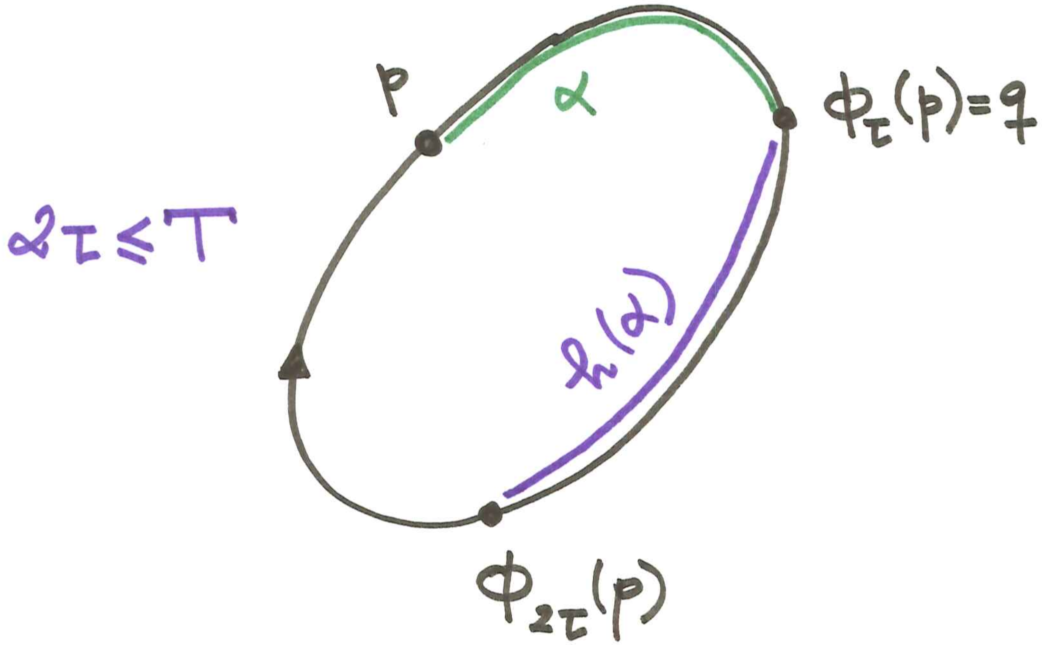
Examples:

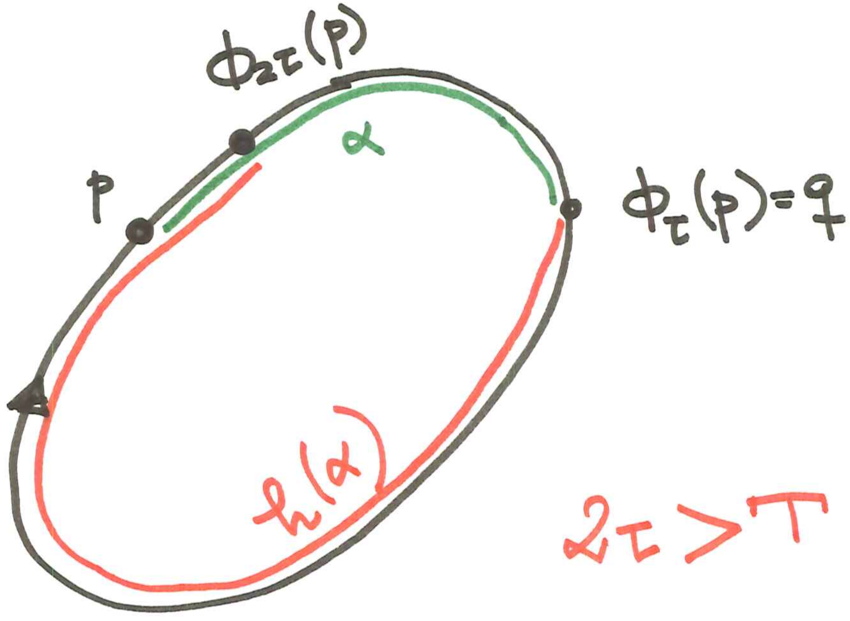
1. $h(x,y) = (x+1, y)$

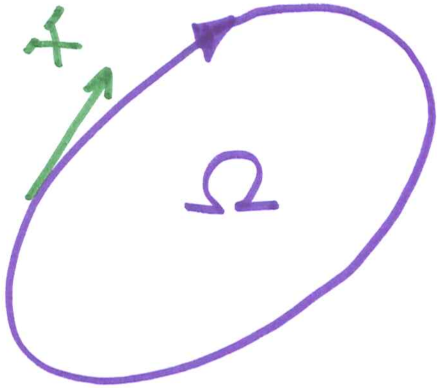




2. Flow $\{\phi_t\}_{t\in\mathbb{R}}$, $\tau > 0$, $h = \phi_\tau$



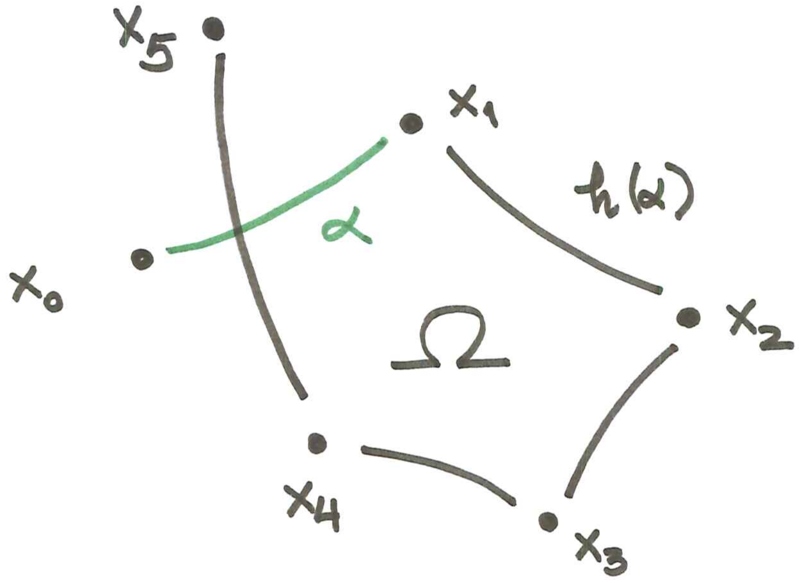




γ closed orbit

X vector field

$d[X, \Omega, 0]$



α translation arc

$h^4(\alpha) \cap \alpha \neq \phi$

$f := id - h$

$d[f, \Omega, 0]$

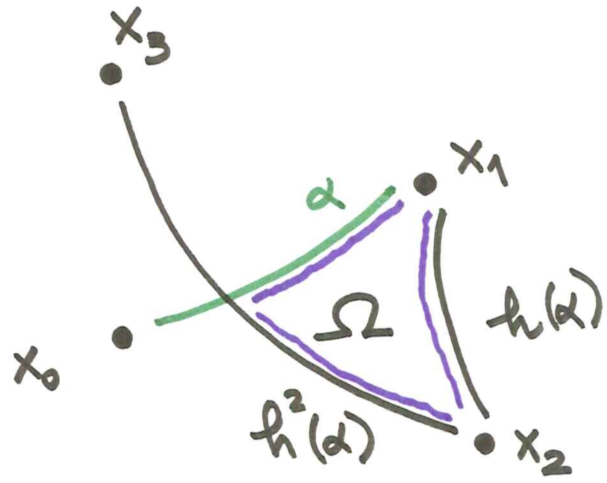
Brouwer's Lemma on translations arcs

$h \in \mathcal{E}_+$, α translation arc

$\exists n \geq 2 :$ $h^n(\alpha) \cap \alpha \neq \emptyset$

$\Rightarrow \exists \underset{\sim}{\gamma}$ Jordan curve $\subset$

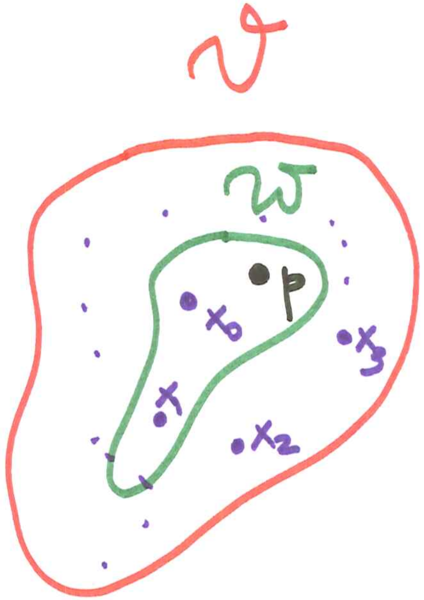
$\alpha \cup h(\alpha) \cup \ldots \cup h^n(\alpha)$



$d[\mathrm{id}-h, \Omega, 0] = 1$

Proofs: Fathi, Franks, M. Brown

Index of stable fixed points



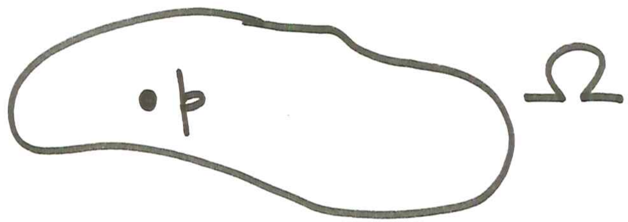
$h : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$, $\mathcal{U}$ open, $p \in \mathcal{U}$

$p = h(p)$ Lyapunov stable

$\forall \mathcal{V}(p)\ \exists \mathcal{W}(p) : h^n(\mathcal{W}) \subseteq \mathcal{V}$ if $n \geq 0$

Th. (Dancer + O., 1994)

$h \in \mathcal{E}_+$, p stable fixed point



$\Longrightarrow$

$d[id - h, \Omega, 0] = 1$ if $Fix(h) \cap \overline{\Omega} = \{p\}$

Instability criteria

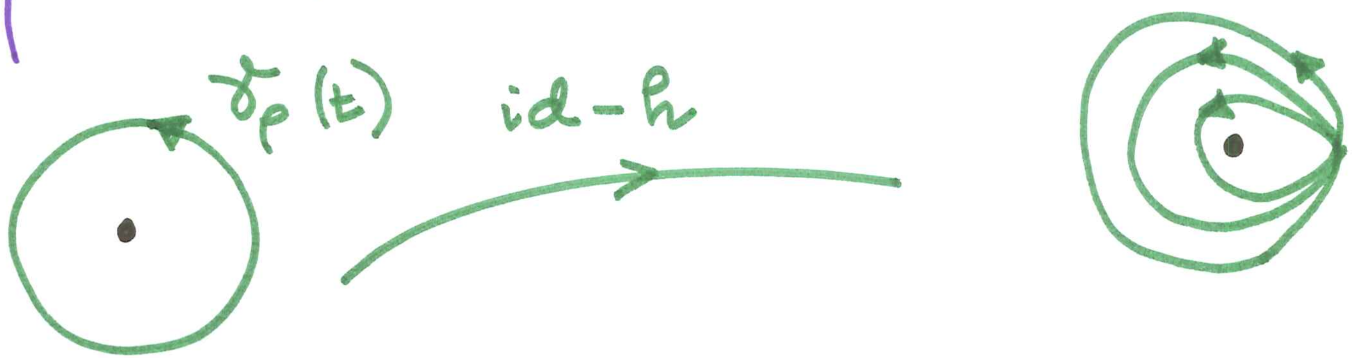
$$z_{n+1} = z_n + z_n^3 + R(z_n, \bar{z}_n)$$

$$R(z, \bar{z}) = o(|z|^3) \text{ as } z \to 0$$

$z = 0$ is unstable

$$(id - h)(z, \bar{z}) = z^3 + R(z, \bar{z})$$

$$\gamma_\rho(t) = \rho e^{2\pi i t}, \quad t \in [0, 1]$$



$$d[id - h, \Omega_\rho, 0] = 3$$

Persistence of stable fixed points

$h \in \mathcal{H}_+$

$\Lambda \subset \mathbb{R}^2$ compact, $h(\Lambda) = \Lambda$

Λ is **persistent** if $\forall \varepsilon > 0 \ \exists \delta > 0 : \tilde{h} \in \mathcal{H}_+, \|h - \tilde{h}\|_\infty \leq \delta$

$\Rightarrow \exists \tilde{\Lambda} \subset \mathbb{R}^2$ compact, $\tilde{h}(\tilde{\Lambda}) = \tilde{\Lambda}$

$$D_H(\Lambda, \tilde{\Lambda}) \leq \varepsilon$$

Corollary $h \in \mathcal{H}_+$, $p = h(p)$ stable + isolated $\Rightarrow$

$\Lambda = \{p\}$ is persistent

True if $h \in \mathcal{H}_-$ [Ruiz del Portal]

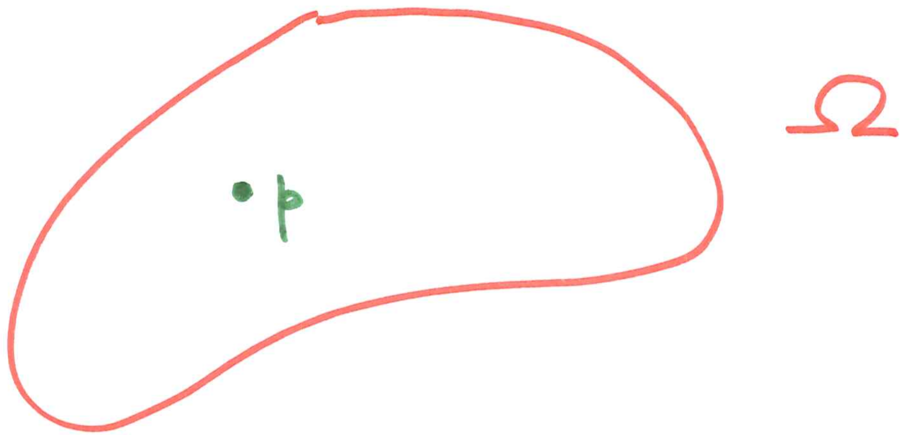
False in dimension 3 [Krasnoselskii & Zabreiko

Bonatti & Villadelprat]

Proof of Theorem:

$h \in \mathcal{E}_+$, $p = h(p)$ stable $\Longrightarrow$

$$d[id - h, \Omega, 0] = 1 \quad \text{if} \quad \mathrm{Fix}(h) \cap \overline{\Omega} = \{p\}$$



Preliminary facts (any dimension)

① Th. valid if p asymptotically stable

[Browder, Krasnoselskii]

② p not asymptotically stable $\Longrightarrow$

$\forall\, \mathcal{U}(p) \quad \exists\, q \in \mathcal{U},\ q \neq p,\ q$ recurrent

$$q \in L_\omega(q)$$

Preliminary facts (dimension 2)

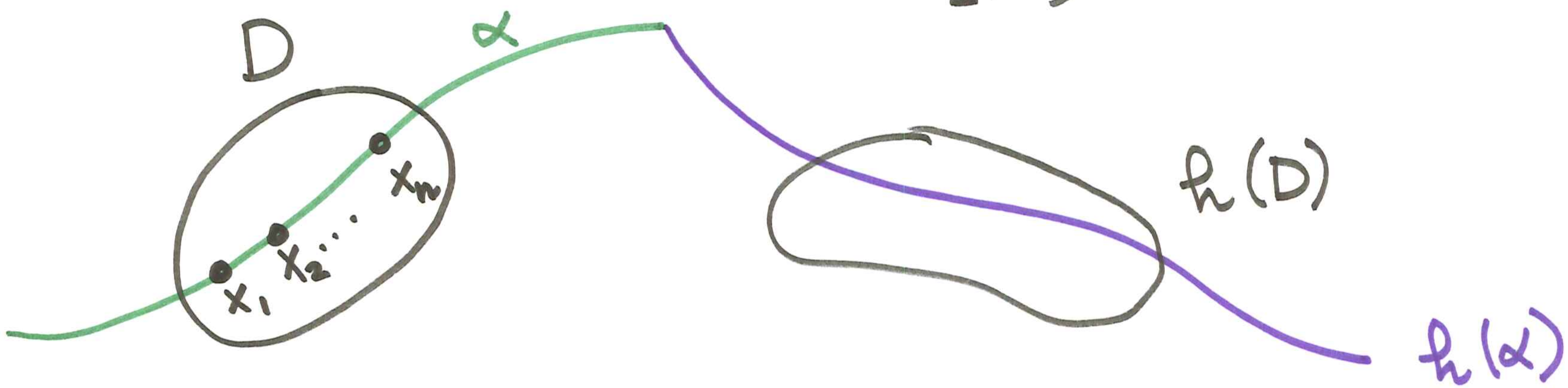
① $\exists\, W$ open + simply connected, $h(W) \subseteq W$

$\mathrm{Fix}(h) \cap W = \{p\}$ [Siegel & Moser]

② $h \in \mathcal{E}_+$, $D \cap h(D) = \phi$

$x_1, \dots, x_n \in D \Longrightarrow \exists$ translation arc α :

$x_1, \dots, x_n \in \alpha$



Proof: p stable not asymptotically

$\mathcal{W}$ open + simply connected, $h(\mathcal{W}) \subseteq \mathcal{W}$, $\mathrm{Fix}(h) \cap \mathcal{W} = \{p\}$

$$\begin{tikzcd} \mathcal{W} \arrow[r, "h"] \arrow[d, phantom, "\cong" rotate=-90] & \mathcal{W} \arrow[d, phantom, "\cong" rotate=-90] \\ \mathbb{R}^2 \arrow[r, dashed, "H"'] & \mathbb{R}^2 \end{tikzcd}$$

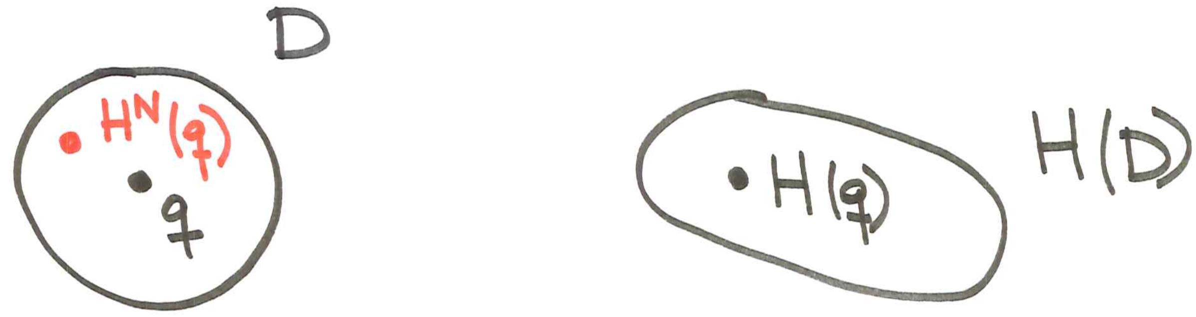
$H \in \mathcal{E}_+$, $\mathrm{Fix}(H) = \{p\}$

$\exists q \neq p$, $q \in L_\omega(q, H)$

recurrent

1

2 $\exists N \geq 2$:



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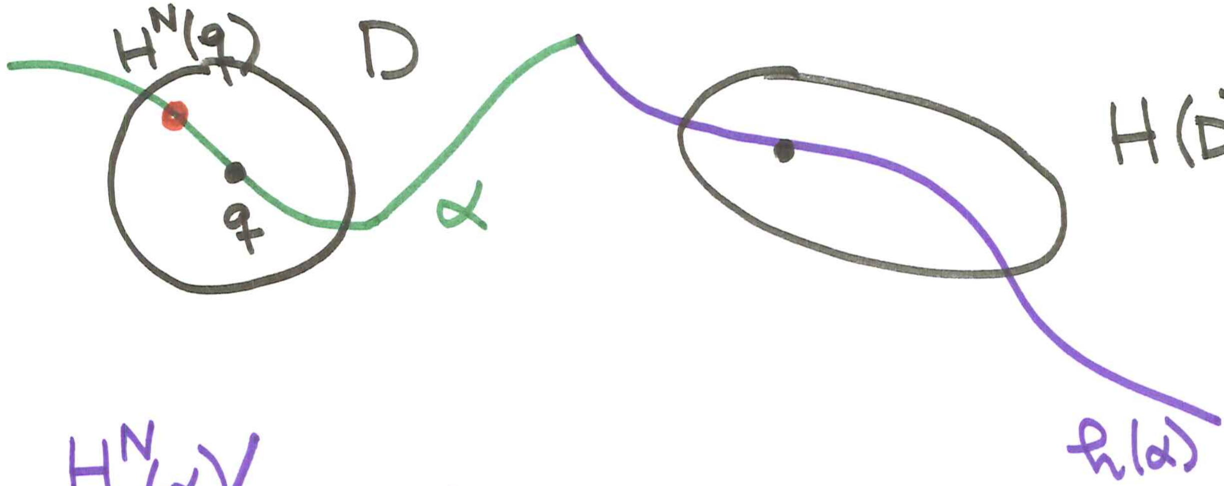

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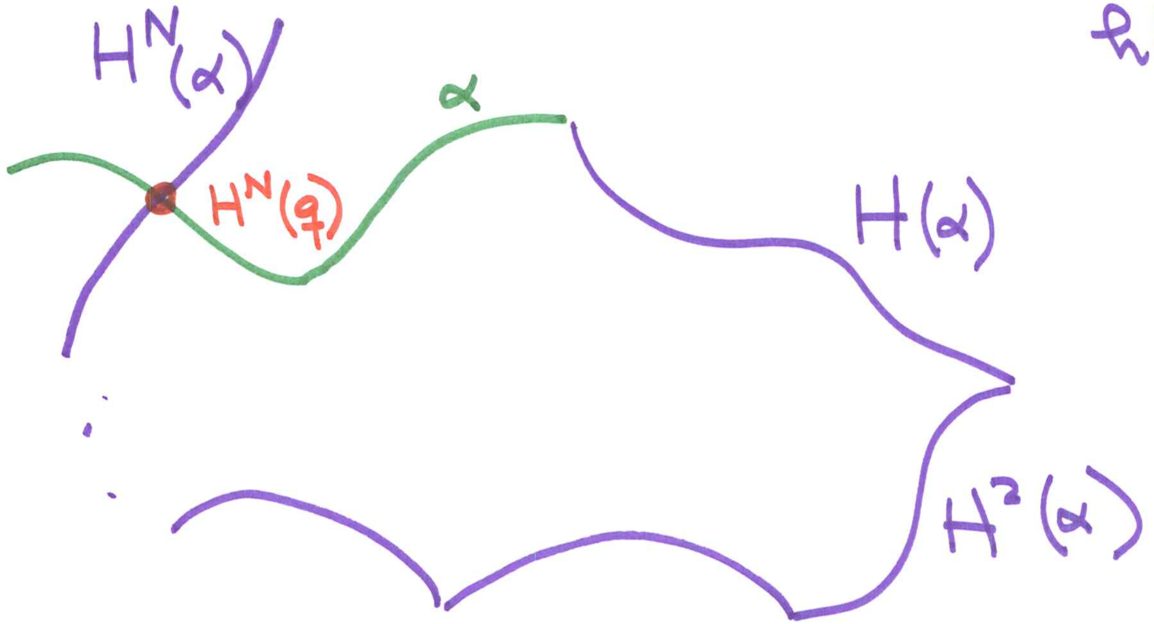
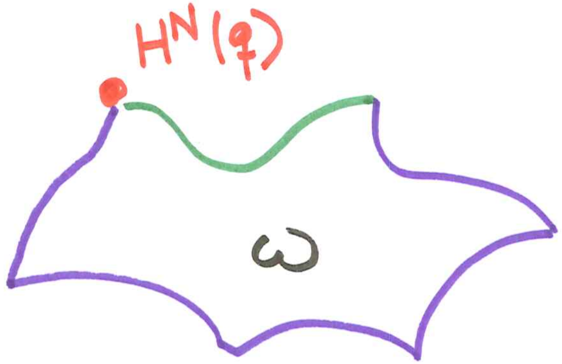


Brouwer's Lemma



$$d[\mathrm{id}-H,\omega,0]=1$$

$$\mathrm{Fix}(H)=\{p\} \Longrightarrow d[\mathrm{id}-H,\omega,0]$$

$$= d[\mathrm{id}-h,\Omega,0]$$

Excision + Multiplication Th.

Stable Cantor Sets

$h : \mathbb{R}^2 \to \mathbb{R}^2$ homeomorphism

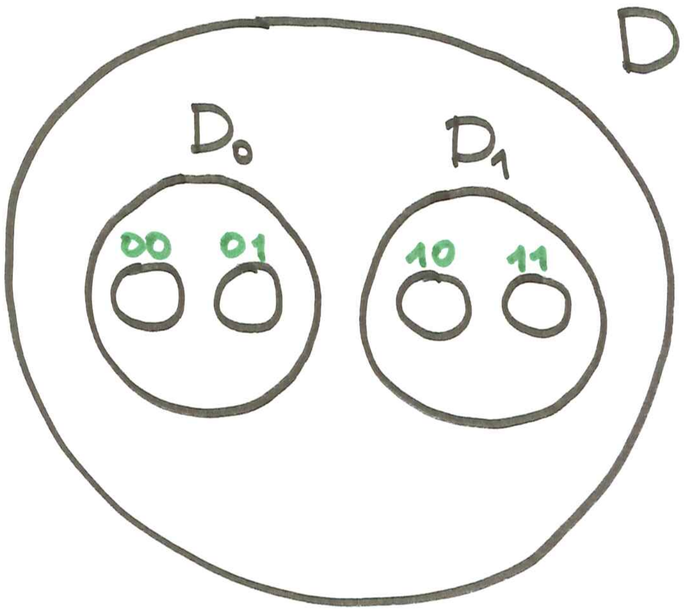
$\Lambda \subset \mathbb{R}^2$ Cantor set, $h(\Lambda) = \Lambda$

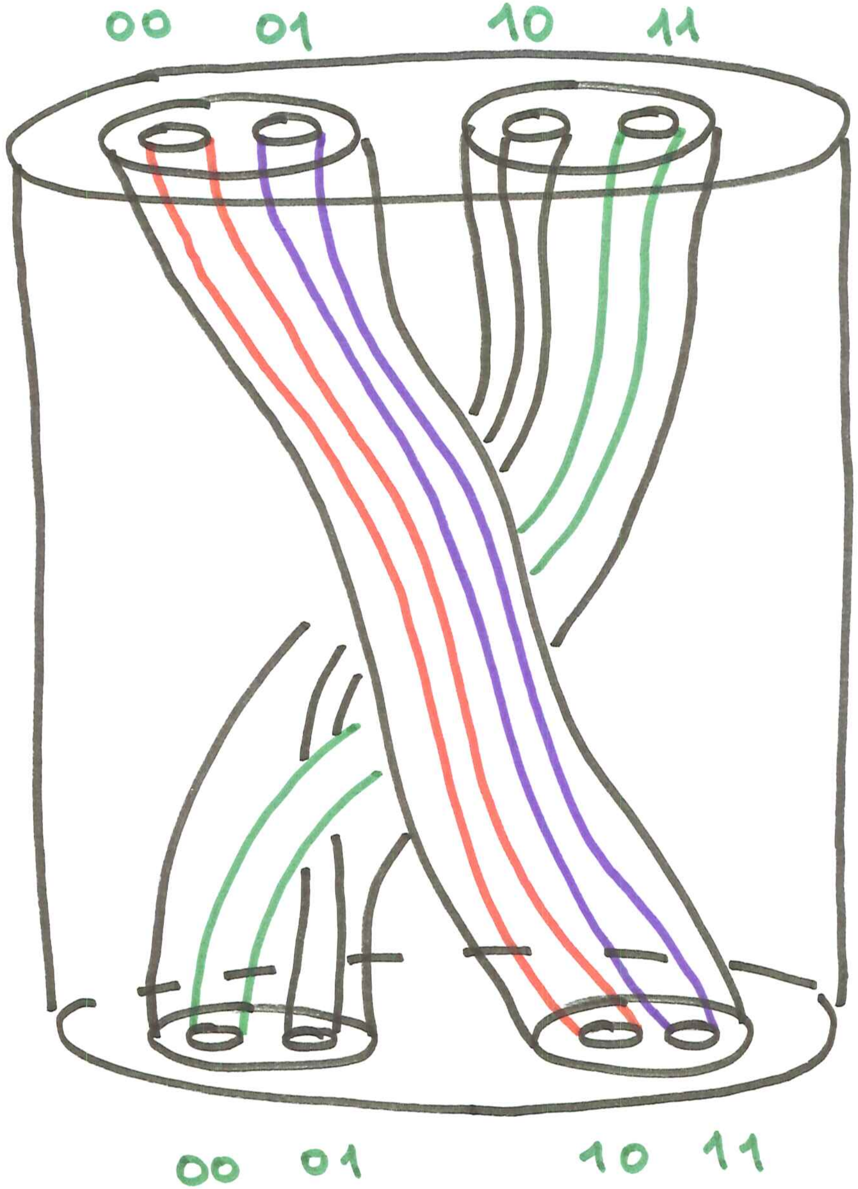
Λ transitive $\left(\exists p \in \Lambda : L_\omega(p) = \Lambda \right)$

Λ is <u>stable</u> if

$\forall \mathcal{U}(\Lambda) \ \exists \mathcal{V}(\Lambda) : \ h^n(\mathcal{V}) \subseteq \mathcal{U} \ \forall n \geqslant 0$

Example:





D

$D_0 \cup D_1$

$D_{00} \cup D_{01} \cup D_{10} \cup D_{11}$

...

Λ Cantor Set

Bell & Meyer (1995)

Λ Cantor set + transitive + stable $\Longrightarrow$

Λ can be approximated by periodic points

Proof: Buescu & Stewart

Λ Cantor + transitive + stable $\Rightarrow$ Λ minimal

Cartwright + Littlewood $K \subset \mathbb{R}^2$ continuum

$\mathbb{R}^2 \setminus K$ connected, $H \in \mathcal{H}_+$, $H(K)=K$

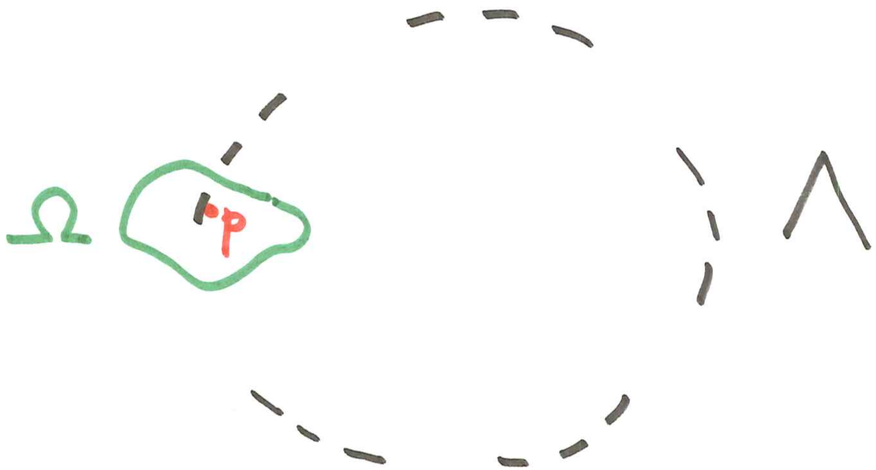
$\Longrightarrow$ $\exists$ fixed point on K

O. + Ruiz-Herrera (2012) $\quad h \in \mathcal{H}$

Λ Cantor set + transitive + stable $\Longrightarrow$

$\forall \delta > 0 \ \forall p \in \Lambda \quad \exists\, \Omega,\ p \in \Omega,\ \exists N \geq 1$

$D_H(\{p\}, \Omega) \leq \delta, \quad d[id - h^N, \Omega, 0] = 1$



Corollary 1 Bell & Meyer

Corollary 2 $\nexists$ Cantor + transitive + stable $\Longrightarrow$

$\nexists$ persistent

Proof $\|h - \tilde{h}\|_\infty$ small, $d[id - h^N, \Omega, 0]$

$$= d[id - \tilde{h}^N, \Omega, 0]$$

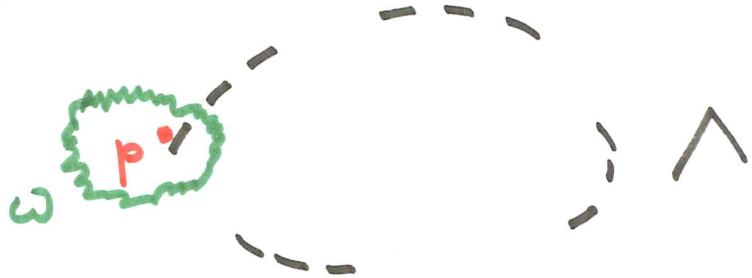
False in 3d

Proof: $p \in \Lambda$, $\delta > 0$

① Λ minimal + stable $\Longrightarrow$

$\exists\, \omega \subset \mathbb{R}^2$ open + simply connected + small /

$$h^N(\omega) \subseteq \omega$$



② $\exists$ recurrent point on ω for h^{2N}

③

$$\begin{tikzcd} \omega \arrow[r, "h^{2N}"] \arrow[d, phantom, "\cong" {rotate=90}] & \omega \arrow[d, phantom, "\cong" {rotate=90}] \\ \mathbb{R}^2 \arrow[r, dashed] & \mathbb{R}^2 \end{tikzcd}$$

Brouwer's Lemma