RANDOM HOMEOMORPHISMS OF AN INTERVAL

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Introduction:

We investigate the properties of the systems of randomly applied orientation preserving homeomorphisms of the compact interval $[0, 1]$. Such a system can be considered as a skew product with a mixed topological-measure structure. In the base we do not need any topology (although sometimes we have it), but we assume that we have there an ergodic measure preserving transformation of a probability space. In the fiber, which is an interval, we have orientation preserving homeomorphisms, depending in a measurable way on the point in the base.

We are interested in the existence of almost global attractors which are graphs of measurable functions from the base to the fiber. When we speak of an attractor, we mean a set towards which almost all orbits converge, and the convergence is considered fiberwise (only in the direction of a fiber). This agrees with the philosophy saying that the phase space is really only the fiber space (here, the interval).

Those systems and their attractors can be looked upon from various points of view (random systems, Strange Nonchaotic Attractors, Iterated Function Systems, nonautonomous systems, etc.).

Our main result is a detailed description of the behavior of a certain one-parameter family of piecewise linear random homeomorphisms. However, we precede it with some general results, which can be applied to very general random systems of interval homeomorphisms.

Boundaries of basins of attraction:

Let us start with a very general situation. Let Ω be some space (later there will be an invariant measure on it), $R:\Omega\to\Omega$ a map, $I=[0,1]$, $G:\Omega\times I\to\Omega\times I$ a skew product: $G(\vartheta,x)=(R(\vartheta),g_\vartheta(x))$, and let π_2 be the projection from $\Omega\times I$ to I. We assume that each g_ϑ is an orientation preserving homeomorphism of I onto itself.

The question is: if the level 0 set $\Omega\times\{0\}$ is an attractor, what can we say about the boundary of the basin of attraction?

We take the approach of Bonifant and Milnor [BM].

Let $\varphi_{n,m}(\vartheta)$ be the unique number such that

$$G^n(\vartheta,\varphi_{n,m}(\vartheta))=\left(R^n(\vartheta),\frac{1}{m}\right).$$

This defines the function $\varphi_{n,m}:\Omega\to I$. Then we define a function $\varphi:\Omega\to I$ by

$$\varphi(\vartheta)=\lim_{m\to\infty}\lim_{N\to\infty}\inf_{n\geq N}\varphi_{n,m}(\vartheta). \tag{1}$$

Note that $\inf_{n\geq N}\varphi_{n,m}(\vartheta)$ is increasing in N and decreasing in m, so the limits above exist.

Lemma 1. *If $x<\varphi(\vartheta)$ then*

$$\lim_{n\to\infty}\pi_2(G^n(\vartheta,x))=0. \tag{2}$$

If $x>\varphi(\vartheta)$ then (2) does not hold.

Lemma 2. *For a given $\vartheta\in\Omega$ assume that there exists $\eta>0$ and λ_n $(n=0,1,2,\dots)$ such that*

$$g_{R^n(\vartheta)}(x)\leq\lambda_n x$$

for all n and $x<\eta$, and

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log\lambda_k<0.$$

Then $\varphi(\vartheta)>0$.

Let us now assume additionally that Ω is equipped with an R-invariant ergodic probability measure μ, the maps g_ϑ depend on ϑ in a measurable way and they are all differentiable at 0. Let Λ be the exponent at level 0, that is,

$$\Lambda=\int_\Omega g'_\vartheta(0)\,d\mu(\vartheta).$$

By the Birkhoff Ergodic Theorem, for almost every ϑ we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log g'_{R^k(\vartheta)}(0)=\Lambda. \tag{3}$$

Theorem 3. *Assume that $\Lambda < 0$ and that at least one of the following assumptions is satisfied:*

(i) the set $\{g_\vartheta : \vartheta \in \Omega\}$ is finite,

(ii) all functions g_ϑ are concave,

(iii) all functions g_ϑ are twice differentiable and there exists a constant C such that $g''_\vartheta(x)/g'_\vartheta(x) \le C$ for all ϑ, x.

Then there exists a measurable function $\varphi : \Omega \to I$, positive almost everywhere, such that for every $\vartheta \in \Omega$ (2) holds if $x < \varphi(\vartheta)$ and does not hold if $x > \varphi(\vartheta)$.

Remark 4. *If (2) holds if $x < \varphi(\vartheta)$ and does not hold if $x > \varphi(\vartheta)$, then it is easy to see that the graph of φ is G-invariant.*

Two directions of time:

Let us consider a skew product like above, under an additional assumption that the map in the base is invertible. Then we can investigate what happens when the time goes to $+\infty$ and what happens when it goes to $-\infty$. To be in agreement with the theory of Strange Nonchaotic Attractors, we will think of the phenomena from the preceding section as occurring as the time goes to $-\infty$. Thus, we need new notation.

As before Ω is a space with a probability measure μ. Now, $S : \Omega \to \Omega$ is an invertible measurable map (with S^{-1} also measurable), for which μ is invariant and ergodic. The map $F : \Omega \times I \to \Omega \times I$ a skew product, given by $F(\vartheta, x) = (S(\vartheta), f_\vartheta(x))$, and each f_ϑ is an orientation preserving homeomorphism of I onto itself.

We assume that the maps f_ϑ are differentiable at 0 and 1, and define

$$\Lambda_0 = \int_\Omega f'_\vartheta(0)\, d\mu(\vartheta), \qquad \Lambda_1 = \int_\Omega f'_\vartheta(1)\, d\mu(\vartheta).$$

If both Λ_0 and Λ_1 are positive, then as the time goes to $-\infty$, the levels 0 and 1 are attracting. In many cases we can use Theorem 3 to conclude that their basins of attraction are nontrivial. However, there is no guarantee that the boundaries of those basins coincide. For this we need some kind of contraction in the fibers as the time goes to $+\infty$. Since the fiber maps are homeomorphisms, we cannot get contractions on closed intervals $[0, 1]$. However, sometimes there is a kind of contraction on the open intervals $(0, 1)$. One example of such a situation is given in the paper [BM]. There all maps f_ϑ have positive Schwarzian derivative. Later we will give a completely different example with two piecewise linear maps. However, there is no standard method of proving forward contraction for homeomorphisms. Therefore in our general theorem that follows, we make it one of the assumptions. In particular, we will use the following terminology, independently whether S is invertible or not.

Definition 5. *The skew product $F : \Omega \times I \to \Omega \times I$ is* essentially contracting *if for almost all $\vartheta \in \Omega$ and all $x, y \in (0, 1)$, the distance*

$$|\pi_2(F^n(\vartheta, x)) - \pi_2(F^n(\vartheta, y))|$$

goes to 0 as $n \to \infty$.

If $\psi : \omega \to I$ is a measurable function, then we define the measure μ_ψ, concentrated on the graph of ψ, as the lifting of the measure μ, that is,

$$\mu_\psi(A) = \mu\{\vartheta \in \Omega : (\vartheta, \psi(\vartheta)) \in A\}.$$

Theorem 6. *For a skew product F as above, assume that*

(I) $\Lambda_0, \Lambda_1 > 0$,

(II) either the set $\{f_\vartheta : \vartheta \in \Omega\}$ is finite, or all f_ϑ are diffeomorphisms of class C^2 with $|f''_\vartheta|/(f'_\vartheta)^2$ bounded uniformly in ϑ and x,

(III) F is essentially contracting.

Then there exists a measurable function $\varphi : \Omega \to (0,1)$ with the following properties:

(a) for almost every $\vartheta \in \Omega$, if $x < \varphi(\vartheta)$ then

$$\lim_{n\to\infty} \pi_2(F^{-n}(\vartheta, x)) = 0 \tag{4}$$

and if $x > \varphi(\vartheta)$ then

$$\lim_{n\to\infty} \pi_2(F^{-n}(\vartheta, x)) = 1, \tag{5}$$

(b) the graph of φ is F-invariant,

(c) for almost every $\vartheta \in \Omega$ and every $x \in (0,1)$,

$$\lim_{n\to\infty} |\pi_2(F^n(\vartheta, x)) - \varphi(S^n(\vartheta))| = 0,$$

(d) for almost every $\vartheta \in \Omega$ and for every compact set $A \subset (0,1)$ and $\varepsilon > 0$ there exists N such that for every $n \geq N$

$$\pi_2(F^n(\{S^{-n}(\vartheta)\} \times A) \subset (\varphi(\vartheta) - \varepsilon, \varphi(\vartheta) + \varepsilon). \tag{6}$$

(e) if Ω is a metric compact space and F is continuous, then for almost every $\vartheta \in \Omega$ and every $x \in (0,1)$, the measures

$$\frac{1}{n} \sum_{k=0}^{n-1} F^k_*(\delta_{(\vartheta,x)})$$

converge (as $n \to \infty$) in the weak-$*$ topology to the measure μ_φ.

The next theorem holds whether S (and therefore, F) is invertible or not. We assume in it that there is topology in Ω in which μ is a Borel measure.

Theorem 7. *Assume that F is essentially contracting. Then there is at most one ergodic probability measure invariant for F that projects to μ under $(\pi_1)_*$ and such that the measure of $\Omega \times \{0,1\}$ is 0.*

Bernoulli shift in the base:

Let us assume now that (S, Ω, μ) is a Bernoulli shift on a finite alphabet. We can consider a two-sided shift (σ, Σ, μ) or a two-sided one $(\sigma_+, \Sigma_+, \mu_+)$. We will write the points of Σ and Σ_+ as $\underline{\omega} = (\omega_n)_{n=-\infty}^{\infty}$ or $\underline{\omega} = (\omega_n)_{n=0}^{\infty}$ respectively. We will also assume that the maps $f_{\underline{\omega}}$ depend only on ω_0 (so there are only finitely many of them). The interpretation is that we are choosing those map randomly and independently each time.

There is a natural projection $P : \Sigma \to \Sigma_+$. It is a semiconjugacy and it sends the measure μ to μ_+.

In this context, let us look closer at the definition of the function φ.

Lemma 8. *If* $\underline{\omega} = (\omega_n)_{n=-\infty}^{\infty}$, *then* $\varphi(\underline{\omega})$ *depends only on* ω_n *with* $n < 0$.

Theorem 9. *There exists a probability measure* ν *on* $(0, 1)$ *such that*

$$(P \times \mathrm{id}_I)_*(\mu_\varphi) = \mu_+ \times \nu.$$

Piecewise linear homeomorphisms:

Now we consider a one-parameter family of random homeomorphisms of an interval, for which we can prove that the theory presented above applies.

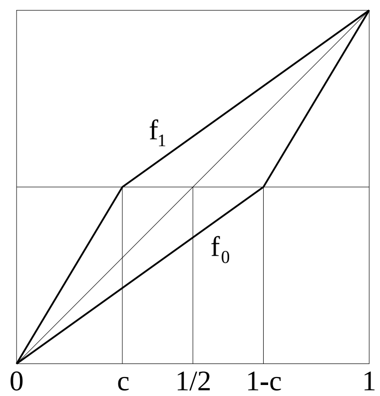
The system in the base will be the Bernoulli shift with probabilities $(1/2, 1/2)$. The corresponding interval homeomorphisms, $f_0, f_1 : I \to I$ will be piecewise liner with two pieces. Additionally, their graphs will be symmetric with respect to $(1/2, 1/2)$, that is, $f_1(x) = 1 - f_0(1 - x)$. For each map the point at which it is not linear can be considered as a critical point. As always, the situation is simpler if there is only one critical value, and by the symmetry, this common critical value has to be $1/2$. Since our maps are orientation preserving homeomorphisms, we have $f_0(0) = f_1(0) = 0$ and $f_0(1) = f_1(1) = 1$.

These conditions determine a one-parameter family of pairs of maps

$$f_0(x) = \begin{cases} ax & \text{if } 0 \le x \le 1 - c, \\ 1 - b(1 - x) & \text{if } 1 - c \le x \le 1, \end{cases}$$

$$f_1(x) = \begin{cases} bx & \text{if } 0 \le x \le c, \\ 1 - a(1 - x) & \text{if } c \le x \le 1. \end{cases}$$

where $a = \frac{1}{2(1-c)}$, $b = \frac{1}{2c}$, and $0 < c < 1/2$. Observe that the harmonic mean of the slopes a and b is 1, and that $0 < a < 1 < b$.



We will apply f_j, $j=0,1$, when the 0-th coordinate of $\underline{\omega}\in\Sigma$ (or in Σ_+) is j. That is, we consider skew products $F:\Sigma\times I\to\Sigma\times I$ given by $F(\underline{\omega},x)=(\sigma(\underline{\omega}),f_{\omega_0}(x))$, where $\underline{\omega}=(\omega_n)_{n=-\infty}^{\infty}$, and $F_+:\Sigma_+\times I\to\Sigma_+\times I$ given by $F_+(\underline{\omega},x)=(\sigma_+(\underline{\omega}),f_{\omega_0}(x))$, where $\underline{\omega}=(\omega_n)_{n=0}^{\infty}$.

We want to apply Theorem 6. Therefore we need to check that its assumptions are satisfied by F. Assumption (I) is satisfied because $ab=\frac{1}{4c(1-c)}>1$. Assumption (II) is satisfied because there are only 2 maps f_ϑ. Thus, we have to prove that F is essentially contracting. As we mentioned earlier, this is a nontrivial thing to do.

The main idea is to find a homeomorphism from $(0,1)$ to $\mathbb{R}$ such that in the new metric in $(0,1)$, which we get by transporting back the natural metric from $\mathbb{R}$, both maps f_0 and f_1 are contractions. In fact, they will be very weak contractions (on the most of the space they will be isometries), so we need more work in order to prove that F is essentially contracting.

Fix $\underline{\omega}\in\Sigma$. For $x_0\in[0,1]$ we will write $x_n=\pi_2(F^n(\underline{\omega},x_0))$. Set

$$\Gamma=\left\{\underline{\omega}\in\Sigma:\lim_{n\to\infty}\frac{1}{n}\#\{k\in\{0,1,\dots,n-1\}:\omega_k=0\}=\frac{1}{2}\right\}.$$

By the Birkhoff Ergodic Theorem, $\mu(\Gamma)=1$.

Theorem 10. *Let $\underline{\omega}\in\Gamma$ and let $x_0,y_0\in(0,1)$. Then* $\lim_{n\to\infty}|x_n-y_n|=0$.

Measures:

We continue to investigate F and F_+, this time from the point of view of invariant measures. The relevant invariant measures for F and F_+ are those that project to μ and μ_+. There are two trivial ergodic ones: $\mu\times\delta_0$ and $\mu\times\delta_1$ (in the one-sided case, $\mu_+\times\delta_0$ and $\mu_+\times\delta_1$).

By Theorem 7, there is at most one nontrivial measure of this type. Such measure for F is μ_φ, which appears in Theorem 6 (e). It is clear that the projection from $\Sigma\times I$ to the first coordinate is an isomorphism of the systems $(\Sigma\times I,F,\mu_\varphi)$ and (Σ,σ,μ). In particular, this shows that μ_φ is ergodic for F.

Now we consider F_+. Here the situation is completely different. Denote the Lebesgue measure on I by λ. The following theorem can be interpreted as the Lebesgue measure being invariant for our random system of maps. The proof is straightforward and specific for our family.

Theorem 11. *The measure $\mu_+\times\lambda$ is invariant for F_+.*

Theorem 12. *For almost every $x\in I$ the preimage $\varphi^{-1}(x)$ is dense in Σ. In particular, the graph of φ is dense in $\Sigma\times I$.*

Two-sided vs. one sided case:

By Theorems 6 (c) and 10, the map F has a fiberwise attractor which is a graph of a measurable invariant function from the base to the fiber space. We will show that this is not the case if we consider F_+, even if we skip the assumption of invariance.

Theorem 13. *There is no measurable function* $\varphi_+ : \Sigma_+ \to (0,1)$ *whose graph is an attractor for* F_+ *in the sense that for almost every* $\underline{\omega} \in \Sigma_+$ *and every* $x_0 \in (0,1)$ *we have*

$$\lim_{n\to\infty} |x_n - \varphi_+(\sigma_+^n(\underline{\omega}))| = 0.$$

In such a way we get an excellent illustration of the *Mystery of the Vanishing Attractor*. For an invertible system an attractor exists, but it vanishes when we pass to the noninvertible system. This happens in spite of the fact that in the definition of an attractor we only look at forward orbits, and that in the base the future is completely independent of the past.

One can try to explain this paradox by saying that for F_+ also there is an attractor, but it is the whole space. This is true, but normally when thinking of an attractor one considers subsets much smaller than the whole space. Another explanation is that when trying to find an attractor for F_+, which is a graph, we try to specify one point in $(0,1)$ for each $\underline{\omega} \in \Sigma_+$, without specifying x_0. However, when we know the past, we basically know x_0, and with the knowledge of x_0 and $\underline{\omega} \in \Sigma_+$ we know x_n for all $n \geq 0$. Again, this is a kind of explanation (due to M. Rams), but still the question why in order to have a nice description of the future we need the past, if the past and the future are independent, remains a little mysterious.