On periodic solutions of 2–periodic Lyness difference equations

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We study the *set of periods* of the 2-periodic Lyness' equations

$$u_{n+2} = \frac{a_n + u_{n+1}}{u_n}, \tag{1}$$

where

$$a_n = \begin{cases} a & \text{for} \quad n = 2\ell + 1, \\ b & \text{for} \quad n = 2\ell, \end{cases} \tag{2}$$

and being $(u_1, u_2) \in \mathcal{Q}^+$; $\ell \in \mathbb{N}$ and $a > 0, b > 0$.

This can be done using the *composition map:*

$$F_{b,a}(x,y) := (F_b \circ F_a)(x,y) = \left(\frac{a+y}{x}, \frac{a+bx+y}{xy}\right), \tag{3}$$

where F_a and F_b are the Lyness maps: $F_\alpha(x,y) = \left(y, \frac{\alpha+y}{x}\right)$. Indeed:

$$(u_1,u_2) \xrightarrow{F_a} (u_2,u_3) \xrightarrow{F_b} (u_3,u_4) \xrightarrow{F_a} (u_4,u_5) \xrightarrow{F_b} (u_5,u_6) \xrightarrow{F_a} \cdots$$

The map $F_{b,a}$:

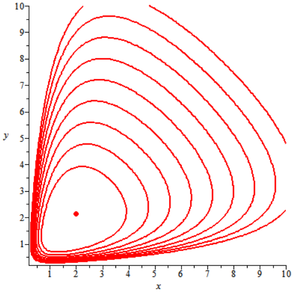
- Is a **QRT** map whose first integral is (Quispel, Roberts, Thompson; 1989):

$$V_{b,a}(x,y) = \frac{(bx+a)(ay+b)(ax+by+ab)}{xy},$$

see also (Janowski, Kulenović, Nurkanović; 2007) and (Feuer, Janowski, Ladas; 1996).

- Has a unique fixed point $(x_c, y_c) \in \mathcal{Q}^+$, which is the unique global minimum of $V_{b,a}$ in $\mathcal{Q}^+$.
- Setting $h_c := V_{b,a}(x_c, y_c)$, for $h > h_c$ the level sets $\{V_{b,a} = h\} \cap \mathcal{Q}^+$ are the closed curves.

$$\mathcal{C}_h^+ := \{(bx+a)(ay+b)(ax+by+ab) - hxy = 0\} \cap \mathcal{Q}^+ \text{ for } h > h_c.$$



The dynamics of $F_{b,a}$ restricted to $\mathcal{C}_h^+$ is *conjugate to a rotation* with associated *rotation number* $\theta_{b,a}(h)$.

Theorem A

Consider the family $F_{b,a}$ with $a, b > 0$.

(i) *If $(a, b) \neq (1, 1)$, then $\exists\, p_0(a, b) \in \mathbb{N}$ s.t. for any $p > p_0(a, b)$, $\exists$ at least an oval $\mathcal{C}_h^+$ filled by p–periodic orbits.*

(ii) *The set of periods arising in the family $\{F_{b,a},\ a > 0, b > 0\}$ restricted to $\mathcal{Q}^+$ contains all prime periods except 2, 3, 4, 6, 10.*

Corollary.

Consider the 2–periodic Lyness' recurrence for $a, b > 0$ and positive initial conditions u_1 and u_2.

(i) *If $(a, b) \neq (1, 1)$, then $\exists\, p_0(a, b) \in \mathbb{N}$, s.t. for any $p > p_0(a, b)$ $\exists$ continua of initial conditions giving 2p–periodic sequences.*

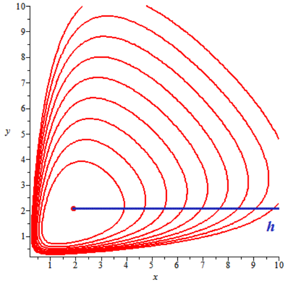
(ii) *The set of prime periods arising when $(a, b) \in (0, \infty)^2$ and positive initial conditions are considered contains all the even numbers except 4, 6, 8, 12, 20.*

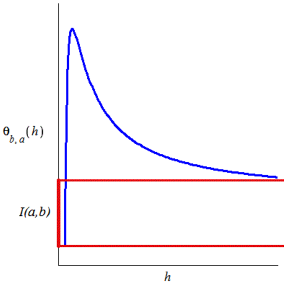
If $a \neq b$, then it does not appear any odd period, except 1.

The value $p_0(a, b)$ is *computable* for an open and dense set in the parameter space.

To compute the allowed periods, the main issues to take into account are:

- The fact that the rotation number function $\theta_{b,a}(h)$ is continuous in $[h_c, +\infty)$.
- The fact that *generically* $\theta_{b,a}(h_c) \neq \lim_{h\to+\infty} \theta_{b,a}(h) \Rightarrow \exists\, I(a,b)$, a *rotation interval*.





Proposition B.

$$\lim_{h\to h_c^+} \theta_{b,a}(h) = \sigma(a,b) := \frac{1}{2\pi} \arccos\left(\frac{1}{2}\left[-2 + \frac{1}{x_c\, y_c}\right]\right), \quad \text{and} \quad \lim_{h\to+\infty} \theta_{b,a}(h) = \frac{2}{5}.$$

Corollary

$$\text{Set } I(a,b) := \left\langle \sigma(a,b), \tfrac{2}{5} \right\rangle.$$

- If $\sigma(a,b) \neq 2/5$ $\forall \theta \in I(a,b)$, $\exists$ an oval $\mathcal{C}_h^+$ s.t. $F_{b,a}(\mathcal{C}_h^+)$ is *conjugate to a rotation*, with a rotation number $\theta_{b,a}(h) = \theta$.
- In particular, $\forall$ irreducible $q/p \in I(a,b)$, $\exists$ *periodic orbits* of $F_{b,a}$ of prime period p.

The periods of the family $F_{b,a}$.

Using the previous results with the family $a = b^2$ we found that:

$$\bigcup_{b>0} I(b^2, b) = \left(\frac{1}{3}, \frac{1}{2}\right) \subset \bigcup_{a>0,\, b>0} I(a,b) \subset \bigcup_{a>0,\, b>0} \mathrm{Image}\left(\theta_{b,a}\left(h_c, +\infty\right)\right).$$

Proposition.

- *For each θ in $(1/3, 1/2)$ $\exists$ $a, b > 0$ and an oval $\mathcal{C}_h^+$, s.t. $F_{b,a}(\mathcal{C}_h^+)$ is conjugate to a rotation with rotation number $\theta_{b,a}(h) = \theta$.*
- *In particular, $\forall$ irreducible $q/p \in (1/3, 1/2)$, $\exists$ p-periodic orbits of $F_{b,a}$*

We'll know some periods of $\{F_{b,a},\ a, b > 0\}$

$\Leftrightarrow$

We know which are the irreducible fractions in $(1/3, 1/2)$

Lemma (Cima, Gasull, M; 2007)

Given (c,d); Let $p_1 = 2, p_2 = 3, p_3, \ldots, p_n, \ldots$ be all the prime numbers.

- *Let p_{m+1} be the smallest prime number satisfying that $p_{m+1} > \max(3/(d-c), 2)$,*
- *Given any prime number p_n, $1 \le n \le m$, let s_n be the smallest natural number such that $p_n^{s_n} > 4/(d-c)$.*
- *Set $p_0 := p_1^{s_1 - 1} p_2^{s_2 - 1} \cdots p_m^{s_m - 1}$.*

Then, for any $p > p_0$ $\exists$ an irreducible fraction q/p s.t. $q/p \in (c,d)$.

Proof of Theorem A (ii):

- We apply the above result to $(1/3, 1/2)$. $\forall p \in \mathbb{N}$, s.t. $p > p_0$

$$p_0 := 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 12\,252\,240,$$

$\exists$ an irreducible fraction $q/p \in (1/3, 1/2)$.

- A finite checking determines which values of $p \le p_0$ s.t. $q/p \in (1/3, 1/2)$, resulting that there appear irreducible fractions with all the denominators except 2, 3, 4, 6 and 10.

- Proposition C $\Longrightarrow$ $\exists$ $a, b > 0$ s.t. $\exists$ an oval with rotation number $\theta_{b,a}(h) = q/p$, thus giving rise to p–periodic orbits of $F_{b,a}$ for all allowed p.

- Still it must be proved that 2, 3, 4, 6 and 10 are forbidden, since

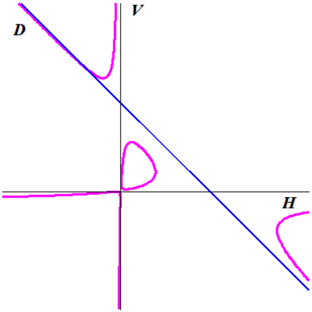
$I(a,b) \subseteq \mathrm{Image}\left(\theta_{b,a}\left(h_c, +\infty\right)\right)$ ∎

Continuity and asymptotic behavior of $\theta_{b,a}(h)$.

The curves $\mathcal{C}_h$, in homogeneous coordinates $[x : y : t] \in \mathbb{C}P^2$, are

$$\widetilde{\mathcal{C}}_h = \{(bx + at)(ay + bt)(ax + by + abt) - hxyt = 0\}.$$

The points $H = [1 : 0 : 0]$; $V = [0 : 1 : 0]$; $D = [b : -a : 0]$ are common to all curves



Proposition

If $a > 0$ and $b > 0$, and for all $h > h_c$, the curves $\widetilde{\mathcal{C}}_h$ are elliptic.

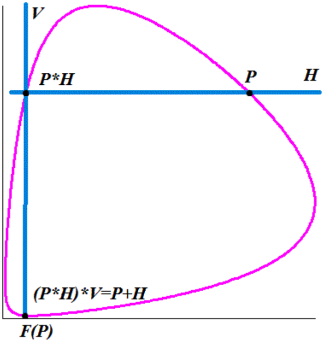
$F_{b,a}$ extends to $\mathbb{C}P^2$ as $\widetilde{F}_{b,a}([x : y : t]) = \left[ayt + y^2 : at^2 + bxt + yt : xy\right]$.

Lemma. Relation between the dynamics of $F_{b,a}$ and the *group structure* of $\mathcal{C}_h$ (*)

For each h s.t. $\widetilde{\mathcal{C}}_h$ is elliptic,

$$\widetilde{F}_{b,a}|_{\widetilde{\mathcal{C}}_h}(P) = P + H$$

Where $+$ is the addition of the group law of $\widetilde{\mathcal{C}}_h$ taking the infinite point V as the zero element.



Observe that

$$F^n(P) = P + nH,$$

so $\widetilde{\mathcal{C}}_h$ is full of p-periodic orbits $\Leftrightarrow$

$$pH = V$$

i.e. H is a *torsion point* of $\widetilde{\mathcal{C}}_h$.

(*) Birational maps preserving elliptic curves can be explained using its group structure (Jogia, Roberts, Vivaldi; 2006).

Instead of looking to a normal form for F we look for a normal form for $\widetilde{\mathcal{C}}_h$.

$$\begin{array}{ccc} \left(\widetilde{\mathcal{C}}_h, +, V\right) & \xrightarrow{\cong} & \left(\widehat{\mathcal{E}}_L, +, \widehat{V}\right) \\ \widetilde{F}_{|_{\widetilde{\mathcal{C}}_h}} : P \mapsto P + H & \longrightarrow & \widehat{G}_{|_{\mathcal{E}_L}} : P \mapsto P + \widehat{H} \end{array}$$

Where $\widehat{\mathcal{E}}_L$ is the *Weierstrass Normal Form* which in the affine plane is:

$$\mathcal{E}_L = \{\, y^2 = 4\,x^3 - g_2\,x - g_3 \,\}$$

with $g_i := g_i(a, b, h)$.

WHY?

1. Because we can *parameterize* it using the Weierstrass $\wp$ function...
2. ...that gives an *integral expression* for the *rotation number function*.

$$2\Theta(L) = \frac{\displaystyle\int_{x(L)}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4s^3 - g_2\,s - g_3}}}{\displaystyle\int_{e_1}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4s^3 - g_2\,s - g_3}}} \qquad \text{where} \qquad \theta_{b,a}(h) \sim \Theta(L)$$

3. The *asymptotics* of this integral expression can be studied.

This scheme was used in (Bastien, Rogalski; 2004).

The Weierstrass normal form of C_h is

$$\mathcal{E}_L = \{ y^2 = 4x^3 - g_2 x - g_3 \}$$

where

$$g_2 = \frac{1}{192}\left(L^8 + \sum_{i=4}^{7} p_i(\alpha,\beta)L^i\right) \text{ and } g_3 = \frac{1}{13824}\left(-L^{12} + \sum_{i=6}^{11} q_i(\alpha,\beta)L^i\right),$$

being

$$\begin{aligned}
p_7(a,b) &= -4(\alpha+\beta+1), \\
p_6(a,b) &= 2\left(3(\alpha-\beta)^2 + 2(\alpha+\beta)+3\right), \\
p_5(a,b) &= -4(\alpha+\beta-1)\left(\alpha^2 - 4\beta\alpha + \beta^2 - 1\right), \\
p_4(a,b) &= (\alpha+\beta-1)^4 .
\end{aligned}$$

and

$$\begin{aligned}
q_{11}(a,b) &= 6(\alpha+\beta+1), \\
q_{10}(a,b) &= 3\left(-5\alpha^2 + 2\alpha\beta - 5\beta^2 - 6\alpha - 6\beta - 5\right) \\
q_9(a,b) &= 4\left(5\alpha^3 - 12\alpha^2\beta - 12\alpha\beta^2 + 5\beta^3 + 3\alpha^2 - 3\alpha\beta + 3\beta^2 + 3\alpha + 3\beta + 5\right) \\
q_8(a,b) &= 3\left(-5\alpha^4 + 16\alpha^3\beta - 30\alpha^2\beta^2 + 16\alpha\beta^3 - 5\beta^4 + 4\alpha^3\right. \\
&\quad \left. -12\alpha^2\beta - 12\alpha\beta^2 + 4\beta^3 + 2\alpha^2 - 8\alpha\beta + 2\beta^2 + 4\alpha + 4\beta - 5\right) \\
q_7(a,b) &= 6\left(\alpha^2 - 4\alpha\beta + \beta^2 - 1\right)(\alpha+\beta-1)^3 \\
q_6(a,b) &= -(\alpha+\beta-1)^6
\end{aligned}$$

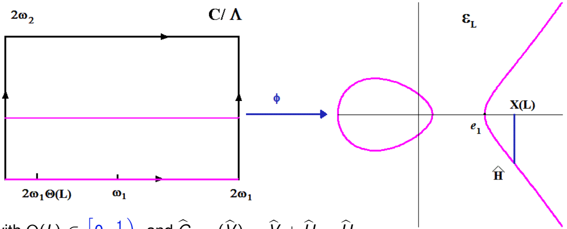
where $\alpha = a/b^2$ and b/a^2 and $L \to +\infty \Leftrightarrow h \to +\infty$.

Since $\widehat{\mathcal{E}}_L \cong \mathbb{T}^2 = \mathbb{C}/\Lambda$, the *Weierstrass ℘ function relative to a lattice* Λ gives the parametrization of $\widehat{\mathcal{E}}_L$.

$$\begin{array}{llll} \phi : & \mathbb{T}^2 = \mathbb{C}/\Lambda & \longrightarrow & \widehat{\mathcal{E}}_L \\ & z & \longrightarrow & \left\{ \; [\wp(z) : \wp'(z) : 1] \text{ if and } z \notin \Lambda; \quad [0:1:0] = \widehat{V} \text{ if } z \in \Lambda, \right. \end{array}$$

Hence $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ since $y^2 = 4\,x^3 - g_2\,x - g_3$, and integrating on $[0, u]$:

$$(*) \quad \mathbf{u} = \int_{\wp(\mathbf{u})}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4s^3 - g_2 s - g_3}}$$



- $\widehat{G}_{|\mathcal{E}_L}$ is a *rotation* with $\Theta(L) \in \left[0, \frac{1}{2}\right)$, and $\widehat{G}_{|\mathcal{E}_L}(\widehat{V}) = \widehat{V} + \widehat{H} = \widehat{H}$
- $\widehat{H}$ has *negative ordinate* $\Rightarrow$ is given by $u = 2\omega_1\Theta(L)$ and its abscissa is $X(L) = \wp(2\omega_1\Theta(L))$. Hence from (*):

$$2\omega_1\Theta(L) = \int_{X(L)}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4s^3 - g_2 s - g_3}} \quad \overset{e_1 = \wp(\omega_1)}{\Rightarrow} \quad 2\Theta(L) = \frac{\displaystyle\int_{X(L)}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4s^3 - g_2\,s - g_3}}}{\displaystyle\int_{e_1}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4s^3 - g_2\,s - g_3}}} \sim \frac{4}{5}.$$

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THANK YOU!