Modified Lotka-Volterra maps and their interior periodic points

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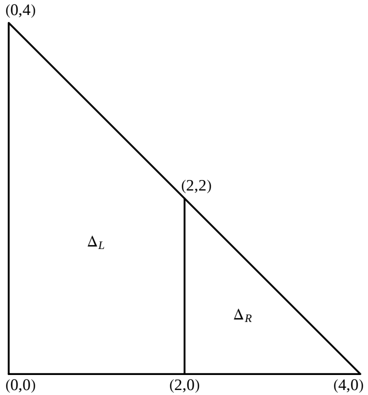
Theorem (Maličký 2012)

Let P be a lower saddle fixed point of the map F^n. Then there is an interior fixed point Q of F^n with the same period and itinerary, where the itinerary is considered with respect to the sets Δ_L and Δ_R.



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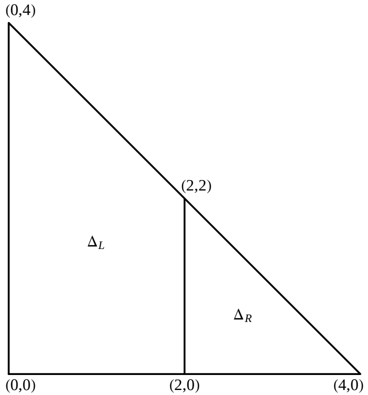

Itinerary

For a fixed point P of the map F^n it is sufficient to consider its itinerary W as a sequence $(w_i)_{i=0}^{n-1}$ defined by

$$w_i = \begin{cases} L & \text{if } F^i(P) \in \Delta_L \,, \\ R & \text{if } F^i(P) \in \Delta_R \,. \end{cases}$$

Such a sequence we will write in a shorten form

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It is natural to express the triangle Δ as the union

$$\Delta = \Delta_L \cup \Delta_R \ ,$$

where

$$\Delta_L = \{ [x, y] \in \Delta : x \leq 2 \} \quad \text{and}$$
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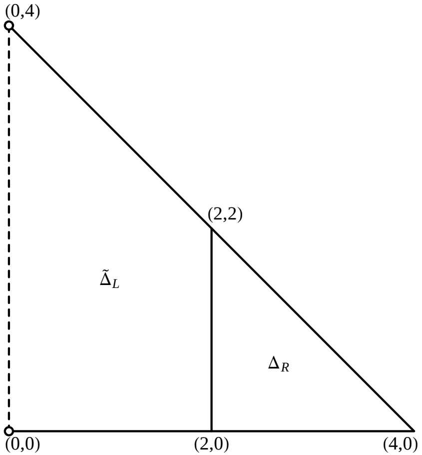
Put also

$$\widetilde{\Delta}_L = \{\, [x,\ y] \in \Delta : 0 < x \leq 2\} \text{ and}$$
$$\widetilde{\Delta} = \Delta \setminus \{[0,0]\}.$$



Inverse maps

The map F is not invertible, but F restricted to $\widetilde{\Delta}_L$ and Δ_R is. The inverse maps of these restrictions are given by

$$F_L^{-1} : \widetilde{\Delta} \to \widetilde{\Delta}_L\,, \quad [x,\, y] \mapsto \left[2 - \sqrt{4 - x - y},\, \frac{y}{2-\sqrt{4-x-y}}\right]$$

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Lower fixed point of F^n

Note that $F : [x, 0] \mapsto [f(x), 0]$, where
$f : \langle 0, 4 \rangle \to \langle 0, 4 \rangle$, $f(x) = x(4 - x)$ is the logistic map, which is conjugate with the tent map
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Jacobi matrix

Let $P = [x_0, 0] \in \Delta$ be a fixed point of the map F^n. In this case $P = \left[4\sin^2 \frac{k\pi}{2^n \pm 1}, 0\right]$. Then the Jacobi matrix of the map F^n at the point P has a form

$$\begin{pmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \mp 2^n & \mu \\ 0 & \prod_{i=0}^{n-1} x_i \end{pmatrix},$$

where

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Main result

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Modifications

Assume that for any $x \in (0,\ 4)$ we have an increasing homeomorphism φ_x of the interval $\langle 0,\ 4-x\rangle$ onto itself. Moreover let the function $\varphi(x,y) = \varphi_x(y)$ be continuous in the domain

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To obtain such above family φ_x, choose for $0 < x < 4$ a family of increasing homeomorphisms ψ_x of the interval $\langle 0,\ 1\rangle$ such that the function $\psi(x, y) = \psi_x(y)$ is continuous in $(0, 4) \times \langle 0,\ 1\rangle$ and put

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Repulsive and saddle points fixed points

Definition

Let $G^n[x,y] = [g_n(x,y), h_n(x,y)]$ and $P = [x_0, 0]$ be a lower fixed point of the map G^n. The point P is called a repulsive (respectively saddle) point if there is $\delta > 0$ such that

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If the above inequalities can be replaced by

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with $k > 1$ (respectively $0 < k < 1$) than P is called strictly repulsive (respectively strict saddle) point.

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$$h_n(x,y) > ky \ \ (\text{ respectively } h_n(x,y) < ky)$$

with $k > 1$ (respectively $0 < k < 1$) than P is called strictly repulsive (respectively strict saddle) point.

Main result for modified Lotka–Volterra maps

Theorem

Let $P \neq [0,0]$ be a lower saddle fixed point of the map G^n. Then there is an interior fixed point Q of G^n with the same period and itinerary.

Formula for λ_2

We have

$$\widetilde{\lambda}_2 = \prod_{i=0}^{n-1} x_i \varphi'_{x_i}(0) = \prod_{i=0}^{n-1} x_i \psi'_{x_i}(0) \ ,$$

or equivalently

$$\widetilde{\lambda}_2 = \prod_{i=0}^{n-1} x_i \frac{\partial \varphi}{\partial y}(x_i, 0) = \prod_{i=0}^{n-1} x_i \frac{\partial \psi}{\partial y}(x_i, 0) \ ,$$

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(i) Let $0 \leq a \leq 2$ and $\psi_x(y) = ay + (1-a)y^2$. Then we obtain $\varphi_x(y) = ay + \frac{(1-a)y^2}{4-x}$.

(ii) Let $\psi_x(y) = \frac{\sqrt{2y+x^2}-\sqrt{y+x^2}}{\sqrt{2+x^2}-\sqrt{1+x^2}}$. Then we obtain

$$\varphi_x(y) = \frac{\sqrt{2y(4-x)+x^2(4-x)^2} - \sqrt{y(4-x)+x^2(4-x)^2}}{\sqrt{2+x^2}-\sqrt{1+x^2}}.$$

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Let $0 \leq a \leq 2$ and $G : \Delta \to \Delta$ be defined by

$$G[x,y] = \begin{cases} [0,0] & \text{if } x = 4, \\ \left[x\left(4 - x - ay - \frac{(1-a)y^2}{4-x}\right), x\left(ay + \frac{(1-a)y^2}{4-x}\right)\right] & \text{otherwise} . \end{cases}$$

In this case

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