ON THE RATIONALITY OF THE MULTIDIMENSIONAL RECURSIVE SERIES

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Intoduction

In this note we give a formula for the generating function of the solution of a multidimensional difference equation under the assumption that the generating function of the initial data is known. We also state the necessary and sufficient condition for rationality of the generating function.

Richard Stanley in his book «Enumerative combinatorics» gives a hierarchy of «the most useful» classes of the generating functions (GF): D-finite $\supset$ algebraic $\supset$ rational.

De Moivre considered the recursive series as the power series $F(z) = f(0) + f(1)z + \ldots + f(k)z^k + \ldots$ with the constant coefficients $f(0), f(1), \ldots$ that make recursive sequence $\{f(n)\}, n = 0, 1, 2, \ldots$ satisfying the difference equation

$$c_0 f(x+m) + c_1 f(x+m-1) + \ldots + c_i f(x+m-i) + \ldots + c_m f(x) = 0,$$

with some constant coefficients $c_i \in \mathbb{C}$, where $0 \leq i \leq m$.

In 1722 he proved that the power series $F(z)$ are rational functions.

Let $C = \{\alpha\}$, where $\alpha = (\alpha_1, \dots, \alpha_n)$, be a finite subset of the positive octant $\mathbb{Z}^n_+$ of the integer lattice $\mathbb{Z}^n$, $f : \mathbb{Z}^n_+ \to \mathbb{C}$ and let $m = (m_1, m_2, \dots, m_n) \in C$. Moreover for all $\alpha \in C$ the condition

$$\alpha_1 \leqslant m_1, \dots, \alpha_n \leqslant m_n \qquad (*)$$

be fulfilled.

The problem Cauchy

The problem Cauchy is to find the solution $f(x)$ of the difference equation (we use a multidimensional notation)

$$\sum_{\alpha \in C} c_\alpha f(x + \alpha) = 0, \qquad (1)$$

which coincides with the some given function $\varphi : X_m \to \mathbb{C}$ on the set $X_m = \mathbb{Z}^n_+ \setminus \left(m + \mathbb{Z}^n_+\right)$ («initial data»).

It's well known that this Cauchy problem has a unique solution.

- M. Bousquet-Mélou, M. Petkovšek, Linear recurrences with constant coefficients: the multivariate case, DM, 225, 51-75.
- E. Leinartas, Multiple Laurent Series and Difference Equations. Siberian Mathematical Journal, 2004, Volume 45, Number 2, 321-326.
- E. Leinartas, Multiple Laurent series and fundamental solutions of linear difference equations, Siberian Mathematical Journal, Vol. 48, No. 2, pp. 268–272.

Let $J = (j_1, ..., j_n)$, where $j_k \in \{0, 1\}$, $k = 1, ..., n$, is an ordered set of zeros and ones. With every such set J we associate the face Γ_J of the n-dimensional integer parallelepiped

$$\Pi_m = \{x \in \mathbb{Z}^n : 0 \leqslant x_k \leqslant m_k, k = 1, ..., n\}$$

as follows:

$$\Gamma_J = \{x \in \Pi_m : x_k = m_k, \text{ if } j_k = 1, \text{ and } x_k < m_k, \text{ if } j_k = 0\}.$$

GF of «inital data»

The function

$$\Phi(z) = \sum_{x \in X_m} \frac{\varphi(x)}{z^{x+I}}$$

is the generation function of the initial data of the difference equation (1).

GF of «inital data» can be represented as the sum

$$\Phi(z) = \sum_{J} \Phi_J(z),$$

where

$$\Phi_J(z) = \sum_{\tau \in \Gamma_J} \Phi_{\tau,J}(z), \ \Phi_{\tau,J}(z) = \sum_{y \geqslant 0} \frac{\varphi(\tau + Jy)}{z^{\tau + Jy + I}}.$$

Theorem

The generating function $F(z) = \sum_{x \in \mathbb{Z}_+^n} \frac{\varphi(x)}{z^{x+1}}$ *of the solution of the difference equation (1) is*

$$F(z)P(z) = \sum_{J} \sum_{\tau \in \Gamma_J} \Phi_{\tau,J}(z) P_\tau(z), \text{ where } P_\tau(z) = \sum_{\substack{\alpha \leqslant m \\ \alpha \nleqslant \tau}} c_\alpha z^\alpha$$

and $P(z) = \sum_{\alpha \in C} c_\alpha z^\alpha$ *is the characteristic polynomial of the difference equation (1).*

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Corollary

The generating function $F(z)$ *of the solution of the difference equation (1) is rational if and only if the generating function* $\Phi(z)$ *of the initial data is rational.*

Example

Bloom's srtings

Bloom studies the number of singles in all the 2^x x-length bit strings, where a single is any isolated 1 or 0, i.e., any run of length 1. Let $r(x, y)$ be the number of n-length bit strings beginning with 0 and having y singles.

- D.M.Bloom, Singles in a Sequence of Coin Tosses, The College Mathematics Journal, 29(1998), 307-344.

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The Cauchy problem

The sequence $r(x, y)$ satisfies the difference equation
$r(x+2, y+1) - r(x+1, y+1) - r(x+1, y) - r(x, y+1) + r(x, y) = 0.$
with the «initail data»
$\varphi(0,0) = 1, \varphi(1,0) = 0, \varphi(x,0) = \varphi(x-1,0) + \varphi(x-2,0), x \leqslant 2,$
$\varphi(1,1) = 1, \varphi(0,y) = 0, y \leqslant 1$ and $\varphi(1,y) = 0, y \leqslant 2.$

Computation

$$\Phi_{0,0} = \frac{1}{zw}, \qquad P_{0,0} = z^2w - zw - z - w,$$

$$\Phi_{1,0} = 0, \qquad P_{1,0} = z^2w - zw - w,$$

$$\Phi_{0,1} = 0, \qquad P_{0,1} = z^2w - zw - z,$$

$$\Phi_{1,1} = \frac{1}{z^2w^2}, \qquad P_{1,1} = z^2w,$$

$$\Phi_{2,0} = \frac{1}{zw(z^2 - z - 1)}, \qquad P_{2,0} = z^2w - zw - w,$$

It's easy!

$$F(z) = \frac{z-1}{z^2w - zw - z - w + 1}$$