Hyperbolicity in dissipative polygonal billiards

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Outline

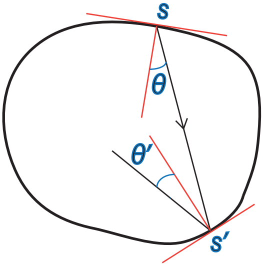
Billiard dynamics

A billiard is a mechanical system consisting of a point-particle moving freely inside a planar region D (**billiard table**) and being reflected off the perimeter of the region ∂D according to some **reflection law**.

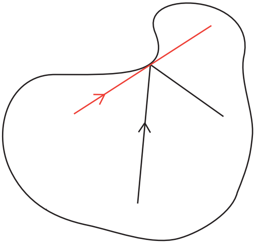
Example (Billiards in the world)

- Light in mirrors
- Acoustics in closed rooms, echoes
- Lorentz gas model for electricity (small electron bounces between large molecules)
- games: billiard, pool, snooker, pinballs, flippers

Examples of billiard tables



Convex

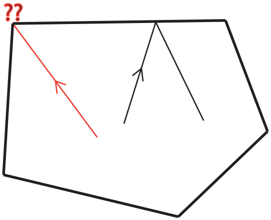


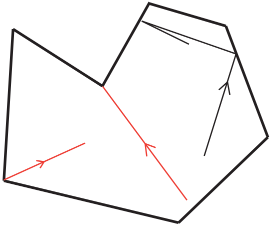
Non-convex

Smooth billiards

billiard map $\Phi\colon (s, \theta) \mapsto (s', \theta')$

Examples of billiard tables





Convex

Non-convex

Polygonal billiards

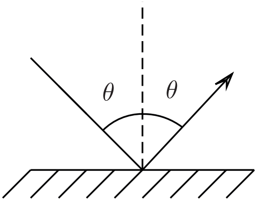
More examples

Triangle

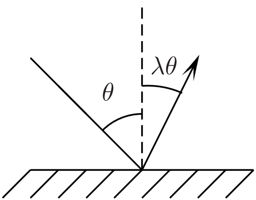
Rectangle

Z-shaped

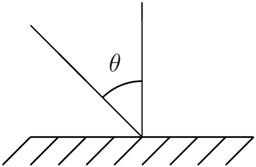
Examples of reflection laws



Conservative
classical
standard
specular
elastic
$\lambda = 1$



Linear contraction
dissipative
non-elastic
pinball
$0 < \lambda < 1$



Slap
$\lambda = 0$

Billiard map

Singularities/Discontinuity points

$$V = \{\text{corners and tangencies}\} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$S^+ = V \cup \Phi^{-1}(V)$$

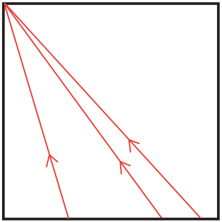
Billiard map

Piecewise smooth

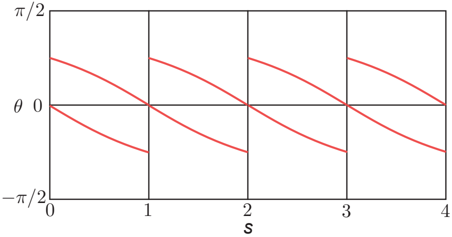
$$\Phi\colon M \to \overline{M}$$
$$(s, \theta) \mapsto (s', \theta')$$

Phase space

$$M = \partial D \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus S^+$$



Polygonal convex billiard table (it has corners)



Phase space M

Standard reflection law

The billiard map Φ preserves the measure $\cos\theta\, d\theta\, ds$

Conservative system

No attractors

Contractive reflection law

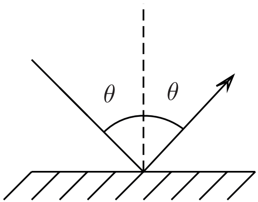
The area is contracted

Dissipative system

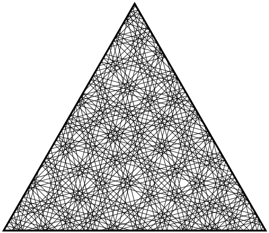
There are attractors

Outline

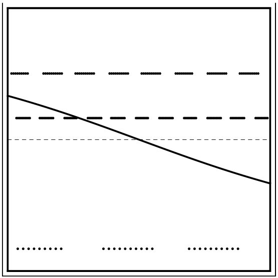
Conservative convex polygonal billiards



Specular reflection law



Configuration space
regular triangle



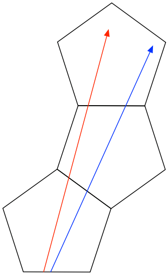
Phase space
(one component)

Conservative polygons are not chaotic

Zero Lyapunov exponents:

$$LE(x,\alpha,v) = \lim_{t\to+\infty} \frac{1}{t} \log \|D\Phi^t(x,\alpha)\, v\| = 0$$

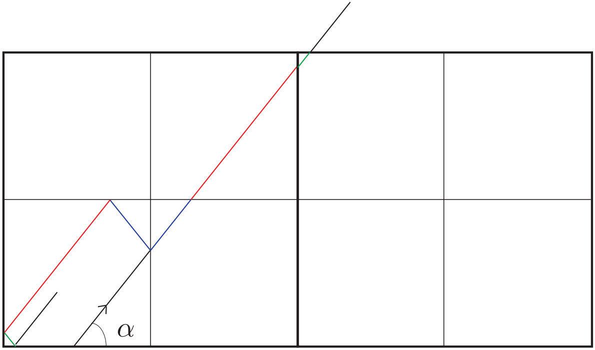
where Φ^t is the billiard flow, $\mathcal{M} = (D \times S^1)/\sim$ is the phase space



"Unfolding" the polygonal table along the orbit
Linear divergence of straight lines

The square

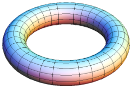
Reduction to a torus flow with direction $\alpha \in S^1$

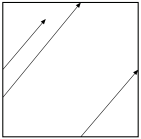


Unfolding the square table

A billiard is integrable if

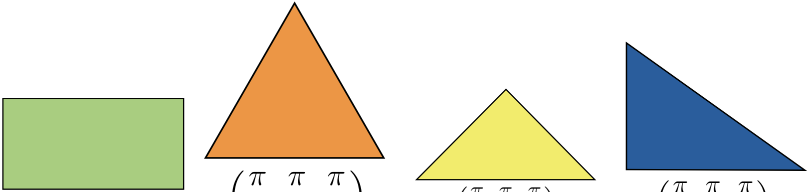
1. $\mathcal{M} = \bigcup_\alpha \mathcal{M}_\alpha$ (3-dim)
2. $\mathcal{M}_\alpha = \mathbb{T}^2$
3. $\Phi^t(\mathcal{M}_\alpha) = \mathcal{M}_\alpha$
4. $\Phi^t|_{\mathcal{M}_\alpha}$ linear flow, i.e. $\Phi^t(x, \alpha) = (x + t(\cos\alpha, \sin\alpha), \alpha)$





Linear flow on a torus is either **periodic** or **quasi-periodic** (minimal, ergodic)

The only integrable polygons are:



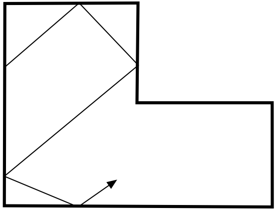
$$\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) \quad \left(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}\right) \quad \left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}\right)$$

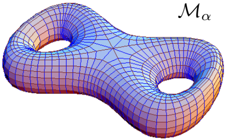
A billiard is quasi-integrable if

1. $\mathcal{M} = \bigcup_\alpha \mathcal{M}_\alpha$
2. $\Phi^t(\mathcal{M}_\alpha) = \mathcal{M}_\alpha$
3. $\Phi^t|_{\mathcal{M}_\alpha}$ linear
4. $\mathcal{M}_\alpha$ has genus $g > 1$

Example

$g(\mathcal{M}_\alpha) = 2$ for the L-shaped polygon





L-shaped billiard table

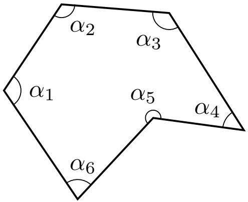
Example

k-regular polygon with $\alpha = \frac{k-2}{k}\pi = \frac{m}{n}\pi$ with m and n co-prime,

$$g(\mathcal{M}_\alpha) = 1 + \frac{n}{2}\left(k - 2 - \frac{k}{n}\right)$$

Rational polygons

A polygon is **rational** if every internal angle $\alpha_i \in \pi\mathbb{Q}$



$$\alpha_i = \frac{m_i}{n_i}\pi$$

For a polygon with k-sides, N least common multiplier of n_i
$g(\mathcal{M}_\alpha) = 1 + \frac{N}{2}\left(k - 2 - \sum_{i=1}^{k} \frac{1}{n_i}\right)$

Theorem (Veech)

Rational polygons are either integrable or quasi-integrable

Relation with Teichmuller spaces, quadratic differentials, interval exchange transformations...

General polygons

Theorem (Kerchoff-Masur-Smillie)

Polygonal billiards are generically ergodic

- Not known if a given irrational polygon is ergodic
- Not known if every triangle has a periodic orbit

Outline

Dissipative convex polygonal billiards

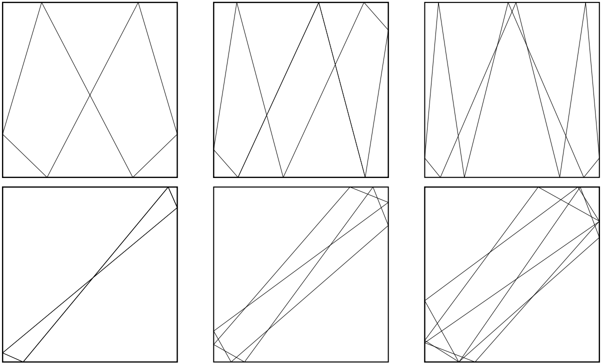
Let

- Dissipative reflection law $\theta \mapsto \lambda\theta$, $0 < \lambda < 1$
- $U = \{|\theta| < \lambda\pi/2\}$ is an invariant set
- the full-measure set of points whose orbit remains forever in the domain is
$$U^+ = \{x \in U : \Phi_\lambda^n(x) \notin S^+, n \geq 0\}$$
- The attractor of Φ_λ is the invariant set:
$$\Omega_\lambda = \overline{\bigcap_{n\geq 0} \Phi_\lambda^n(U^+)}$$

It might have several components

Periodic orbits

For many billiards there are many periodic orbits. E.g.



Square, $\lambda = 0.6$.

Parabolic attractor:

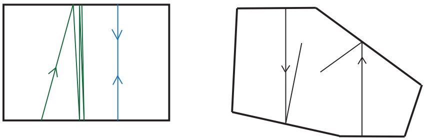
$$\mathcal{P} = \{\text{Period 2 orbits}\} \subset \{\theta = 0\}$$

Basin of attraction:

$$W^s(\mathcal{P}) = \{(s, \theta) \in M : \operatorname{dist}(\Phi_\lambda^n(s, \theta), \mathcal{P}) \to 0\}$$

Remark

$\mathcal{P} \neq \emptyset$ iff there are parallel sides facing each other



Dominated splitting

Let Σ be a Φ_λ-invariant set (e.g. periodic orbits, horseshoes, attractors)

Theorem (Markarian-Pujals-Sambarino)

For every polygon, Σ has **dominated splitting***: there is a non-trivial continuous invariant splitting $T\Sigma = E \oplus F$, $0 < \mu < 1$ and $c > 0$ st on Σ*

$$\frac{\|D\Phi_\lambda^n|_E\|}{\|D\Phi_\lambda^n|_F\|} \leq c\mu^n, \quad n \geq 0$$

Dominated splitting is weaker than **uniform hyperbolicity**

$$\|D\Phi_\lambda^n|_E\| \leq c\mu^n \qquad \|D\Phi_\lambda^{-n}|_F\| \leq c\mu^n$$

Uniform hyperbolicity

Theorem

If the polygon has

1. *no parallel sides facing each other,*
2. *OR parallel sides facing each other AND* $\exists C > 0$ *st orbits in* Σ *do not bounce more than* C *times between parallel sides,*

then Σ *is uniformly hyperbolic*

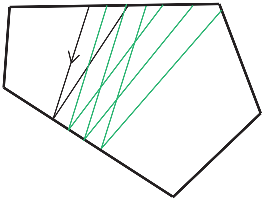
Example

Odd-sided regular polygons do not have parallel sides

Example (periodic orbits)

- Period $= 2$, parabolic $\mathcal{P}$
- Period > 2, hyperbolic

Local unstable manifold of periodic points is inside $\{\theta = const\}$. Global is cut in local pieces due to singularities.



Even-sided regular polygons

Let D be a regular polygon with $2N$ sides ($N \geq 3$)

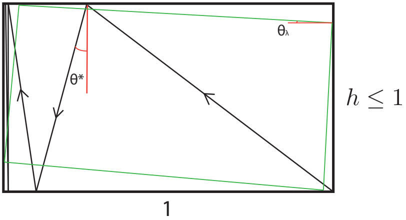
Theorem

If $\Sigma \neq \mathcal{P}$ and $\lambda < \frac{1}{2}$, then Σ is uniformly hyperbolic

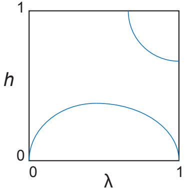
Rectangles

Proposition

1. *If* $\Sigma \neq \mathcal{P}$ *and* $h \sum_{n \geq 0} \tan\left(\lambda^{n+1}(1-\lambda)\frac{\pi}{2}\right) > 1$, *then* Σ *is uniformly hyperbolic*
2. *If* $\theta_\lambda \leq \theta_*$, *then* $\mathcal{P}$ *attracts every orbit* ($\Omega_\lambda = \mathcal{P}$)

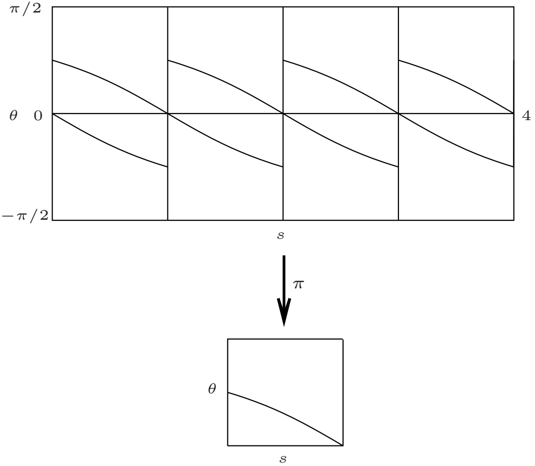


Configuration space



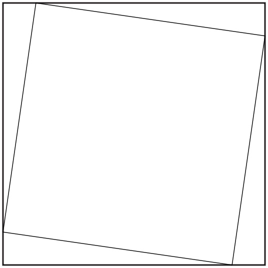
Parameter space

The square billiard $h = 1$

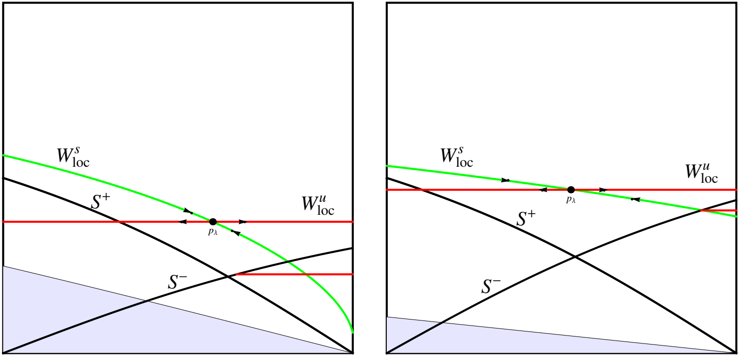


Phase space reduction - reduced billiard map

The square billiard $h = 1$



Period 4 orbit = fixed point of reduced map

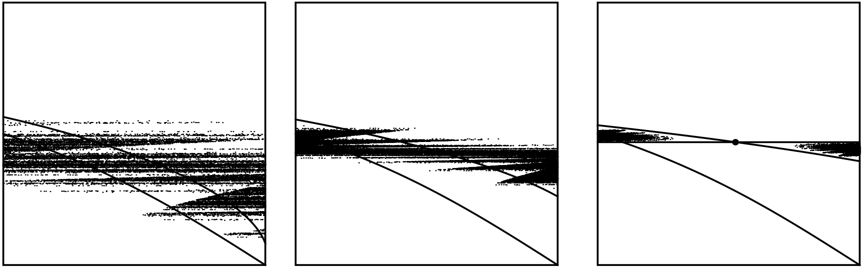


$\lambda < \lambda_2$ $\qquad$ $\lambda = \lambda_2 \simeq 0.8736$

Invariant manifolds of the fixed point p_λ
Transverse homoclinic intersection for $0 < \lambda < \lambda_2$
Existence of horseshoe

Proposition

For $0 < \lambda < \lambda_2$, Φ_λ *has positive topological entropy*



$\lambda = 0.615$ $\lambda = 0.75$ $\lambda = 0.88$

Square billiard: non-trivial attractor

Sinai-Ruelle-Bowen measures

A Φ_λ-invariant measure μ that has absolutely continuous conditional measures on unstable manifolds.

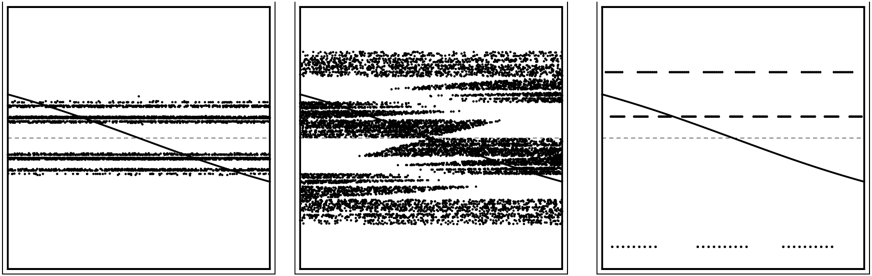
Remark

"The relevance of SRB measures lies in the fact that they are the invariant measures more related to volume for dissipative systems, helping to explain how local instability on attractors gives coherent statistics for orbits starting at the basin of attraction." (Lai-Sang Young)

Equivalently, there is a positive **Lebesgue measure** set of $x \in M$ st

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(\Phi_\lambda^j(x)) = \int_M \varphi\, d\mu, \quad \varphi \in C^0(M,\mathbb{R})$$

Regular triangle



$\lambda = 0.3$ $\lambda = 0.7$ $\lambda = 1$

Reduced regular triangle: non-trivial attractor

Theorem (Arroyo-Markarian-Sanders)

The regular triangle has a transitive attractor with an ergodic SRB invariant measure for $\lambda < 1/3$.

Strongly dissipative polygons

Theorem

For sufficiently small λ regular polygons with $2N+1$ sides

1. *have uniformly hyperbolic attractors with finitely-many SRB measures and dense hyperbolic periodic orbits*
2. *are ergodic iff $N = 1, 2$*

Theorem

Generic polygons have uniformly hyperbolic attractors with finitely-many SRB measures and dense hyperbolic periodic orbits for sufficiently small λ.

Idea of proof: For $\lambda \simeq 0$, Φ_λ is close to slap map Φ_0 (1-dim piecewise affine expanding map), and satisfies conditions:

1. uniform hyperbolicity
2. the smallest expansion rate along unstable direction is $> p$
3. $\Phi_\lambda(W^u_{loc})$ is cut by singularities S^+ in no more than p pieces

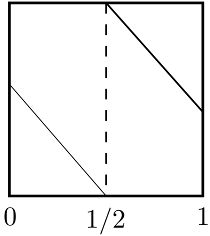
Theorem

Then the attractor has finitely-many SRB measures and dense hyperbolic periodic orbits

Proof: Version of Pesin result

Slap maps $\lambda = 0$

If there are no parallel sides, the slap map is a piecewise affine expanding map of the interval. Thus, it has expanding attractors [Markarian-Pujals-Sambarino]



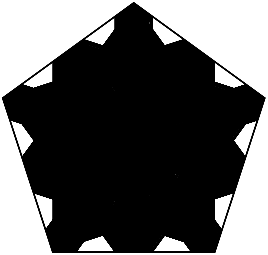
$$\beta = -1/\cos\left(\frac{\pi}{2N+1}\right)$$

$$\Phi_0(s) = \beta\left(s - \tfrac{1}{2}\right) \mod 1$$

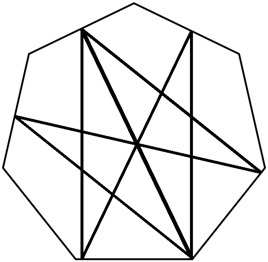
Slap map of regular polygon with $2N+1$ sides

Proposition

For regular polygons, only the triangle and the pentagon have an ergodic slap map (with respect to the invariant measure on the attractors)



Pentagon $\lambda = 0$



Heptagon $\lambda = 0$