Periodic point free continuous self–maps on graphs and surfaces

Jaume Llibre

Universitat Autònoma de Barcelona

ICDEA 2012

July 23 to 27, 2012

A discrete dynamical system $(\mathbb{M}, f)$ is formed by a topological space $\mathbb{M}$ and a continuous map $f : \mathbb{M} \to \mathbb{M}$.

A discrete dynamical system $(\mathbb{M}, f)$ is formed by a topological space $\mathbb{M}$ and a continuous map $f : \mathbb{M} \to \mathbb{M}$.

A point x is called fixed if $f(x) = x$,

A discrete dynamical system $(\mathbb{M}, f)$ is formed by a topological space $\mathbb{M}$ and a continuous map $f : \mathbb{M} \to \mathbb{M}$.

A point x is called fixed if $f(x) = x$,

and periodic of period k if $f^k(x) = x$ and $f^i(x) \neq x$ if $0 < i < k$.

A discrete dynamical system $(\mathbb{M}, f)$ is formed by a topological space $\mathbb{M}$ and a continuous map $f : \mathbb{M} \to \mathbb{M}$.

A point x is called fixed if $f(x) = x$,

and periodic of period k if $f^k(x) = x$ and $f^i(x) \neq x$ if $0 < i < k$.

We denote the set of periods of all the periodic points of f by $\mathrm{Per}(f)$.

The following two results are well known:

Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map of degree d. If $\text{Per}(f) = \emptyset$ (i.e. if f is periodic point free), then $d = 1$.

The following two results are well known:

Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map of degree d. If $\text{Per}(f) = \emptyset$ (i.e. if f is periodic point free), then $d = 1$.

Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a continuous map of degree d. If $\text{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1 and d.

The following two results are well known:

Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map of degree d. If $\text{Per}(f) = \emptyset$ (i.e. if f is periodic point free), then $d = 1$.

Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a continuous map of degree d. If $\text{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1 and d.

Ll. Alsedà, S. Baldwin, J. Llibre, R. Swanson and W. Szlenk, Minimal sets of periods for torus maps via Nielsen numbers, Pacific J. of Math. **169** (1995), 1–32.

Our objective is two extend the previous results to arbitrary continuous self–maps on graphs and surfaces.

Our objective is two extend the previous results to arbitrary continuous self–maps on graphs and surfaces.

The tools used for proving these results can be applied for studying the periodic point free continuous self–maps of many other compact absolute neighborhood retract spaces.

A graph is a union of vertices (points) and edges, which are homeomorphic to the closed interval, and have mutually disjoint interiors. The endpoints of the edges are vertexes (not necessarily different) and the interiors of the edges are disjoint from the vertices.

GRAPH THEOREM

Let $\mathbb{G}$ be a connected compact graph such that $\dim_{\mathbb{Q}} H_1(\mathbb{G}, \mathbb{Q}) = r$, and let $f : \mathbb{G} \to \mathbb{G}$ be a continuous map.

GRAPH THEOREM

Let $\mathbb{G}$ be a connected compact graph such that $\dim_{\mathbb{Q}} H_1(\mathbb{G}, \mathbb{Q}) = r$, and let $f : \mathbb{G} \to \mathbb{G}$ be a continuous map.

If $\text{Per}(f) = \emptyset$,

GRAPH THEOREM

Let $\mathbb{G}$ be a connected compact graph such that $\dim_{\mathbb{Q}} H_1(\mathbb{G}, \mathbb{Q}) = r$, and let $f : \mathbb{G} \to \mathbb{G}$ be a continuous map.

If $\text{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1 and 0, this last with multiplicity $r - 1$.

GRAPH THEOREM

Let $\mathbb{G}$ be a connected compact graph such that $\dim_{\mathbb{Q}} H_1(\mathbb{G}, \mathbb{Q}) = r$, and let $f : \mathbb{G} \to \mathbb{G}$ be a continuous map.

If $\text{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1 and 0, this last with multiplicity $r - 1$.

Of course, as a corollary of the GRAPH THEOREM it follows the mentioned result for the circle:

GRAPH THEOREM

Let $\mathbb{G}$ be a connected compact graph such that $\dim_{\mathbb{Q}} H_1(\mathbb{G}, \mathbb{Q}) = r$, and let $f : \mathbb{G} \to \mathbb{G}$ be a continuous map.

If $\text{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1 and 0, this last with multiplicity $r - 1$.

Of course, as a corollary of the GRAPH THEOREM it follows the mentioned result for the circle:

Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map of degree d. If $\text{Per}(f) = \emptyset$, then $d = 1$.

An orientable connected compact surface WITHOUT boundary of genus $g \geq 0$, $\mathbb{M}_g$, is homeomorphic to the sphere if $g = 0$, to the torus if $g = 1$, or to the connected sum of g copies of the torus if $g \geq 2$.

An orientable connected compact surface WITHOUT boundary of genus $g \geq 0$, $\mathbb{M}_g$, is homeomorphic to the sphere if $g = 0$, to the torus if $g = 1$, or to the connected sum of g copies of the torus if $g \geq 2$.

An orientable connected compact surface WITH boundary of genus $g \geq 0$, $\mathbb{M}_{g,b}$, is homeomorphic to $\mathbb{M}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closures. In what follows $\mathbb{M}_{g,0} = \mathbb{M}_g$.

ORIENTABLE SURFACE THEOREM

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map. The degree of f is d if $b = 0$.

ORIENTABLE SURFACE THEOREM

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map. The degree of f is d if $b = 0$.

If $\text{Per}(f) = \emptyset$,

ORIENTABLE SURFACE THEOREM

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map. The degree of f is d if $b = 0$.

If $\text{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1, d and 0, this last with multiplicity $2g - 2$ if $b = 0$ and $g \geq 1$;

ORIENTABLE SURFACE THEOREM

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map. The degree of f is d if $b = 0$.

If $\text{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1, d and 0, this last with multiplicity $2g - 2$ if $b = 0$ and $g \geq 1$; and 1 and 0, this last with multiplicity $2g + b - 2 \geq 0$ if $b > 0$.

ORIENTABLE SURFACE THEOREM

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map. The degree of f is d if $b = 0$.

If $\text{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1, d and 0, this last with multiplicity $2g - 2$ if $b = 0$ and $g \geq 1$; and 1 and 0, this last with multiplicity $2g + b - 2 \geq 0$ if $b > 0$.

Of course, as a corollary of the ORIENTABLE SURFACE THEOREM it follows the mentioned result for the 2-dimensional torus:

ORIENTABLE SURFACE THEOREM

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map. The degree of f is d if $b = 0$.

If $\text{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1, d and 0, this last with multiplicity $2g - 2$ if $b = 0$ and $g \geq 1$; and 1 and 0, this last with multiplicity $2g + b - 2 \geq 0$ if $b > 0$.

Of course, as a corollary of the ORIENTABLE SURFACE THEOREM it follows the mentioned result for the 2-dimensional torus:

Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a continuous map of degree d. If $\text{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1 and d.

A non–orientable connected compact surface WITHOUT boundary of genus $g \geq 1$, $\mathbb{N}_g$, is homeomorphic to the real projective plane if $g = 1$, or to the connected sum of g copies of the real projective plane if $g > 1$.

A non–orientable connected compact surface WITHOUT boundary of genus $g \geq 1$, $\mathbb{N}_g$, is homeomorphic to the real projective plane if $g = 1$, or to the connected sum of g copies of the real projective plane if $g > 1$.

A non–orientable connected compact surface WITH boundary of genus $g \geq 1$, $\mathbb{N}_{g,b}$, is homeomorphic to $\mathbb{N}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closures. In what follows $\mathbb{N}_{g,0} = \mathbb{N}_g$.

NON-ORIENTABLE SURFACE THEOREM

Let $\mathbb{N}_{g,b}$ be a non–orientable connected compact surface of genus $g \geq 1$ with $b \geq 0$ boundary components, and let $f : \mathbb{N}_{g,b} \to \mathbb{N}_{g,b}$ be a continuous map.

NON-ORIENTABLE SURFACE THEOREM

Let $\mathbb{N}_{g,b}$ be a non–orientable connected compact surface of genus $g \geq 1$ with $b \geq 0$ boundary components, and let $f : \mathbb{N}_{g,b} \to \mathbb{N}_{g,b}$ be a continuous map.

If $\text{Per}(f) = \emptyset$,

NON-ORIENTABLE SURFACE THEOREM

Let $\mathbb{N}_{g,b}$ be a non–orientable connected compact surface of genus $g \geq 1$ with $b \geq 0$ boundary components, and let $f : \mathbb{N}_{g,b} \to \mathbb{N}_{g,b}$ be a continuous map.

If $\text{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1 and 0, this last with multiplicity $g + b - 2 \geq 0$.

Let $\mathbb{M}$ be a graph $\mathbb{G}$, or an orientable surface $\mathbb{M}_{g,b}$, or a non-orientable surface $\mathbb{N}_{g,b}$.

Let $\mathbb{M}$ be a graph $\mathbb{G}$, or an orientable surface $\mathbb{M}_{g,b}$, or a non-orientable surface $\mathbb{N}_{g,b}$.

We denote by $H_k(\mathbb{M}, \mathbb{Q})$ the homological spaces with coefficients in $\mathbb{Q}$.

Let $\mathbb{M}$ be a graph $\mathbb{G}$, or an orientable surface $\mathbb{M}_{g,b}$, or a non-orientable surface $\mathbb{N}_{g,b}$.

We denote by $H_k(\mathbb{M}, \mathbb{Q})$ the homological spaces with coefficients in $\mathbb{Q}$.

Of course, $k = 0, 1$ if $\mathbb{M}$ is a graph,

Let $\mathbb{M}$ be a graph $\mathbb{G}$, or an orientable surface $\mathbb{M}_{g,b}$, or a non-orientable surface $\mathbb{N}_{g,b}$.

We denote by $H_k(\mathbb{M}, \mathbb{Q})$ the homological spaces with coefficients in $\mathbb{Q}$.

Of course, $k = 0, 1$ if $\mathbb{M}$ is a graph, or $k = 0, 1, 2$ if $\mathbb{M}$ is a surface.

Let $\mathbb{M}$ be a graph $\mathbb{G}$, or an orientable surface $\mathbb{M}_{g,b}$, or a non-orientable surface $\mathbb{N}_{g,b}$.

We denote by $H_k(\mathbb{M}, \mathbb{Q})$ the homological spaces with coefficients in $\mathbb{Q}$.

Of course, $k = 0, 1$ if $\mathbb{M}$ is a graph, or $k = 0, 1, 2$ if $\mathbb{M}$ is a surface.

Each of these spaces is a finite dimensional linear space over $\mathbb{Q}$.

Let $\mathbb{M}$ be a graph $\mathbb{G}$, or an orientable surface $\mathbb{M}_{g,b}$, or a non-orientable surface $\mathbb{N}_{g,b}$.

We denote by $H_k(\mathbb{M}, \mathbb{Q})$ the homological spaces with coefficients in $\mathbb{Q}$.

Of course, $k = 0, 1$ if $\mathbb{M}$ is a graph, or $k = 0, 1, 2$ if $\mathbb{M}$ is a surface.

Each of these spaces is a finite dimensional linear space over $\mathbb{Q}$.

Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ it induces linear maps $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$ on the homological spaces of $\mathbb{M}$.

Let $\mathbb{M}$ be a graph $\mathbb{G}$, or an orientable surface $\mathbb{M}_{g,b}$, or a non-orientable surface $\mathbb{N}_{g,b}$.

We denote by $H_k(\mathbb{M},\mathbb{Q})$ the homological spaces with coefficients in $\mathbb{Q}$.

Of course, $k = 0, 1$ if $\mathbb{M}$ is a graph, or $k = 0, 1, 2$ if $\mathbb{M}$ is a surface.

Each of these spaces is a finite dimensional linear space over $\mathbb{Q}$.

Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ it induces linear maps $f_{*k} : H_k(\mathbb{M},\mathbb{Q}) \to H_k(\mathbb{M},\mathbb{Q})$ on the homological spaces of $\mathbb{M}$.

Every linear map f_{*k} is given by an $n_k \times n_k$ matrix with integer entries, where n_k is the dimension of $H_k(\mathbb{M},\mathbb{Q})$.

Let n be the topological dimension of a compact polyhedron $\mathbb{M}$.

Let n be the topological dimension of a compact polyhedron $\mathbb{M}$. Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ its Lefschetz number $L(f)$ is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \text{trace}(f_{*k}).$$

Let n be the topological dimension of a compact polyhedron $\mathbb{M}$. Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ its Lefschetz number $L(f)$ is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \text{trace}(f_{*k}).$$

One of the main results connecting the algebraic topology with the fixed point theory is the Lefschetz Fixed Point Theorem which establishes the existence of a fixed point if $L(f) \neq 0$.

Let n be the topological dimension of a compact polyhedron $\mathbb{M}$. Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ its Lefschetz number $L(f)$ is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \text{trace}(f_{*k}).$$

One of the main results connecting the algebraic topology with the fixed point theory is the Lefschetz Fixed Point Theorem which establishes the existence of a fixed point if $L(f) \neq 0$.

If we consider the Lefschetz number of f^m, in general, it is not true that $L(f^m) \neq 0$ implies that f has a periodic point of period m;

Let n be the topological dimension of a compact polyhedron $\mathbb{M}$. Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ its Lefschetz number $L(f)$ is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}(f_{*k}).$$

One of the main results connecting the algebraic topology with the fixed point theory is the Lefschetz Fixed Point Theorem which establishes the existence of a fixed point if $L(f) \neq 0$.

If we consider the Lefschetz number of f^m, in general, it is not true that $L(f^m) \neq 0$ implies that f has a periodic point of period m; it only implies the existence of a periodic point of period a divisor of m.

From the Lefschetz Fixed Point Theorem it follows immediately the next result.

From the Lefschetz Fixed Point Theorem it follows immediately the next result.

PROPOSITION 1

Let $\mathbb{M}$ be a polyhedron. A necessary condition in order that a map $f : \mathbb{M} \to \mathbb{M}$ be periodic point free (i.e. $\text{Per}(f) = \emptyset$) is that all Lefschetz numbers $L(f^m)$ be zero for $m = 1, 2, 3, \ldots$.

From the Lefschetz Fixed Point Theorem it follows immediately the next result.

PROPOSITION 1

Let $\mathbb{M}$ be a polyhedron. A necessary condition in order that a map $f : \mathbb{M} \to \mathbb{M}$ be periodic point free (i.e. $\text{Per}(f) = \emptyset$) is that all Lefschetz numbers $L(f^m)$ be zero for $m = 1, 2, 3, \ldots$.

We define that a continuous self-map f of $\mathbb{M}$ is Lefschetz periodic point free if $L(f^m) = 0$ for $m = 1, 2, 3, \ldots$

The Lefschetz zeta function of f is defined as

$$\mathcal{Z}_f(t) = \exp\left(\sum_{m\geq 1} \frac{L(f^m)}{m}\, t^m\right).$$

The Lefschetz zeta function of f is defined as

$$\mathcal{Z}_f(t) = \exp\left(\sum_{m\geq 1} \frac{L(f^m)}{m} t^m\right).$$

The Lefschetz zeta function $\mathcal{Z}_f(t)$ is a generating function for all the Lefschetz numbers of all iterates of f.

When $\mathbb{M}$ is a polyhedron there is the following alternative way to compute the Lefschetz zeta function

When $\mathbb{M}$ is a polyhedron there is the following alternative way to compute the Lefschetz zeta function

$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(Id_k - t\, f_{*k})^{(-1)^{k+1}},$$

where $n = \dim \mathbb{M}$ and Id_k is the identity map of $H_k(\mathbb{M}, \mathbb{Q})$, and by convention $\det(Id_k - t\, f_{*k}) = 1$ if $n_k = 0$.

When $\mathbb{M}$ is a polyhedron there is the following alternative way to compute the Lefschetz zeta function

$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(Id_k - t\, f_{*k})^{(-1)^{k+1}},$$

where $n = \dim \mathbb{M}$ and Id_k is the identity map of $H_k(\mathbb{M}, \mathbb{Q})$, and by convention $\det(Id_k - t\, f_{*k}) = 1$ if $n_k = 0$.

Note that the Lefschetz zeta function is a rational function with integers coefficients,

When $\mathbb{M}$ is a polyhedron there is the following alternative way to compute the Lefschetz zeta function

$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(Id_k - t\, f_{*k})^{(-1)^{k+1}},$$

where $n = \dim \mathbb{M}$ and Id_k is the identity map of $H_k(\mathbb{M}, \mathbb{Q})$, and by convention $\det(Id_k - t\, f_{*k}) = 1$ if $n_k = 0$.

Note that the Lefschetz zeta function is a rational function with integers coefficients, so the power series defining it converges.

When $\mathbb{M}$ is a polyhedron there is the following alternative way to compute the Lefschetz zeta function

$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(Id_k - t\, f_{*k})^{(-1)^{k+1}},$$

where $n = \dim \mathbb{M}$ and Id_k is the identity map of $H_k(\mathbb{M}, \mathbb{Q})$, and by convention $\det(Id_k - t\, f_{*k}) = 1$ if $n_k = 0$.

Note that the Lefschetz zeta function is a rational function with integers coefficients, so the power series defining it converges.

Moreover, the Lefschetz zeta function with a finite number of integers (the coefficients of the rational function) keeps the information of the infinite sequence $\{L(f^m)\}_{m \in \mathbb{N}}$ for $m = 1, 2, \ldots$.

From the definition of Lefschetz zeta function and Proposition 1 it follows immediately the next result.

From the definition of Lefschetz zeta function and Proposition 1 it follows immediately the next result.

PROPOSITION 2

A necessary condition in order that a map $f : \mathbb{M} \to \mathbb{M}$ be periodic point free is that the Lefschetz zeta function $\mathcal{Z}_f(t) = 1$.

Let $\mathbb{G}$ be a connected compact graph such that $\dim_{\mathbb{Q}} H_1(\mathbb{G}, \mathbb{Q}) = r$, and let $f : \mathbb{G} \to \mathbb{G}$ be a continuous map.

Let $\mathbb{G}$ be a connected compact graph such that $\dim_{\mathbb{Q}} H_1(\mathbb{G}, \mathbb{Q}) = r$, and let $f : \mathbb{G} \to \mathbb{G}$ be a continuous map.

Then, f_{*1} is an $r \times r$ matrix, and f_{*0} is the 1×1 matrix (1) because $\mathbb{G}$ is connected.

Let $\mathbb{G}$ be a connected compact graph such that $\dim_{\mathbb{Q}} H_1(\mathbb{G}, \mathbb{Q}) = r$, and let $f : \mathbb{G} \to \mathbb{G}$ be a continuous map.

Then, f_{*1} is an $r \times r$ matrix, and f_{*0} is the 1×1 matrix (1) because $\mathbb{G}$ is connected.

Therefore, if $p(\lambda)$ is the characteristic polynomial of the matrix f_{*1}, we have

$$\mathcal{Z}_f(t) = \prod_{k=0}^{1} \det(Id_k - t\, f_{*k})^{(-1)^{k+1}} = \frac{\det(Id - tf_{*1})}{1-t}.$$

If $\text{Per}(f) = \emptyset$, by Proposition 2 we must have $\mathcal{Z}_f(t) = 1$.

If $\text{Per}(f) = \emptyset$, by Proposition 2 we must have $\mathcal{Z}_f(t) = 1$. Therefore, from the following equalities the characteristic polynomial of f_1^* must be $p(\lambda) = (-1)^r(\lambda^r - \lambda^{r-1})$, because then

$$\begin{aligned}
\mathcal{Z}_f(t) = & \frac{\det(Id - tf_{*1})}{1-t} = \frac{t^r \det\left(\frac{1}{t} Id - f_{*1}\right)}{1-t} \\
= & \frac{(-1)^r t^r \det\left(f_{*1} - \frac{1}{t} Id\right)}{1-t} = \frac{(-1)^r t^r p\left(\frac{1}{t}\right)}{1-t} \\
= & \frac{(-1)^{2r} t^r \left(\frac{1}{t^r} - \frac{1}{t^{r-1}}\right)}{1-t} = 1.
\end{aligned}$$

If $\text{Per}(f) = \emptyset$, by Proposition 2 we must have $\mathcal{Z}_f(t) = 1$. Therefore, from the following equalities the characteristic polynomial of f_1^* must be $p(\lambda) = (-1)^r(\lambda^r - \lambda^{r-1})$, because then

$$\begin{aligned}
\mathcal{Z}_f(t) &= \frac{\det(Id - tf_{*1})}{1-t} = \frac{t^r \det\left(\frac{1}{t} Id - f_{*1}\right)}{1-t} \\
&= \frac{(-1)^r t^r \det\left(f_{*1} - \frac{1}{t} Id\right)}{1-t} = \frac{(-1)^r t^r p\left(\frac{1}{t}\right)}{1-t} \\
&= \frac{(-1)^{2r} t^r \left(\frac{1}{t^r} - \frac{1}{t^{r-1}}\right)}{1-t} = 1.
\end{aligned}$$

Clearly the zeros of the characteristic polynomial $(-1)^r\lambda^{r-1}(\lambda - 1)$ are 1 and 0, this last with multiplicity $r-1$.

If $\mathrm{Per}(f) = \emptyset$, by Proposition 2 we must have $\mathcal{Z}_f(t) = 1$. Therefore, from the following equalities the characteristic polynomial of f_1^* must be $p(\lambda) = (-1)^r(\lambda^r - \lambda^{r-1})$, because then

$$\begin{aligned}\mathcal{Z}_f(t) = & \frac{\det(Id - tf_{*1})}{1-t} = \frac{t^r \det\left(\frac{1}{t}Id - f_{*1}\right)}{1-t} \\ = & \frac{(-1)^r t^r \det\left(f_{*1} - \frac{1}{t}Id\right)}{1-t} = \frac{(-1)^r t^r p\left(\frac{1}{t}\right)}{1-t} \\ = & \frac{(-1)^{2r} t^r \left(\frac{1}{t^r} - \frac{1}{t^{r-1}}\right)}{1-t} = 1.\end{aligned}$$

Clearly the zeros of the characteristic polynomial $(-1)^r\lambda^{r-1}(\lambda - 1)$ are 1 and 0, this last with multiplicity $r-1$. Hence the GRAPH THEOREM is proved.

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map.

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map.

The degree of f is d if $b = 0$.

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map.

The degree of f is d if $b = 0$.

We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$,

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map.

The degree of f is d if $b = 0$.

We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$,

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map.

The degree of f is d if $b = 0$.

We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$, $n_2 = 1$ if $b = 0$,

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map.

The degree of f is d if $b = 0$.

We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$, $n_2 = 1$ if $b = 0$, and $n_2 = 0$ if $b > 0$;

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map.

The degree of f is d if $b = 0$.

We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$, $n_2 = 1$ if $b = 0$, and $n_2 = 0$ if $b > 0$; and the induced linear maps $f_{*0} = (1)$, $f_{*2} = (d)$ if $b = 0$,

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map.

The degree of f is d if $b = 0$.

We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$, $n_2 = 1$ if $b = 0$, and $n_2 = 0$ if $b > 0$; and the induced linear maps $f_{*0} = (1)$, $f_{*2} = (d)$ if $b = 0$, and $f_{*2} = 0$ if $b > 0$.

Assume $b = 0$.

Assume $b = 0$.

Since $\mathrm{Per}(f) = \emptyset$, by Proposition 2 we must have $\mathcal{Z}_f(t) = 1$.

Assume $b = 0$.

Since $\text{Per}(f) = \emptyset$, by Proposition 2 we must have $\mathcal{Z}_f(t) = 1$.

The characteristic polynomial of f_1^* must be $p(\lambda) = \lambda^{2g} - (d+1)\lambda^{2g-1} + d\lambda^{2g-2}$, because then $\mathcal{Z}_f(t) =$ is

$$\prod_{k=0}^{2} \det(Id_k - t\, f_{*k})^{(-1)^{k+1}} = \frac{\det(Id - tf_{*1})}{(1-t)(1-dt)}$$

$$= \frac{t^{2g} \det\left(\frac{1}{t} Id - f_{*1}\right)}{1-(d+1)t+dt^2} = \frac{t^{2g} \det\left(f_{*1} - \frac{1}{t} Id\right)}{1-(d+1)t+dt^2} = \frac{t^{2g} p\left(\frac{1}{t}\right)}{1-(d+1)t+dt^2}$$

$$= \frac{t^{2g}\left(\frac{1}{t^{2g}} - (d+1)\frac{1}{t^{2g-1}} + d\frac{1}{t^{2g-2}}\right)}{1-(d+1)t+dt^2} = 1.$$

Clearly the zeros of the characteristic polynomial $\lambda^{2g} - (d+1)\lambda^{2g-1} + d\lambda^{2g-2} = \lambda^{2g-2}(\lambda - 1)(\lambda - d)$ are 1, d and 0, this last with multiplicity $2g-2$.

Clearly the zeros of the characteristic polynomial $\lambda^{2g} - (d+1)\lambda^{2g-1} + d\lambda^{2g-2} = \lambda^{2g-2}(\lambda - 1)(\lambda - d)$ are 1, d and 0, this last with multiplicity $2g-2$.

Hence the ORIENTABLE SURFACE THEOREM is proved when $b = 0$.

The proof of the ORIENTABLE SURFACE THEOREM when $b > 0$,

The proof of the ORIENTABLE SURFACE THEOREM when $b > 0$,

or the proof of the NON-ORIENTABLE SURFACE THEOREM are similar, and we do not do them in this talk.

The proof of the ORIENTABLE SURFACE THEOREM when $b > 0$,

or the proof of the NON-ORIENTABLE SURFACE THEOREM are similar, and we do not do them in this talk.

The talk is based in the article:

J. Llibre, Periodic point free continuous self–maps on graphs and surfaces, Topology and its Applications **159** (2012), 2228–2231.

The end

THANK YOU VERY MUCH FOR YOUR ATTENTION