Periodic point free continuous self–maps on graphs and surfaces

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We denote the set of periods of all the periodic points of f by $\text{Per}(f)$.

The following two results are well known:

Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map of degree d. If $\text{Per}(f) = \emptyset$ (i.e. if f is periodic point free), then $d = 1$.

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Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a continuous map of degree d. If $\text{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1 and d.

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Ll. Alsedà, S. Baldwin, J. Llibre, R. Swanson and W. Szlenk, Minimal sets of periods for torus maps via Nielsen numbers, Pacific J. of Math. **169** (1995), 1–32.

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The tools used for proving these results can be applied for studying the periodic point free continuous self–maps of many other compact absolute neighborhood retract spaces.

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A graph is a union of vertices (points) and edges, which are homeomorphic to the closed interval, and have mutually disjoint interiors. The endpoints of the edges are vertexes (not necessarily different) and the interiors of the edges are disjoint from the vertices.

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GRAPH THEOREM

Let $\mathbb{G}$ be a connected compact graph such that $\dim_{\mathbb{Q}} H_1(\mathbb{G}, \mathbb{Q}) = r$, and let $f : \mathbb{G} \to \mathbb{G}$ be a continuous map.

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An orientable connected compact surface WITH boundary of genus $g \geq 0$, $\mathbb{M}_{g,b}$, is homeomorphic to $\mathbb{M}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closures. In what follows $\mathbb{M}_{g,0} = \mathbb{M}_g$.

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Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map. The degree of f is d if $b = 0$.

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If $\text{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1, d and 0, this last with multiplicity $2g - 2$ if $b = 0$ and $g \geq 1$;

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Of course, as a corollary of the ORIENTABLE SURFACE THEOREM it follows the mentioned result for the 2-dimensional torus:

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Each of these spaces is a finite dimensional linear space over $\mathbb{Q}$.

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Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ it induces linear maps $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$ on the homological spaces of $\mathbb{M}$.

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Every linear map f_{*k} is given by an $n_k\times n_k$ matrix with integer entries, where n_k is the dimension of $H_k(\mathbb{M},\mathbb{Q})$.

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If we consider the Lefschetz number of f^m, in general, it is not true that $L(f^m) \neq 0$ implies that f has a periodic point of period m;

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If we consider the Lefschetz number of f^m, in general, it is not true that $L(f^m) \neq 0$ implies that f has a periodic point of period m; it only implies the existence of a periodic point of period a divisor of m.

From the Lefschetz Fixed Point Theorem it follows immediately the next result.

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PROPOSITION 1
Let $\mathbb{M}$ be a polyhedron. A necessary condition in order that a map $f : \mathbb{M} \to \mathbb{M}$ be periodic point free (i.e. $\text{Per}(f) = \emptyset$) is that all Lefschetz numbers $L(f^m)$ be zero for $m = 1, 2, 3, \ldots$.

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We define that a continuous self-map f of $\mathbb{M}$ is Lefschetz periodic point free if $L(f^m) = 0$ for $m = 1, 2, 3, \ldots$

The Lefschetz zeta function of f is defined as

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The Lefschetz zeta function $\mathcal{Z}_f(t)$ is a generating function for all the Lefschetz numbers of all iterates of f.

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$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(Id_k - t\, f_{*k})^{(-1)^{k+1}},$$

where $n = \dim \mathbb{M}$ and Id_k is the identity map of $H_k(\mathbb{M}, \mathbb{Q})$, and by convention $\det(Id_k - t\, f_{*k}) = 1$ if $n_k = 0$.

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Note that the Lefschetz zeta function is a rational function with integers coefficients,

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Note that the Lefschetz zeta function is a rational function with integers coefficients, so the power series defining it converges.

Moreover, the Lefschetz zeta function with a finite number of integers (the coefficients of the rational function) keeps the information of the infinite sequence $\{L(f^m)\}_{m\in\mathbb{N}}$ for $m = 1, 2, \ldots$.

From the definition of Lefschetz zeta function and Proposition 1 it follows immediately the next result.

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PROPOSITION 2
A necessary condition in order that a map $f : \mathbb{M} \to \mathbb{M}$ be periodic point free is that the Lefschetz zeta function $\mathcal{Z}_f(t) = 1$.

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Then, f_{*1} is an $r \times r$ matrix, and f_{*0} is the 1×1 matrix (1) because $\mathbb{G}$ is connected.

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Then, f_{*1} is an $r \times r$ matrix, and f_{*0} is the 1×1 matrix (1) because $\mathbb{G}$ is connected.

Therefore, if $p(\lambda)$ is the characteristic polynomial of the matrix f_{*1}, we have

$$\mathcal{Z}_f(t) = \prod_{k=0}^{1} \det(Id_k - t\, f_{*k})^{(-1)^{k+1}} = \frac{\det(Id - tf_{*1})}{1-t}.$$

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If $\mathrm{Per}(f) = \emptyset$, by Proposition 2 we must have $\mathcal{Z}_f(t) = 1$.

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If $\mathrm{Per}(f) = \emptyset$, by Proposition 2 we must have $\mathcal{Z}_f(t) = 1$. Therefore, from the following equalities the characteristic polynomial of f_1^* must be $p(\lambda) = (-1)^r(\lambda^r - \lambda^{r-1})$, because then

$$\begin{aligned}
\mathcal{Z}_f(t) &= \frac{\det(Id - tf_{*1})}{1-t} = \frac{t^r \det\left(\frac{1}{t} Id - f_{*1}\right)}{1-t} \\
&= \frac{(-1)^r t^r \det\left(f_{*1} - \frac{1}{t} Id\right)}{1-t} = \frac{(-1)^r t^r p\left(\frac{1}{t}\right)}{1-t} \\
&= \frac{(-1)^{2r} t^r \left(\frac{1}{t^r} - \frac{1}{t^{r-1}}\right)}{1-t} = 1.
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Clearly the zeros of the characteristic polynomial $(-1)^r\lambda^{r-1}(\lambda - 1)$ are 1 and 0, this last with multiplicity $r - 1$.

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Hence the GRAPH THEOREM is proved.

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map.

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The degree of f is d if $b = 0$.

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The degree of f is d if $b = 0$.

We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\ldots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$,

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Since $\text{Per}(f) = \emptyset$, by Proposition 2 we must have $\mathcal{Z}_f(t) = 1$.

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The characteristic polynomial of f_1^* must be $p(\lambda) = \lambda^{2g} - (d+1)\lambda^{2g-1} + d\lambda^{2g-2}$, because then $\mathcal{Z}_f(t) =$ is

$$\prod_{k=0}^{2} \det(Id_k - t\, f_{*k})^{(-1)^{k+1}} = \frac{\det(Id - tf_{*1})}{(1-t)(1-dt)}$$

$$= \frac{t^{2g}\det\left(\frac{1}{t}Id - f_{*1}\right)}{1-(d+1)t+dt^2} = \frac{t^{2g}\det\left(f_{*1} - \frac{1}{t}Id\right)}{1-(d+1)t+dt^2} = \frac{t^{2g}p\left(\frac{1}{t}\right)}{1-(d+1)t+dt^2}$$

$$= \frac{t^{2g}\left(\frac{1}{t^{2g}} - (d+1)\frac{1}{t^{2g-1}} + d\frac{1}{t^{2g-2}}\right)}{1-(d+1)t+dt^2} = 1.$$

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Clearly the zeros of the characteristic polynomial $\lambda^{2g} - (d+1)\lambda^{2g-1} + d\lambda^{2g-2} = \lambda^{2g-2}(\lambda-1)(\lambda-d)$ are 1, d and 0, this last with multiplicity $2g-2$.

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Hence the ORIENTABLE SURFACE THEOREM is proved when $b = 0$.

The proof of the ORIENTABLE SURFACE THEOREM when $b > 0$,

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The talk is based in the article:

J. Llibre, Periodic point free continuous self–maps on graphs and surfaces, Topology and its Applications **159** (2012), 2228–2231.

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The end

THANK YOU VERY MUCH FOR YOUR ATTENTION