On Rational Difference Equations with Periodic Coefficients

E. Drymonis$^1$ Y. Kostrov$^2$ Z. Kudlak$^3$

$^1$University of Rhode Island
Kingston, RI 02881
mdrymonis@math.uri.edu

$^2$Xavier University
New Orleans, LA 70125
ykostrov@xula.edu

$^3$Mount Saint Mary College
Newburgh, NY 12550
zachary.kudlak@msmc.edu

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Introduction

Definition:

A difference equation is a recurrence relation of the form
\[ x_{n+1} = f(x_n, x_{n-1}, \ldots). \]

For this talk, we will consider \( x_{n+1} = f(x_n, x_{n-1}) \), where \( f \) is a rational function.

When nonnegative initial conditions \( x_{-1} \) and \( x_0 \) are given in such a way that the denominator is nonzero, we say that the sequence \( \{x_n\}_{n=-1}^{\infty} \) is a solution to the difference equation, if the sequence satisfies the given relation.
Theorem 1 (Amleh, Camouzis, Ladas)

Let \( I \) be a set of real numbers and let

\[ f : I \times I \to I \]

be a function \( f(z_1, z_2) \) which increases in both variables. Then for every solution, \( \{x_n\}_{n=-1}^{\infty} \), of \( x_{n+1} = f(x_n, x_{n-1}) \), the subsequences \( \{x_{2n}\}_{n=0}^{\infty} \) and \( \{x_{2n+1}\}_{n=-1}^{\infty} \) do exactly one of the following:

(i) Eventually they are both monotonically increasing.

(ii) Eventually they are both monotonically decreasing.

(iii) One of them is monotonically increasing and the other is monotonically decreasing.
Theorem 2 (Camouzis, Ladas)

Let $I$ be a set of real numbers and suppose that

$$f : I \times I \rightarrow I$$

be a function $f(z_1, z_2)$ which decreases in $z_1$ and increases in $z_2$.

Then for every solution, \( \{x_n\}_{n=-1}^{\infty} \), of $x_{n+1} = f(x_n, x_{n-1})$, the subsequences \( \{x_{2n}\}_{n=0}^{\infty} \) and \( \{x_{2n+1}\}_{n=-1}^{\infty} \) are either

(i) both monotonically increasing,

(ii) both monotonically decreasing,

(iii) or eventually one subsequence is increasing and the other is decreasing.
We consider the second order difference equation of the form:

\[ x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, 2, \ldots \] (1)
Autonomous Equation

We consider the second order difference equation of the form:

\[
x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, 2, \ldots
\]

Note:

This equation was studied extensively in the following:


The Equation $x_{n+1} = \frac{\alpha_n}{1+x_nx_{n-1}}$

Equation (2)

$x_{n+1} = \frac{\alpha_n}{1 + x_nx_{n-1}}$, $n = 0, 1, 2, \ldots$ (2)

- The autonomous case, when $\alpha_n = \alpha$, was studied by Amleh, Camouzis and Ladas in [1].
- They showed that every solution was bounded for all values of $\alpha > 0$ and for all nonnegative initial conditions.
- They showed that every solution converged to a finite limit for $0 \leq \alpha < 2$ and for all initial nonnegative conditions.
- They conjectured that every solution converges for all values of $\alpha > 0$. 
Every solution of $x_{n+1} = \frac{\alpha}{1 + x_n x_{n-1}}$ converges

We have confirmed the conjecture by Amleh, Camouzis, and Ladas, namely,

**Theorem 3**

*Let $\alpha > 0$. Every solution to the equation $x_{n+1} = \frac{\alpha}{1 + x_n x_{n-1}}$ converges to a finite limit.*
Theorem 4

If $k > 0$, and $\{\alpha_n\}$ is a nonnegative sequence of real numbers with period-$k$, then every solution to the equation $x_{n+1} = \frac{\alpha_n}{1 + x_n x_{n-1}}$ is bounded.
Period-2 Convergence

Theorem 5

If \( \{\alpha_n\} = \{\alpha_0, \alpha_1, \alpha_0, \alpha_1, \ldots\} \), where \( \alpha_0, \alpha_1 \) are distinct, nonnegative real numbers, then every solution to the equation

\[
\alpha_n = \frac{\alpha_n}{1 + x_n x_{n-1}}
\]

converges to a unique prime period-two solution.
Sketch of Proof

- We begin by defining a new sequence

\[ z_{n+1} = x_{2n+1}x_{2n+2} \]  \hspace{1cm} (3)

\[ z_{n+1} = \frac{\alpha_0\alpha_1}{(1 + x_{2n}x_{2n-1})(1 + x_{2n+1}x_{2n})} \]  \hspace{1cm} (4)

\[ z_{n+1} = \frac{\alpha_0\alpha_1}{(1 + z_n)(1 + z_{n-1})}. \]  \hspace{1cm} (5)

- We then show that every solution, \( \{z_n\} \), to this difference equation converges.

- We use the change of variable \( z_n = \frac{\sqrt{\alpha_0\alpha_1}}{y_n} - 1 \) to transforms Eq. (5) into

\[ y_{n+1} = \frac{\sqrt{\alpha_0\alpha_1}}{1 + y_ny_{n-1}}. \]  \hspace{1cm} (6)

- And thus, the even and odd subsequences of the \( \{x_n\} \) solution converge to distinct limits if \( \alpha_0 \neq \alpha_1 \).
Advantageous Behavior

Definition

A difference equation with coefficients from a periodic environment, which converges to a periodic limit is said to be advantageous if the arithmetic mean of the periodic limits is greater than the limit of the autonomous case, with coefficients equal to the arithmetic mean of the periodic parameters.
Advantageous Behavior

**Definition**

A difference equation with coefficients from a periodic environment, which converges to a periodic limit is said to be **advantageous** if the arithmetic mean of the periodic limits is greater than the limit of the autonomous case, with coefficients equal to the arithmetic mean of the periodic parameters.

**Theorem 6**

If \( \{\alpha_n\} \) is a prime period-two sequence, then the equation

\[
x_{n+1} = \frac{\alpha_n}{1 + x_n x_{n-1}}
\]

is advantageous, in the sense that the average of the periodic limits is greater than the limit with the average of the coefficients.
The Advantageous Behavior of $x_{n+1} = \frac{\alpha_n}{1+x_nx_{n-1}}$

Figure: The first 50 terms, where $\alpha_0 = 0.5$, $\alpha_1 = 10.7$ compared to the autonomous equation with $\alpha = \frac{0.5 + 10.7}{2} = 5.6.$
Proof of Advantageous Behavior

- Define $a = \frac{\alpha_0 + \alpha_1}{2}$.

- Consider the autonomous equation

  $$y_{n+1} = \frac{a}{1 + y_n y_{n-1}}, \quad n = 0, 1, \ldots$$

- In [1], it is shown that this solution converges to $\bar{y}$, the unique positive solution to $\bar{y}^3 + \bar{y} - a = 0$.

- Define the equation $f(y) = y^3 + y - a$. 

Proof of Advantageous Behavior

- The \( \{z_n\} \) sequence has a unique positive equilibrium \( \bar{z} \) which is the positive root to the equation

\[
\bar{z}^3 + 2\bar{z}^2 + \bar{z} - \alpha_0\alpha_1 = 0
\]

- \( \{x_{2n+1}\} \) converges to \( \frac{\alpha_0}{1+\bar{z}} \).
- \( \{x_{2n}\} \) converges to \( \frac{\alpha_1}{1+\bar{z}} \).
- \( L = \frac{\frac{\alpha_0}{1+\bar{z}} + \frac{\alpha_1}{1+\bar{z}}}{2} = \frac{a}{1+\bar{z}} \)
Proof of Advantageous Behavior

We want to show that $f(L) > 0$.

$$f(L) = \frac{a^3}{(1 + \bar{z})^3} + \frac{a}{1 + \bar{z}} - a$$

$$= \frac{a(\alpha_0 - \alpha_1)^2}{4(1 + \bar{z})^3} \geq 0$$

This shows that when the coefficients have period-2, then the average of their limiting sequence will always be larger than a constant coefficient sequence with parameter with the same average.
The Equation $x_{n+1} = \frac{\alpha_n}{(1+x_n)x_{n-1}}$

We now consider the equation

$$x_{n+1} = \frac{\alpha_n}{(1 + x_n)x_{n-1}}, \quad n \geq 0$$

(7)

where $\{\alpha_n\}_{n=0}^{\infty}$ is a periodic sequence.

**Autonomous Case**

Amleh, Camouzis, and Ladas showed that the autonomous case of this equation possesses an invariant, namely,

$$x_{n-1} + x_n + x_{n-1}x_n + \alpha \left( \frac{1}{x_{n-1}} + \frac{1}{x_n} \right) = \text{constant}, \quad \forall n \geq 0.$$  

(8)

This implies that every solution of this equation is bounded from above and from below by positive constants.
Non-autonomous case

**Theorem 7**

Let \( \{\alpha_n\}_{n=0}^\infty = \{\alpha_0, \alpha_1, \alpha_0, \alpha_1, \ldots\} \) be a period-two sequence. Then, Equation (7) possesses an invariant, namely,

\[
x_{n-1} + x_n + x_{n-1}x_n + \frac{\alpha_n}{x_{n-1}} + \frac{\alpha_{n+1}}{x_n} = \text{constant}, \quad \forall n \geq 0. \tag{9}
\]
Non-autonomous case

Theorem 7

Let \( \{\alpha_n\}_{n=0}^{\infty} = \{\alpha_0, \alpha_1, \alpha_0, \alpha_1, \ldots\} \) be a period-two sequence. Then, Equation (7) possesses an invariant, namely,

\[
x_{n-1} + x_n + x_{n-1}x_n + \frac{\alpha_n}{x_{n-1}} + \frac{\alpha_{n+1}}{x_n} = \text{constant}, \forall n \geq 0.
\]  

(9)

Corollary 8

When \( \{\alpha_n\} \) is a period-two sequence, then every solution to Equation (7) is bounded by positive constants.
Non-autonomous case

**Theorem 7**

Let \( \{\alpha_n\}_{n=0}^\infty = \{\alpha_0, \alpha_1, \alpha_0, \alpha_1, \ldots\} \) be a period-two sequence. Then, Equation (7) possesses an invariant, namely,

\[
x_{n-1} + x_n + x_{n-1}x_n + \frac{\alpha_n}{x_{n-1}} + \frac{\alpha_{n+1}}{x_n} = \text{constant}, \forall n \geq 0.
\]  

(9)

**Corollary 8**

When \( \{\alpha_n\} \) is a period-two sequence, then every solution to Equation (7) is bounded by positive constants.

**Note:**

This partially answers an open question posed by Amleh, Camouzis, and Ladas in [1].
The Invariant of $x_{n+1} = \frac{\alpha_n}{(1+x_n)x_{n-1}}$

Figure: Showing the invariant cycles of the first 500 terms, $\alpha_0 = 2.5$, $\alpha_1 = 15.1$, $x_{-1} = 1.1$, $x_0 = 10.3$. 
Are there invariants for higher periods?

Figure: Showing the invariant cycles of the first 1000 terms, \( \alpha_0 = 1.1 \), \( \alpha_1 = 1.3 \), \( \alpha_2 = 1.0 \), \( x_{-1} = 1.1 \), \( x_0 = 1.0 \).
Are there invariants for higher periods?

Figure: Showing the invariant cycles of the first 1000 terms, $\alpha_0 = 1.1$, $\alpha_1 = 1.3$, $\alpha_2 = 1.0$, $x_{-1} = 1.1$, $x_0 = 2.0$. 
Are there invariants for higher periods?

Figure: Showing the invariant cycles of the first 100,000 terms, $\alpha_0 = 1.1$, $\alpha_1 = 1.3$, $\alpha_2 = 1.0$, $x_{-1} = 1.1$, $x_0 = 2.0$. 
The Equation \( x_{n+1} = \frac{\beta_n x_n x_{n-1}}{1 + x_n x_{n-1}} \)

We now consider the equation

\[
x_{n+1} = \frac{\beta_n x_n x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, 2, \ldots \tag{10}
\]

and \( \{\beta_n\}_{n=0}^{\infty} \) is a periodic sequence.

**Autonomous Case**

Amleh, Camouzis, and Ladas have shown that when \( \beta_n = \beta \), then every solution to Equation (10) converges to a finite limit.
Non-Autonomous Case

\[ x_{n+1} = \frac{\beta_n x_n x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, 2, \ldots \]

**Theorem 9**

*Every solution to Equation (10) is bounded when the coefficient \( \beta_n \) is periodic.*
Period-2 case

Consider
\[ x_{n+1} = \frac{\beta_n x_n x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, 2, \ldots \]

where \( \{\beta_n\} \) is a prime period-two sequence, \( \{\beta_0, \beta_1, \beta_0, \beta_1, \ldots\} \).

**Theorem 10**

Let \( B = \beta_0 \cdot \beta_1 \). Then:

(i.) For \( B < 4 \), every solution of \( x_{n+1} = \frac{\beta_n x_n x_{n-1}}{1 + x_n x_{n-1}} \) will converge to 0.

(ii.) For \( B \geq 4 \), every solution of \( x_{n+1} = \frac{\beta_n x_n x_{n-1}}{1 + x_n x_{n-1}} \) will converge to a period-2 solution.
Proof of Theorem 10

- \( z_{n+1} = 2^{n+1} x_{2n+1} x_{2n+2} \)

- \( z_{n+1} = \frac{B z_n z_{n-1}}{(1 + z_n)(1 + z_{n-1})} \)

- **Claim:** Every solution to \( z_{n+1} = \frac{B z_n z_{n-1}}{(1 + z_n)(1 + z_{n-1})} \) converges.

- Let us define a function \( f(x, y) \) such that \( z_{n+1} = f(z_n, z_{n-1}) \).

- \( \{z_n\}_{n=-1}^{\infty} \) converges according to the Amleh-Camouzis-Ladas Theorem.
Sketch of the proof

- Suppose that \( \lim_{n \to \infty} z_n = \bar{z} \).

- \( \bar{z}(1 + \bar{z})^2 = B\bar{z}^2 \)

- \( \bar{z} = 0 \) or \( \bar{z} = \frac{(B - 2) \pm \sqrt{B(B - 4)}}{2} \)

- If \( B < 4 \) then \( \bar{z} = 0 \) is the only equilibrium.

- If \( B = 4 \) then \( \bar{z} = 1 \).

- If \( B > 4 \) then there exist two positive equilibria \( \bar{z}_1 < \bar{z}_2 \).
The Equation $x_{n+1} = \frac{\gamma_n x_{n-1}}{1 + x_n x_{n-1}}$

We next consider the equation

$$x_{n+1} = \frac{\gamma_n x_{n-1}}{1 + x_n x_{n-1}} \quad (11)$$

Where $\{\gamma_n\}_{n=0}^{\infty}$ is a periodic sequence.

**Autonomous Case**

Amleh, Camouzis, and Ladas showed that when $\{\gamma_n\}$ is a constant sequence, every positive solution to Equation (11) is bounded.
Periodicity Destroys Boundedness of $x_{n+1} = \frac{\gamma_n x_{n-1}}{1 + x_n x_{n-1}}$

Assume now that $\{\gamma_n\}_{n=0}^\infty = \{\gamma_0, \gamma_1, \gamma_0, \gamma_1, \ldots\}$.

**Theorem 11**

*When $\{\gamma_n\}$ is a period-two sequence there exist unbounded solutions to equation (11).*

**Conditions for Unboundedness**

The following conditions for initial conditions $x_{-1}$ and $x_0$ and parameters $\gamma_0$ and $\gamma_1$ force an unbounded solution to Equation (11):

\[ x_{-1} < \gamma_0 < 1 < \gamma_1 < x_0 \quad (12) \]

\[ \gamma_0 \cdot \gamma_1 = 1 \quad (13) \]
An Unbounded Solution of $x_{n+1} = \frac{\gamma_n x_{n-1}}{1 + x_n x_{n-1}}$

Figure: The first 100 terms of the solution with $x_{-1} = 0.5$, $\gamma_0 = 0.95$, $\gamma_1 = 0.95^{-1}$, $x_0 = 1.5$. 
Sketch of the proof

\[ z_{n+1} = x_{2n+1} x_{2n+2} \]

\[ z_{n+1} = \frac{\gamma_0 \gamma_1 z_{n-1}}{(1 + z_n)(1 + z_{n-1})} \]
Sketch of the proof

- \( z_{n+1} = x_{2n+1}x_{2n+2} \)

- \( z_{n+1} = \frac{\gamma_0 \gamma_1 z_{n-1}}{(1 + z_n)(1 + z_{n-1})} \)

**Lemma 12**

*When* \( \gamma_0 \cdot \gamma_1 \leq 1 \), *zero is a globally asymptotically stable equilibrium of* \( \{z_n\} \).

Drymonis, Kostrov, Kudlak

On Rational Difference Equations with Periodic Coefficients
Consider
\[ f(x, y) = \frac{\gamma_0 \gamma_1 y}{(1 + x)(1 + y)}, \]

\( f(x, y) \) is decreasing in \( x \) and increasing in \( y \).

There are no period-two solutions when \( \gamma_0 \cdot \gamma_1 \leq 1 \).

The Camouzis-Ladas Theorem applies, and it follows that \( \{z_n\}_{n=-1}^\infty \) converges to a finite limit.

Furthermore,
\[ \bar{z}(1 + 2\bar{z} + \bar{z}^2) = \gamma_0 \gamma_1 \bar{z}, \quad (14) \]
\( \bar{z} = 0 \) is the unique solution when \( \gamma_0 \gamma_1 \leq 1 \).
Sketch of the proof

Let $x_{-1}, x_0, \gamma_0$ and $\gamma_1$ satisfying the following:

\[ x_{-1} < \gamma_0 < 1 < \gamma_1 < x_0 \text{ and } \gamma_0 \cdot \gamma_1 = 1 \]

Then

\[ x_1 = \frac{\gamma_0 x_{-1}}{1 + x_0 x_{-1}} < \gamma_0 x_{-1} < x_{-1} \]

Thus, $\{x_{2n+1}\}$ is decreasing, and must converge to zero.

Since $z_n = x_n \cdot x_{n-1} \to 0$, there exists some $N > 0$ such that for all $n \geq N$, $x_n \cdot x_{n-1} < \gamma_1 - 1$. 
Sketch of the proof

- We have:

\[ x_{2N+1} x_{2N} < \gamma_1 - 1 \]  \hspace{1cm} (15)
\[ \gamma_1 > 1 + x_{2N+1} x_{2N} \]  \hspace{1cm} (16)
\[ \frac{\gamma_1}{1 + x_{2N+1} x_{2N}} > 1. \]  \hspace{1cm} (17)

- Thus

\[ x_{2N+2} = \frac{\gamma_1 x_{2N}}{1 + x_{2N+1} x_{2N}} = \left( \frac{\gamma_1}{1 + x_{2N+1} x_{2N}} \right) x_{2N} > c \cdot x_{2N} \]

- Where \( c > 1 \) is a constant.

- \( \{x_{2n}\} \) is increasing without bound.
When does Equation (11) converge with Period-2 coefficients?

Theorem 13

If $\gamma_0, \gamma_1 \in [0, 1)$ then every positive solution of equation (11) converges to zero.
The Equation $x_{n+1} = \frac{\alpha_n + x_{n-1}}{(1+B_n x_n) x_{n-1}}$

Consider the equation

$$x_{n+1} = \frac{\alpha_n + x_{n-1}}{(1+B_n x_n) x_{n-1}}, \quad n \geq 0 \quad (18)$$

**Autonomous Case**

When $\{\alpha_n\}$ and $\{B_n\}$ are constant sequences, Amleh, Camouzis, and Ladas have shown that every solution to the equation is bounded.
Periodicity Destroys Boundedness of $x_{n+1} = \frac{\alpha_n + x_{n-1}}{(1 + B_n x_n) x_{n-1}}$

**Theorem 14**

*There exist unbounded solutions to*

$$x_{n+1} = \frac{\alpha_n + x_{n-1}}{(1 + B_n x_n) x_{n-1}}, \quad n \geq 0$$  \hspace{1cm} (19)

*when $\{\alpha_n\}$ and $\{B_n\}$ are sequences with period-three.*
An Unbounded Solution of $x_{n+1} = \frac{\alpha_n + x_{n-1}}{(1 + B_n x_n)x_{n-1}}$

Figure: $\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 2, B_0 = 1, B_1 = 2, B_2 = 1$
Sketch of the proof

Assume $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = 2$ and $B_0 = 1$, $B_1 = 2$, $B_2 = 1$. Consider the 3 sub-sequences defined by:

\[
\begin{align*}
x_{3n+1} &= \frac{1}{1 + x_{3n}} \\
x_{3n+2} &= \frac{1 + x_{3n}}{(1 + 2x_{3n+1})x_{3n}} \\
x_{3n+3} &= \frac{2 + x_{3n+1}}{(1 + x_{3n+2})x_{3n+1}}
\end{align*}
\]

It suffices to show that $\lim_{n \to \infty} x_{3n+3} = \infty$.

\[
\begin{align*}
x_{3n+2} &= \frac{(1 + x_{3n})^2}{(3 + x_{3n})x_{3n}} \\
x_{3n+3} &= \left(\frac{1 + 9x_{3n} + 2(x_{3n})^2}{1 + 5x_{3n} + 2(x_{3n})^2}\right)x_{3n}
\end{align*}
\]
Bibliography I


E. Camouzis and G. Ladas.
When does periodicity destroy boundedness in rational equations?

E. Camouzis and G. Ladas.

Rational systems in the plane.
Bibliography III

E. Camouzis, A. Gilbert, M. Heissan, and G. Ladas.
On the boundedness character of the system \( x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{x_n} \)
and \( y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + x_n + y_n} \).

E. Camouzis, E. Drymonis, and G. Ladas.
On the global character of the system \( x_{n+1} = \frac{a}{x_n + y_n} \) and
\( y_{n+1} = \frac{y_n}{B x_n + y_n} \).

E. Camouzis, E. Drymonis, and G. Ladas.
Patterns of boundedness of the rational system
\( x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{A_1 + C_1 y_n} \) and \( y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \).
E. Camouzis, C.M. Kent, G. Ladas, and C.D. Lynd.
On the global character of the solutions of the system
\[ x_{n+1} = \frac{\alpha_1 + y_n}{x_n} \text{ and } y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}. \]

E. Camouzis and G. Ladas.
Global results on rational systems in the plane, part 1.

Patterns of boundedness of the rational system
\[ x_{n+1} = \frac{\alpha_1}{A_1 + B_1 x_n + C_1 y_n} \text{ and } y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}. \]
E. Camouzis and G. Ladas.
Dynamics of rational systems in the plane; with open problems and conjectures.
In preparation.

J. M. Cushing and S. Henson.
Global dynamics of some periodically forced, monotone difference equations.
On the occasion of the 60th birthday of Calvin Ahlbrandt.

E. Camouzis, G. Ladas, and L. Wu.
On the global character of the system $x_{n+1} = \frac{\alpha_1 + \gamma_1 y_n}{x_n}$ and $y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{B_2 x_n + C_2 y_n}$.


Global dynamics of a competitive system of rational difference equations in the plane.

V. L. Kocić and G. Ladas.

M. R. S. Kulenović and M. Nurkanović. 
Asymptotic behavior of a two dimensional linear fractional system of difference equations. 
Dedicated to the memory of Prof. Dr. Naza Tanović-Miller.

M. R. S. Kulenović and M. Nurkanović. 
Asymptotic behavior of a system of linear fractional difference equations. 

Competitive-exclusion versus competitive-coexistence for systems in the plane. 
M. R. S. Kulenović and M. Nurkanović.  
Asymptotic behavior of a competitive system of linear fractional difference equations.  

G. Ladas.  
Open problems on the boundedness of some difference equations.  

On the global attractivity and the periodic character of some difference equations.  
On the occasion of the 60th birthday of Calvin Ahlbrandt.
E. Camouzis and G. Ladas.
Global convergence in difference equations.

E. Camouzis, E. Drymonis, and G. Ladas.
Patterns of boundedness of the rational system
\[ x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{A_1 + B_1 x_n + C_1 y_n} \quad \text{and} \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}. \]