On Rational Difference Equations with Periodic Coefficients

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Introduction

Definition:

A difference equation is a recurrence relation of the form $x_{n+1} = f(x_n, x_{n-1}, \ldots)$.

For this talk, we will consider $x_{n+1} = f(x_n, x_{n-1})$, where f is a rational function.

When nonnegative initial conditions x_{-1} and x_0 are given in such a way that the denominator is nonzero, we say that the sequence $\{x_n\}_{n=-1}^{\infty}$ is a solution to the difference equation, if the sequence satisfies the given relation.

Preliminary

Theorem 1 (Amleh, Camouzis, Ladas)

Let I be a set of real numbers and let

$$f : I \times I \to I$$

be a function $f(z_1, z_2)$ which increases in both variables. Then for every solution, $\{x_n\}_{n=-1}^{\infty}$, of $x_{n+1} = f(x_n, x_{n-1})$, the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ do exactly one of the following:

(i) *Eventually they are both monotonically increasing.*
(ii) *Eventually they are both monotonically decreasing.*
(iii) *One of them is monotonically increasing and the other is monotonically decreasing.*

Preliminary

Theorem 2 (Camouzis, Ladas)

Let I be a set of real numbers and suppose that

$$f : I \times I \to I$$

be a function $f(z_1, z_2)$ which decreases in z_1 and increases in z_2.

Then for every solution, $\{x_n\}_{n=-1}^{\infty}$, of $x_{n+1} = f(x_n, x_{n-1})$, the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ are either

(i) *both monotonically increasing,*
(ii) *both monotonically decreasing,*
(iii) *or eventually one subsequence is increasing and the other is decreasing.*

Autonomous Equation

We consider the second order difference equation of the form:

Equation (1)

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, 2, \dots \tag{1}$$

Autonomous Equation

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$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, 2, \dots \tag{1}$$

Note:

This equation was studied extensively in the following:

1. A.M. Amleh, E. Camouzis, G. Ladas, "On The Dynamics of Rational Difference Equations, Part 1," *International Journal of Difference Equations*, 3(1):1–35, 2008.
2. A.M. Amleh, E. Camouzis, G. Ladas, "On the Dynamics of Rational Difference Equations, Part 2," *International Journal of Difference Equations*, 3(2):195–225, 2008.

The Equation $x_{n+1} = \frac{\alpha_n}{1+x_n x_{n-1}}$

Equation (2)

$$x_{n+1} = \frac{\alpha_n}{1 + x_n x_{n-1}}, \quad n = 0, 1, 2, \dots \tag{2}$$

- The autonomous case, when $\alpha_n = \alpha$, was studied by Amleh, Camouzis and Ladas in [1].
- They showed that every solution was bounded for all values of $\alpha > 0$ and for all nonnegative initial conditions.
- They showed that every solution converged to a finite limit for $0 \le \alpha < 2$ and for all initial nonnegative conditions.
- They conjectured that every solution converges for all values of $\alpha > 0$.

Every solution of $x_{n+1} = \frac{\alpha}{1+x_n x_{n-1}}$ converges

We have confirmed the conjecture by Amleh, Camouzis, and Ladas, namely,

Theorem 3

Let $\alpha > 0$. Every solution to the equation $x_{n+1} = \dfrac{\alpha}{1 + x_n x_{n-1}}$ converges to a finite limit.

Boundedness

Theorem 4

If $k > 0$, and $\{\alpha_n\}$ is a nonnegative sequence of real numbers with period-k, then every solution to the equation $x_{n+1} = \dfrac{\alpha_n}{1 + x_n x_{n-1}}$ is bounded.

Period-2 Convergence

Theorem 5

If $\{\alpha_n\} = \{\alpha_0, \alpha_1, \alpha_0, \alpha_1, \ldots\}$, where α_0, α_1 are distinct, nonnegative real numbers, then every solution to the equation $x_{n+1} = \dfrac{\alpha_n}{1 + x_n x_{n-1}}$ converges to a unique prime period-two solution.

Sketch of Proof

- We begin by defining a new sequence

$$\begin{aligned} z_{n+1} &= x_{2n+1}x_{2n+2} && (3)\\ z_{n+1} &= \frac{\alpha_0\alpha_1}{(1 + x_{2n}x_{2n-1})(1 + x_{2n+1}x_{2n})} && (4)\\ z_{n+1} &= \frac{\alpha_0\alpha_1}{(1 + z_n)(1 + z_{n-1})}. && (5) \end{aligned}$$

- We then show that every solution, $\{z_n\}$, to this difference equation converges.
- We use the change of variable $z_n = \frac{\sqrt{\alpha_0\alpha_1}}{y_n} - 1$ to transforms Eq. (5) into

$$y_{n+1} = \frac{\sqrt{\alpha_0\alpha_1}}{1 + y_n y_{n-1}}. \qquad (6)$$

- And thus, the even and odd subsequences of the $\{x_n\}$ solution converge to distinct limits if $\alpha_0 \neq \alpha_1$.

Advantageous Behavior

Definition

A difference equation with coefficients from a periodic environment, which converges to a periodic limit is said to be **advantageous** if the arithmetic mean of the periodic limits is greater than the limit of the autonomous case, with coefficients equal to the arithmetic mean of the periodic parameters.

Advantageous Behavior

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Theorem 6

If $\{\alpha_n\}$ is a prime period-two sequence, then the equation $x_{n+1} = \dfrac{\alpha_n}{1 + x_n x_{n-1}}$ is **advantageous**, *in the sense that the average of the periodic limits is greater than the limit with the average of the coefficients.*

The Advantageous Behavior of $x_{n+1} = \frac{\alpha_n}{1+x_n x_{n-1}}$

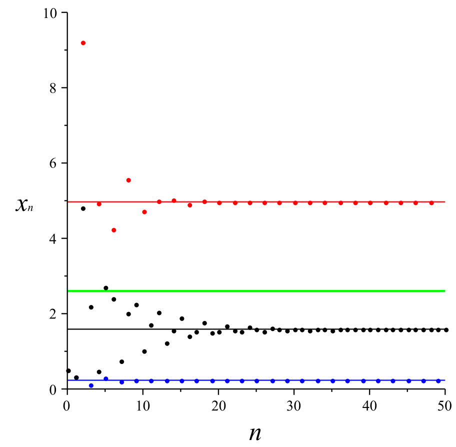

Figure: The first 50 terms, where $\alpha_0 = 0.5$, $\alpha_1 = 10.7$ compared to the autonomous equation with $\alpha = \frac{0.5+10.7}{2} = 5.6$.

Proof of Advantageous Behavior

- Define $a = \frac{\alpha_0+\alpha_1}{2}$.
- Consider the autonomous equation

$$y_{n+1} = \frac{a}{1 + y_n y_{n-1}}, \quad n = 0, 1, \ldots$$

- In [1], it is shown that this solution converges to $\bar{y}$, the unique positive solution to $\bar{y}^3 + \bar{y} - a = 0$.
- Define the equation $f(y) = y^3 + y - a$.

Proof of Advantageous Behavior

- The $\{z_n\}$ sequence has a unique positive equilibrium $\bar{z}$ which is the positive root to the equation

$$\bar{z}^3 + 2\bar{z}^2 + \bar{z} - \alpha_0\alpha_1 = 0$$

- $\{x_{2n+1}\}$ converges to $\frac{\alpha_0}{1+\bar{z}}$.
- $\{x_{2n}\}$ converges to $\frac{\alpha_1}{1+\bar{z}}$.
- $L = \dfrac{\frac{\alpha_0}{1+\bar{z}} + \frac{\alpha_1}{1+\bar{z}}}{2} = \dfrac{a}{1+\bar{z}}$

Proof of Advantageous Behavior

- We want to show that $f(L) > 0$.

$$f(L) = \frac{a^3}{(1+\bar{z})^3} + \frac{a}{1+\bar{z}} - a = \frac{a(\alpha_0 - \alpha_1)^2}{4(1+\bar{z})^3} \geq 0$$

- This shows that when the coefficients have period-2, then the average of their limiting sequence will always be larger than a constant coefficient sequence with parameter with the same average.

The Equation $x_{n+1} = \frac{\alpha_n}{(1+x_n)x_{n-1}}$

We now consider the equation

$$x_{n+1} = \frac{\alpha_n}{(1+x_n)x_{n-1}}, \; n \geq 0 \qquad (7)$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a periodic sequence.

Autonomous Case

Amleh, Camouzis, and Ladas showed that the autonomous case of this equation possesses an invariant, namely,

$$x_{n-1} + x_n + x_{n-1}x_n + \alpha\left(\frac{1}{x_{n-1}} + \frac{1}{x_n}\right) = \text{constant}, \forall n \geq 0. \qquad (8)$$

This implies that every solution of this equation is bounded from above and from below by positive constants.

Non-autonomous case

Theorem 7

Let $\{\alpha_n\}_{n=0}^{\infty} = \{\alpha_0, \alpha_1, \alpha_0, \alpha_1, \ldots\}$ be a period-two sequence. Then, Equation (7) possesses an invariant, namely,

$$x_{n-1} + x_n + x_{n-1}x_n + \frac{\alpha_n}{x_{n-1}} + \frac{\alpha_{n+1}}{x_n} = \text{constant}, \forall n \geq 0. \qquad (9)$$

Non-autonomous case

Theorem 7

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Corollary 8

When $\{\alpha_n\}$ is a period-two sequence, then every solution to Equation (7) is bounded by positive constants.

Non-autonomous case

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Corollary 8

When $\{\alpha_n\}$ is a period-two sequence, then every solution to Equation (7) is bounded by positive constants.

Note:

This partially answers an open question posed by Amleh, Camouzis, and Ladas in [1].

The Invariant of $x_{n+1} = \frac{\alpha_n}{(1+x_n)x_{n-1}}$

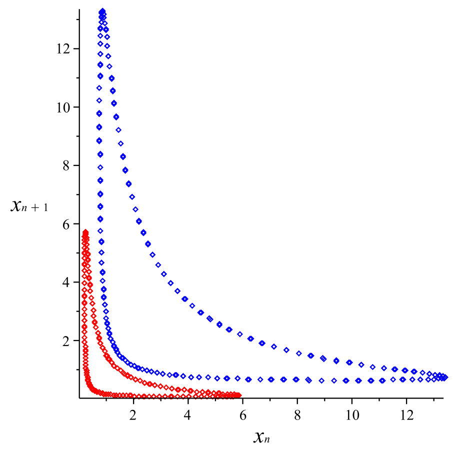

Figure: Showing the invariant cycles of the first 500 terms, $\alpha_0 = 2.5$, $\alpha_1 = 15.1$, $x_{-1} = 1.1$, $x_0 = 10.3$.

Are there invariants for higher periods?

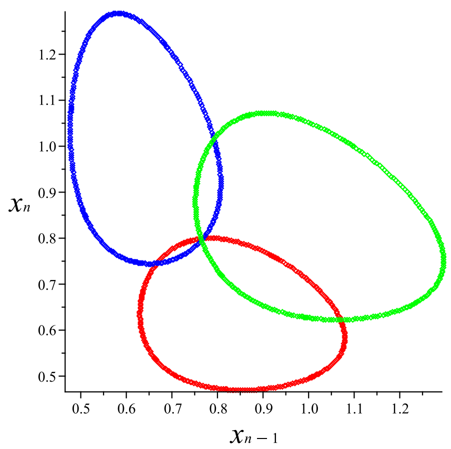

Figure: Showing the invariant cycles of the first 1000 terms, $\alpha_0 = 1.1$, $\alpha_1 = 1.3$, $\alpha_2 = 1.0$, $x_{-1} = 1.1$, $x_0 = 1.0$.

Are there invariants for higher periods?

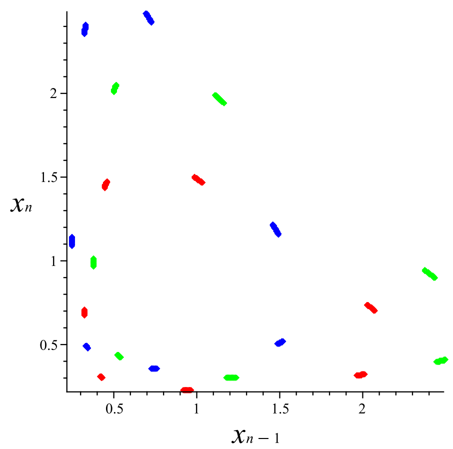

Figure: Showing the invariant cycles of the first 1000 terms, $\alpha_0 = 1.1$, $\alpha_1 = 1.3$, $\alpha_2 = 1.0$, $x_{-1} = 1.1$, $x_0 = 2.0$.

Are there invariants for higher periods?

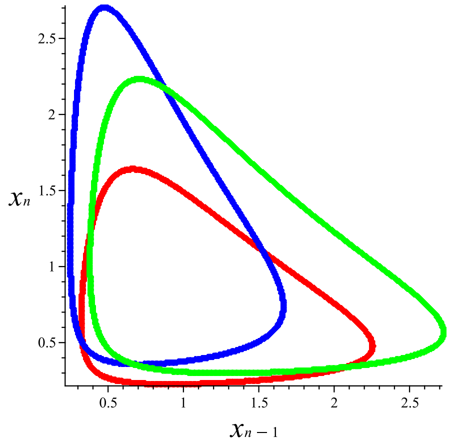

Figure: Showing the invariant cycles of the first 100, 000 terms, $\alpha_0 = 1.1$, $\alpha_1 = 1.3$, $\alpha_2 = 1.0$, $x_{-1} = 1.1$, $x_0 = 2.0$.

The Equation $x_{n+1} = \frac{\beta_n x_n x_{n-1}}{1+x_n x_{n-1}}$

We now consider the equation

Next Equation

$$x_{n+1} = \frac{\beta_n x_n x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, 2, \dots \tag{10}$$

and$\{\beta_n\}_{n=0}^{\infty}$ is a periodic sequence.

Autonomous Case

Amleh, Camouzis, and Ladas have shown that when $\beta_n = \beta$, then every solution to Equation (10) converges to a finite limit.

Non-Autonomous Case

$$x_{n+1} = \frac{\beta_n x_n x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, 2, \dots$$

Theorem 9

Every solution to Equation (10) *is bounded when the coefficient* β_n *is periodic.*

Period-2 case

Consider

$$x_{n+1} = \frac{\beta_n x_n x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, 2, \dots$$

where $\{\beta_n\}$ is a prime period-two sequence, $\{\beta_0, \beta_1, \beta_0, \beta_1, \dots\}$.

Theorem 10

Let $B = \beta_0 \cdot \beta_1$. *Then:*

(i.) *For* $B < 4$, *every solution of* $x_{n+1} = \frac{\beta_n x_n x_{n-1}}{1+x_n x_{n-1}}$ *will converge to* 0.

(ii.) *For* $B \geq 4$, *every solution of* $x_{n+1} = \frac{\beta_n x_n x_{n-1}}{1+x_n x_{n-1}}$ *will converge to a period-2 solution.*

Proof of Theorem 10

- $z_{n+1} = x_{2n+1}x_{2n+2}$
- $z_{n+1} = \frac{Bz_n z_{n-1}}{(1+z_n)(1+z_{n-1})}$
- **Claim:** Every solution to $z_{n+1} = \frac{Bz_n z_{n-1}}{(1+z_n)(1+z_{n-1})}$ converges.
- Let us define a function $f(x,y)$ such that $z_{n+1} = f(z_n, z_{n-1})$.
- $\{z_n\}_{n=-1}^{\infty}$ converges according to the Amleh-Camouzis-Ladas Theorem.

Sketch of the proof

- Suppose that $\lim_{n\to\infty} z_n = \bar{z}$.
- $\bar{z}(1+\bar{z})^2 = B\bar{z}^2$
- $\bar{z} = 0$ or $\bar{z} = \dfrac{(B-2) \pm \sqrt{B(B-4)}}{2}$
- If $B < 4$ then $\bar{z} = 0$ is the only equilibrium.
- If $B = 4$ then $\bar{z} = 1$.
- If $B > 4$ then there exist two positive equilibria $\bar{z}_1 < \bar{z}_2$.

The Equation $x_{n+1} = \frac{\gamma_n x_{n-1}}{1+x_n x_{n-1}}$

We next consider the equation

$$x_{n+1} = \frac{\gamma_n x_{n-1}}{1 + x_n x_{n-1}} \tag{11}$$

Where $\{\gamma_n\}_{n=0}^{\infty}$ is a periodic sequence.

Autonomous Case

Amleh, Camouzis, and Ladas showed that when $\{\gamma_n\}$ is a constant sequence, every positive solution to Equation (11) is bounded.

Periodicity Destroys Boundedness of $x_{n+1} = \frac{\gamma_n x_{n-1}}{1+x_n x_{n-1}}$

Assume now that $\{\gamma_n\}_{n=0}^{\infty} = \{\gamma_0, \gamma_1, \gamma_0, \gamma_1, \ldots\}$.

Theorem 11

When $\{\gamma_n\}$ is a period-two sequence there exist unbounded solutions to equation (11).

Conditions for Unboundedness

The following conditions for initial conditions x_{-1} and x_0 and parameters γ_0 and γ_1 force an unbounded solution to Equation (11):

$$x_{-1} < \gamma_0 < 1 < \gamma_1 < x_0 \tag{12}$$

$$\gamma_0 \cdot \gamma_1 = 1 \tag{13}$$

An Unbounded Solution of $x_{n+1} = \frac{\gamma_n x_{n-1}}{1+x_n x_{n-1}}$

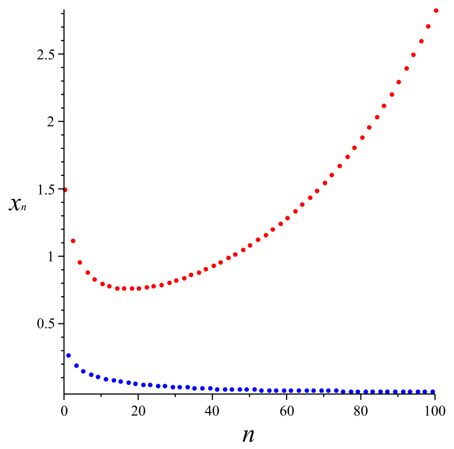

Figure: The first 100 terms of the solution with $x_{-1} = 0.5$, $\gamma_0 = 0.95$, $\gamma_1 = 0.95^{-1}$, $x_0 = 1.5$.

Sketch of the proof

- $z_{n+1} = x_{2n+1}x_{2n+2}$
- $z_{n+1} = \dfrac{\gamma_0\gamma_1 z_{n-1}}{(1+z_n)(1+z_{n-1})}$

Sketch of the proof

- $z_{n+1} = x_{2n+1}x_{2n+2}$
- $z_{n+1} = \dfrac{\gamma_0\gamma_1 z_{n-1}}{(1+z_n)(1+z_{n-1})}$

Lemma 12

When $\gamma_0 \cdot \gamma_1 \leq 1$, zero is a globally asymptotically stable equilibrium of $\{z_n\}$.

- Consider
$$f(x,y) = \frac{\gamma_0\gamma_1 y}{(1+x)(1+y)},$$
- $f(x,y)$ is decreasing in x and increasing in y.
- There are no period-two solutions when $\gamma_0 \cdot \gamma_1 \leq 1$.
- The Camouzis-Ladas Theorem applies, and it follows that $\{z_n\}_{n=-1}^{\infty}$ converges to a finite limit.
- Furthermore,
$$\bar{z}(1+2\bar{z}+\bar{z}^2) = \gamma_0\gamma_1\bar{z}, \tag{14}$$
- $\bar{z} = 0$ is the unique solution when $\gamma_0\gamma_1 \leq 1$.

Sketch of the proof

- Let x_{-1}, x_0, γ_0 and γ_1 satisfying the following:

$$x_{-1} < \gamma_0 < 1 < \gamma_1 < x_0 \text{ and } \gamma_0 \cdot \gamma_1 = 1$$

- Then

$$x_1 = \frac{\gamma_0 x_{-1}}{1 + x_0 x_{-1}} < \gamma_0 x_{-1} < x_{-1}$$

- Thus, $\{x_{2n+1}\}$ is decreasing, and must converge to zero.
- Since $z_n = x_n \cdot x_{n-1} \to 0$, there exists some $N > 0$ such that for all $n \geq N$, $x_n \cdot x_{n-1} < \gamma_1 - 1$.

Sketch of the proof

- We have:

$$\begin{aligned} x_{2N+1}x_{2N} &< \gamma_1 - 1 && (15) \\ \gamma_1 &> 1 + x_{2N+1}x_{2N} && (16) \\ \frac{\gamma_1}{1 + x_{2N+1}x_{2N}} &> 1. && (17) \end{aligned}$$

- Thus

$$x_{2N+2} = \frac{\gamma_1 x_{2N}}{1 + x_{2N+1}x_{2N}} = \left(\frac{\gamma_1}{1 + x_{2N+1}x_{2N}}\right) x_{2N} > c \cdot x_{2N}$$

- Where $c > 1$ is a constant.
- $\{x_{2n}\}$ is increasing without bound.

When does Equation (11) converge with Period-2 coefficients?

$$x_{n+1} = \frac{\gamma_n x_{n-1}}{1 + x_n x_{n-1}}$$

Theorem 13

If $\gamma_0, \gamma_1 \in [0, 1)$ then every positive solution of equation (11) converges to zero.

The Equation $x_{n+1} = \frac{\alpha_n + x_{n-1}}{(1+B_n x_n)x_{n-1}}$

Consider the equation

$$x_{n+1} = \frac{\alpha_n + x_{n-1}}{(1 + B_n x_n)x_{n-1}}, \quad n \geq 0 \qquad (18)$$

Autonomous Case

When $\{\alpha_n\}$ and $\{B_n\}$ are constant sequences, Amleh, Camouzis, and Ladas have shown that every solution to the equation is bounded.

Periodicity Destroys Boundedness of $x_{n+1} = \frac{\alpha_n + x_{n-1}}{(1+B_n x_n)x_{n-1}}$

Theorem 14

There exist unbounded solutions to

$$x_{n+1} = \frac{\alpha_n + x_{n-1}}{(1 + B_n x_n)x_{n-1}}, \quad n \geq 0 \tag{19}$$

when $\{\alpha_n\}$ and $\{B_n\}$ are sequences with period-three.

An Unbounded Solution of $x_{n+1} = \frac{\alpha_n + x_{n-1}}{(1+B_n x_n)x_{n-1}}$

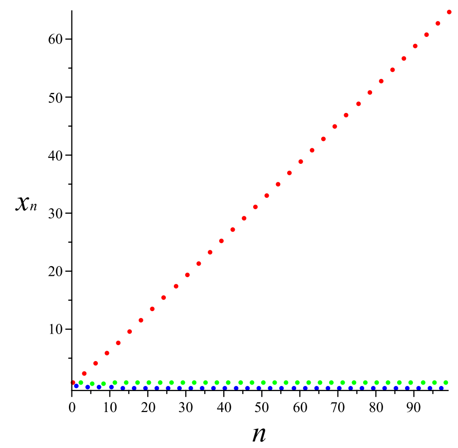

Figure: $\alpha_0 = 0,\ \alpha_1 = 1,\ \alpha_2 = 2,\ B_0 = 1,\ B_1 = 2,\ B_2 = 1$

Sketch of the proof

Assume $\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 2$ and $B_0 = 1, B_1 = 2, B_2 = 1$.
Consider the 3 sub-sequences defined by:

$$\begin{aligned}
x_{3n+1} &= \frac{1}{1 + x_{3n}} \\
x_{3n+2} &= \frac{1 + x_{3n}}{(1 + 2x_{3n+1})x_{3n}} \\
x_{3n+3} &= \frac{2 + x_{3n+1}}{(1 + x_{3n+2})x_{3n+1}}
\end{aligned}$$

It suffices to show that $\lim_{n\to\infty} x_{3n+3} = \infty$.

$$\begin{aligned}
x_{3n+2} &= \frac{(1 + x_{3n})^2}{(3 + x_{3n})x_{3n}} \\
x_{3n+3} &= \left(\frac{1 + 9x_{3n} + 2(x_{3n})^2}{1 + 5x_{3n} + 2(x_{3n})^2}\right) x_{3n}
\end{aligned}$$

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