Hypercyclic and Topologically Mixing Properties of Certain Classes of Abstract Time-Fractional Equations

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Abstract In recent years, considerable effort has been directed toward the topological dynamics of abstract PDEs whose solutions are governed by various types of operator semigroups, fractional resolvent operator families and evolution systems. In this paper, we shall present the most important results about hypercyclic and topologically mixing properties of some special subclasses of the abstract time-fractional equations of the following form:

$$\begin{aligned}&\mathbf{D}_t^{\alpha_n}u(t)+c_{n-1}\mathbf{D}_t^{\alpha_{n-1}}u(t)+\cdots+c_1\mathbf{D}_t^{\alpha_1}u(t)=A\mathbf{D}_t^{\alpha}u(t),\quad t>0,\\&u^{(k)}(0)=u_k,\quad k=0,\cdots,\lceil\alpha_n\rceil-1,\end{aligned}\tag{1}$$

where $n\in\mathbb{N}\setminus\{1\}$, A is a closed linear operator acting on a separable infinite-dimensional complex Banach space E, $c_1,\cdots,c_{n-1}$ are certain complex constants, $0\le\alpha_1<\cdots<\alpha_n$, $0\le\alpha<\alpha_n$, and $\mathbf{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order α ([5]). We slightly generalize results from [24] and provide several applications, including those to abstract higher order differential equations of integer order ([38]).

1 Introduction and Preliminaries

The last two decades have witnessed a growing interest in fractional derivatives and their applications. In this paper, we enquire into the basic hypercyclic and topologically mixing properties of some special subclasses of the abstract time-fractional equations of the form (1), continuing in such a way the research raised in [24]. Our main result is Theorem 2.3, which is the kind of Desch-Schappacher-Webb and Banasiak-Moszyński criteria for chaos

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of strongly continuous semigroups. For further information concerning hypercyclic and topologically mixing properties of single valued operators and abstract PDEs, we refer the reader to [2-4, 6-8, 10-22, 24-25, 33, 36-37]. A fairly complete information on the general theory of operator semigroups, cosine functions and abstract Volterra equations can be obtained by consulting the monographs [1, 9, 22, 35, 38].

Before going any further, it will be convenient to introduce the basic concepts used throughout the paper. We shall always assume that $(E,\|\cdot\|)$ is a separable infinite-dimensional complex Banach space, A and $A_1,\cdots,A_{n-1}$ are closed linear operators acting on E, $n\in\mathbb{N}\setminus\{1\}$, $0\le\alpha_1<\cdots<\alpha_n$ and $0\le\alpha<\alpha_n$. By I is denoted the identity operator on E. Given $s\in\mathbb{R}$, put $\lceil s\rceil:=\inf\{k\in\mathbb{Z}:s\le k\}$. Define $m_j:=\lceil\alpha_j\rceil$, $1\le j\le n$, $m:=m_0:=\lceil\alpha\rceil$, $A_0:=A$ and $\alpha_0:=\alpha$. The dual space of E and the space of continuous linear mappings from E into E are denoted by E^* and $L(E)$, respectively. By $D(A)$, $\mathrm{Kern}(A)$, $R(A)$, $\rho(A)$, $\sigma_p(A)$ and A^*, we denote the domain, kernel, range, resolvent set, point spectrum and adjoint operator of A, respectively. Suppose F is a closed subspace of E. Then the part of A in F, denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}):=\{x\in D(A)\cap F:Ax\in F\}$ and $A_{|F}x:=Ax$, $x\in D(A_{|F})$. In the sequel, we assume that $L(E)\ni C$ is an injective operator satisfying $CA\subseteq AC$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Set $\mathbb{N}_l:=\{1,\cdots,l\}$, $\mathbb{N}_l^0:=\{0,1,\cdots,l\}$, $0^\zeta:=0$, $g_\zeta(t):=t^{\zeta-1}/\Gamma(\zeta)$ ($\zeta>0$, $t>0$) and $g_0:=$ the Dirac δ-distribution. If $\delta\in(0,\pi]$, then we define $\Sigma_\delta:=\{\lambda\in\mathbb{C}:\lambda\ne0,\ |\arg(\lambda)|<\delta\}$. Denote by $\mathcal{L}$ and $\mathcal{L}^{-1}$ the Laplace transform and its inverse transform, respectively.

It is clear that the abstract Cauchy problem (1) is a special case of the following one:

$$\begin{aligned}&\mathbf{D}_t^{\alpha_n}u(t)+A_{n-1}\mathbf{D}_t^{\alpha_{n-1}}u(t)+\cdots+A_1\mathbf{D}_t^{\alpha_1}u(t)=A\mathbf{D}_t^{\alpha}u(t),\quad t>0,\\&u^{(k)}(0)=u_k,\quad k=0,\cdots,\lceil\alpha_n\rceil-1.\end{aligned}\tag{2}$$

In what follows, we shall briefly summarize the most important facts concerning the C-wellposedness of the problem (2).

Definition 1. A function $u\in C^{m_n-1}([0,\infty):E)$ is called a (*strong*) *solution* of (2) iff $A_i\mathbf{D}_t^{\alpha_i}u\in C([0,\infty):E)$ for $0\le i\le n-1$, $g_{m_n-\alpha_n}*(u-\sum_{k=0}^{m_n-1}u_kg_{k+1})\in C^{m_n}([0,\infty):E)$ and (2) holds. The abstract Cauchy problem (2) is said to be *C-wellposed* if:

1. For every $u_0,\cdots,u_{m_n-1}\in\bigcap_{0\le j\le n-1}C(D(A_j))$, there exists a unique solution $u(t;u_0,\cdots,u_{m_n-1})$ of (2).
2. For every $T>0$, there exists $c>0$ such that, for every $u_0,\cdots,u_{m_n-1}\in\bigcap_{0\le j\le n-1}C(D(A_j))$, the following holds:

$$\big\|u(t;u_0,\cdots,u_{m_n-1})\big\|\le c\sum_{k=0}^{m_n-1}\big\|C^{-1}u_k\big\|,\ t\in[0,T].$$

Although not of primary importance in our analysis, the following facts should be stated. The *Caputo fractional derivative* $\mathbf{D}_t^{\alpha_n}u$ is defined for those functions $u\in C^{m_n-1}([0,\infty):E)$ for which $g_{m_n-\alpha_n}*(u-\sum_{k=0}^{m_n-1}u_kg_{k+1})\in C^{m_n}([0,\infty):E)$. If this is the case, then we have $\mathbf{D}_t^{\alpha_n}u(t)=\frac{d^{m_n}}{dt^{m_n}}[g_{m_n-\alpha_n}*(u-\sum_{k=0}^{m_n-1}u_kg_{k+1})]$. Suppose $\beta>0$, $\gamma>0$ and $\mathbf{D}_t^{\beta+\gamma}u$ is defined. Then the equality $\mathbf{D}_t^{\beta+\gamma}u=\mathbf{D}_t^{\beta}\mathbf{D}_t^{\gamma}u$ does not hold in general. The validity of this equality can be proved provided that any of the following conditions holds:

1. $\gamma\in\mathbb{N}$,
2. $\lceil\beta+\gamma\rceil=\lceil\gamma\rceil$, or
3. $u^{(j)}(0)=0$ for $\lceil\gamma\rceil\le j\le\lceil\beta+\gamma\rceil-1$.

Suppose $u(t)\equiv u(t;u_0,\cdots,u_{m_n-1})$, $t\ge0$ is a strong solution of (2), with $f(t)\equiv0$ and initial values $u_0,\cdots,u_{m_n-1}\in R(C)$. Convoluting the both sides of (2) with $g_{\alpha_n}(t)$, and making use of the equality [5, (1.21)], it readily follows that $u(t)$, $t\ge0$ satisfies the following:

$$\begin{aligned}u(\cdot)-\sum_{k=0}^{m_n-1}u_kg_{k+1}(\cdot)&+\sum_{j=1}^{n-1}g_{\alpha_n-\alpha_j}*A_j\Big[u(\cdot)-\sum_{k=0}^{m_j-1}u_kg_{k+1}(\cdot)\Big]\\&=g_{\alpha_n-\alpha}*A\Big[u(\cdot)-\sum_{k=0}^{m-1}u_kg_{k+1}(\cdot)\Big].\end{aligned}\tag{3}$$

Given $i\in\mathbb{N}^0_{m_n-1}$ in advance, set $D_i:=\{j\in\mathbb{N}_{n-1}:m_j-1\ge i\}$. Plugging $u_j=0$, $0\le j\le m_n-1$, $j\ne i$, in (3), one gets:

$$\begin{aligned}&\Big[u(\cdot;0,\cdots,u_i,\cdots,0)-u_ig_{i+1}(\cdot)\Big]\\&\quad+\sum_{j\in D_i}g_{\alpha_n-\alpha_j}*A_j\Big[u(\cdot;0,\cdots,u_i,\cdots,0)-u_ig_{i+1}(\cdot)\Big]\\&\quad+\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\Big[g_{\alpha_n-\alpha_j}*A_ju(\cdot;0,\cdots,u_i,\cdots,0)\Big]\\&\quad=\begin{cases}g_{\alpha_n-\alpha}*Au(\cdot;0,\cdots,u_i,\cdots,0), & m-1<i,\\ g_{\alpha_n-\alpha}*A\Big[u(\cdot;0,\cdots,u_i,\cdots,0)-u_ig_{i+1}(\cdot)\Big], & m-1\ge i,\end{cases}\end{aligned}\tag{4}$$

where u_i appears in the i-th place ($0\le i\le m_n-1$) starting from 0. Suppose now $0<\tau\le\infty$, $0\ne K\in L^1_{loc}([0,\tau))$ and $k(t)=\int_0^tK(s)\,ds$, $t\in[0,\tau)$. Denote $R_i(t)C^{-1}u_i=(K*u(\cdot;0,\cdots,u_i,\cdots,0))(t)$, $t\in[0,\tau)$, $0\le i\le m-1$. Convoluting formally the both sides of (4) with $K(t)$, $t\in[0,\tau)$, one obtains that, for $0\le i\le m_n-1$:

$$\begin{aligned}&\Big[R_i(\cdot)C^{-1}u_i-\big(k*g_i\big)(\cdot)u_i\Big]+\sum_{j\in D_i}g_{\alpha_n-\alpha_j}*A_j\Big[R_i(\cdot)C^{-1}u_i-\big(k*g_i\big)(\cdot)u_i\Big]\\&\quad+\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\Big[g_{\alpha_n-\alpha_j}*A_jR_i(\cdot)C^{-1}u_i\Big]\\&\quad=\begin{cases}\big(g_{\alpha_n-\alpha}*AR_i\big)(\cdot)C^{-1}u_i, & m-1<i,\\ g_{\alpha_n-\alpha}*A\Big[R_i(\cdot)C^{-1}u_i-\big(k*g_i\big)(\cdot)u_i\Big], & m-1\ge i.\end{cases}\end{aligned}$$

Motivated by the above analysis, we introduce the following general definition.

Definition 2. Suppose $0<\tau\le\infty$, $k\in C([0,\tau))$, $C,\ C_1,\ C_2\in L(E)$, C and C_2 are injective. A sequence $((R_0(t))_{t\in[0,\tau)},\cdots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ of strongly continuous operator families in $L(E)$ is called a (*local*, if $\tau<\infty$):

1. *k-regularized C_1-existence propagation family for* (2) if $R_i(0)=(k*g_i)(0)C_1$ and:

$$\begin{aligned}&\Big[R_i(\cdot)x-\big(k*g_i\big)(\cdot)C_1x\Big]+\sum_{j\in D_i}A_j\Big[g_{\alpha_n-\alpha_j}*\Big(R_i(\cdot)x-\big(k*g_i\big)(\cdot)C_1x\Big)\Big]\\&\quad+\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}A_j\big(g_{\alpha_n-\alpha_j}*R_i\big)(\cdot)x\\&\quad=\begin{cases}A\big(g_{\alpha_n-\alpha}*R_i\big)(\cdot)x, & m-1<i,\\ A\Big[g_{\alpha_n-\alpha}*\big(R_i(\cdot)x-\big(k*g_i\big)(\cdot)C_1x\big)\Big](\cdot), & m-1\ge i,\end{cases}\end{aligned}$$

   for any $i=0,\cdots,m_n-1$ and $x\in E$.
2. *k-regularized C_2-uniqueness propagation family for* (2) if $R_i(0)=(k*g_i)(0)C_2$ and:

$$\begin{aligned}&\Big[R_i(\cdot)x-\big(k*g_i\big)(\cdot)C_2x\Big]+\sum_{j\in D_i}g_{\alpha_n-\alpha_j}*\Big[R_i(\cdot)A_jx-\big(k*g_i\big)(\cdot)C_2A_jx\Big]\\&\quad+\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\big(g_{\alpha_n-\alpha_j}*R_i(\cdot)A_jx\big)(\cdot)\\&\quad=\begin{cases}\big(g_{\alpha_n-\alpha}*R_i(\cdot)Ax\big)(\cdot), & m-1<i,\\ g_{\alpha_n-\alpha}*\Big[R_i(\cdot)Ax-\big(k*g_i\big)(\cdot)C_2Ax\Big](\cdot), & m-1\ge i,\end{cases}\end{aligned}$$

   for any $i=0,\cdots,m_n-1$ and $x\in\bigcap_{0\le i\le n-1}D(A_i)$.
3. *k-regularized C-resolvent propagation family for* (2) if $((R_0(t))_{t\in[0,\tau)},\cdots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ is a k-regularized C-uniqueness propagation family for (2), and if for every $t\in[0,\tau)$, $i\in\mathbb{N}^0_{m_n-1}$ and $j\in\mathbb{N}^0_{n-1}$, one has: $R_i(t)A_j\subseteq A_jR_i(t)$, $R_i(t)C=CR_i(t)$ and $CA_j\subseteq A_jC$.

Before proceeding further, we would like to draw the readers attention to the paper [26] for further information concerning some other types of

(C_1, C_2)-existence and uniqueness resolvent families which can be useful in the analysis of (inhomogeneous) abstract Cauchy problems of the form (2). Notice also the following: If A is a subgenerator of a k-regularized C-resolvent propagation family $((R_0(t))_{t\in[0,\tau)}, \cdots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ for (2), then, in general, there do not exist $a_i \in L^1_{loc}([0,\tau))$, $i \in \mathbb{N}^0_{m_n-1}$ and $k_i \in C([0,\tau))$ such that $(R_i(t))_{t\in[0,\tau)}$ is an (a_i, k_i)-regularized C-resolvent family with subgenerator A; cf. [22-23, 26-31] for the basic properties of (a,k)-regularized C-resolvent families and their applications in the study of abstract Cauchy problem (2). The notions of exponential boundedness and analyticity of k-regularized C-resolvent propagation families will be understood in the sense of [26].

In the sequel, we shall consider only global C-resolvent propagation families for (2), i.e., global k-regularized C-resolvent propagation families for (1) with $k(t) \equiv 1$; in the case $C = I$, such a resolvent family is also called a *resolvent propagation family for* (2), or simply a *resolvent propagation family*, if there is no risk for confusion. It will be assumed that every single operator family $(R_i(t))_{t\geq 0}$ of the tuple $((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$ is non-degenerate, i.e., that the supposition $R_i(t)x = 0$, $t \geq 0$ implies $x = 0$. Henceforward we shall assume that there exist complex constants $c_1, \cdots, c_{n-1}$ such that $A_j = c_j I$, $j \in \mathbb{N}_{n-1}$. Then it is also said that the operator A is a *subgenerator* of $((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$. The *integral generator* $\hat{A}$ of $((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$ is defined as the set of all pairs $(x, y) \in E \times E$ such that, for every $i = 0, \cdots, m_n - 1$ and $t \geq 0$, the following holds:

$$\begin{aligned}\Big[R_i(\cdot)x - \big(k * g_i\big)(\cdot)Cx\Big] &+ \sum_{j=1}^{n-1} c_j g_{\alpha_n-\alpha_j} * \Big[R_i(\cdot)x - \big(k * g_i\big)(\cdot)Cx\Big] \\ &+ \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} c_j \big[g_{\alpha_n-\alpha_j+i} * k\big](\cdot)Cx \\ &= \begin{cases} \big[g_{\alpha_n-\alpha} * R_i\big](\cdot)y, & m-1 < i, \\ g_{\alpha_n-\alpha} * \Big[R_i(\cdot)y - \big(k * g_i\big)(\cdot)Cy\Big], & m-1 \geq i. \end{cases}\end{aligned}$$

By a *mild solution* of (3) we mean any function $u \in C([0,\infty) : E)$ such that the following holds:

$$\begin{aligned}u(\cdot) - \sum_{k=0}^{m_n-1} u_k g_{k+1}(\cdot) &+ \sum_{j=1}^{n-1} c_j g_{\alpha_n-\alpha_j} * \Big[u(\cdot) - \sum_{k=0}^{m_j-1} u_k g_{k+1}(\cdot)\Big] \\ &= A\Big(g_{\alpha_n-\alpha} * \Big[u(\cdot) - \sum_{k=0}^{m-1} u_k g_{k+1}(\cdot)\Big]\Big);\end{aligned}$$

a *strong solution* is any function $u \in C([0,\infty) : E)$ satisfying (3). It is clear that every strong solution of (3) is also a mild solution of the same problem; the converse statement is not true, in general. In the sequel, we shall always

assume that, for every $i \in \mathbb{N}^0_{m_n-1}$ with $m - 1 \geq i$, one has: $\mathbb{N}_{n-1} \setminus D_i \neq \emptyset$ and $\sum_{j\in\mathbb{N}_{n-1}\setminus D_i} |c_j|^2 > 0$. Then the problem (3) has at most one mild (strong) solution; cf. [26] for more details.

The proof of following auxiliary lemma follows from an application of [26, Theorem 2.12].

Lemma 1. *1. Suppose A generates an exponentially bounded, analytic C-regularized semigroup of angle $\beta \in (0, \pi/2]$ and A is densely defined. Then A is the integral generator of an exponentially bounded, analytic C-regularized propagation family $((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$ of angle $\min(\frac{\frac{\pi}{2}+\beta}{\alpha_n-\alpha} - \frac{\pi}{2}, \frac{\pi}{2})$, provided that $\frac{\pi}{2} + \beta > \frac{\pi}{2}(\alpha_n - \alpha)$.*
2. Suppose A generates an exponentially bounded C-regularized semigroup and A is densely defined. Then A is the integral generator of an exponentially bounded, analytic C-regularized propagation family $((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$ of angle $\min(\frac{\pi}{2(\alpha_n-\alpha)} - \frac{\pi}{2}, \frac{\pi}{2})$, provided that $\frac{\pi}{2} < \frac{\pi}{2(\alpha_n-\alpha)}$.

We refer the reader to [24, Definition 1.1] for the notion of a global α_n-times C-regularized resolvent family. If $n = 2$, $c_1 = 0$, $\alpha = 0$, and A is a subgenerator of a global C-regularized propagation family $((R_0(t))_{t\geq 0}, \cdots, (R_{m_2-1}(t))_{t\geq 0})$, then it is obvious that $(R_0(t))_{t\geq 0}$ is a global α_2-times C-regularized resolvent family having A as subgenerator. In our recent paper [24], we have considered hypercyclic and topologically mixing properties of fractional C-regularized resolvent families. Therefore, the results of this paper can be viewed as generalizations of corresponding results from [24].

Suppose $\beta > 0$ and $\gamma > 0$. Then the *Mittag-Leffler function* $E_{\beta,\gamma}(z)$ is defined by $E_{\beta,\gamma}(z) := \sum_{n=0}^{\infty} z^n / \Gamma(\beta n + \gamma)$, $z \in \mathbb{C}$. Set, for short, $E_\beta(\cdot) := E_{\beta,1}(\cdot)$. The following asymptotic formulae for the Mittag-Leffler functions ([5], [32]) play a crucial role in our analysis:

$$E_\alpha(z) = \frac{1}{\alpha} e^{z^{1/\alpha}} + \varepsilon_\alpha(z), \ |\arg(z)| < \alpha\pi/2, \tag{5}$$

and

$$E_\alpha(z) = \varepsilon_\alpha(z), \ |\arg(-z)| < \pi - \alpha\pi/2, \tag{6}$$

where

$$\varepsilon_\alpha(z) = \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(1 - \alpha n)} + O\big(|z|^{-N}\big), \ |z| \to \infty. \tag{7}$$

2 Hypercyclicity and Topologically Mixing Property for C-Resolvent Propagation Families

We recall the basic notations used henceforward: E is a separable infinite-dimensional complex Banach space, A is a closed linear operator on E, $n \in$

$\mathbb{N} \setminus \{1\}$, $0 \leq \alpha_1 < \cdots < \alpha_n$, $0 \leq \alpha < \alpha_n$, $A_j = c_j I$ for certain complex constants $c_1, \cdots, c_{n-1}$, $m_j = \lceil \alpha_j \rceil$, $1 \leq j \leq n$, $m = m_0 = \lceil \alpha \rceil$, $A_0 = A$ and $\alpha_0 = \alpha$. We assume, in addition, that $C^{-1}AC = A$ is densely defined and that A is a subgenerator of a global C-resolvent propagation family $((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$. Then we know (see [26]) that A is, in fact, the integral generator of $((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$.

Let $i \in \mathbb{N}^0_{m_n-1}$. Then we denote by $Z_i(A)$ the set which consists of those vectors $x \in E$ such that $R_i(t)x \in R(C)$, $t \geq 0$ and that the mapping $t \mapsto C^{-1}R_i(t)x$, $t \geq 0$ is continuous. Then $R(C) \subseteq Z_i(A)$, and it can be simply proved with the help of [26, Theorem 2.8] that $x \in Z_i(A)$ iff there exists a unique mild solution of (3) with $u_k = \delta_{k,i}x$, $k \in \mathbb{N}^0_{m_n-1}$; if this is the case, the unique mild solution of (3) is given by $u(t;x) := u_i(t;x) := C^{-1}R_i(t)x$, $t \geq 0$.

The Laplace transform can be used to prove the following extension of [24, Lemma 2.1].

Lemma 2. *Suppose $\lambda \in \mathbb{C}$, $x \in E$ and $Ax = \lambda x$. Then $x \in Z_i(A)$ and the unique strong solution of (3) is given by*

$$u_i(t;x) = \mathcal{L}^{-1}\Big(\frac{z^{-i-1} + \sum_{j\in D_i} c_j z^{-\alpha_n-i-1+\alpha_j}}{1 + \sum_{j=1}^{n-1} c_j z^{\alpha_j-\alpha_n} - \lambda z^{\alpha-\alpha_n}}\Big)(t)x,$$

for any $t \geq 0$ and $i \in \mathbb{N}^0_{m_n-1}$.

Set $P_\lambda := \lambda^{\alpha_n-\alpha} + \sum_{j=1}^{n-1} c_j \lambda^{\alpha_j-\alpha}$, $\lambda \in \mathbb{C} \setminus \{0\}$ and

$$F_i(\lambda, t) := \mathcal{L}^{-1}\Big(\frac{z^{-i-1} + \sum_{j\in D_i} c_j z^{-\alpha_n-i-1+\alpha_j}}{1 + \sum_{j=1}^{n-1} c_j z^{\alpha_j-\alpha_n} - P_\lambda z^{\alpha-\alpha_n}}\Big)(t),$$

for any $t \geq 0$, $i \in \mathbb{N}^0_{m_n-1}$ and $\lambda \in \mathbb{C} \setminus \{0\}$.

Definition 3. Let $i \in \mathbb{N}^0_{m_n-1}$, and let $\tilde{E}$ be a closed linear subspace of E. Then it is said that $(R_i(t))_{t\geq 0}$ is:

1. $\tilde{E}$-*hypercyclic* iff there exists $x \in Z_i(A) \cap \tilde{E}$ such that $\{C^{-1}R_i(t)x : t \geq 0\}$ is a dense subset of $\tilde{E}$; such an element is called a $\tilde{E}$-*hypercyclic vector* of $(R_i(t))_{t\geq 0}$;
2. $\tilde{E}$-*topologically transitive* iff for every $y, z \in \tilde{E}$ and for every $\varepsilon > 0$, there exist $x \in Z_i(A) \cap \tilde{E}$ and $t \geq 0$ such that $||y - x|| < \varepsilon$ and $||z - C^{-1}R_i(t)x|| < \varepsilon$;
3. $\tilde{E}$-*topologically mixing* iff for every $y, z \in \tilde{E}$ and for every $\varepsilon > 0$, there exists $t_0 \geq 0$ such that, for every $t \geq t_0$, there exists $x_t \in Z_i(A) \cap \tilde{E}$ such that $||y - x_t|| < \varepsilon$ and $||z - C^{-1}R_i(t)x_t|| < \varepsilon$.

In the case $\tilde{E} = E$, it is also said that a $\tilde{E}$-hypercyclic vector of $(R_i(t))_{t\geq 0}$ is a *hypercyclic vector* of $(R_i(t))_{t\geq 0}$ and that $(R_i(t))_{t\geq 0}$ is *topologically transitive*, resp. *topologically mixing*.

Suppose $C = I$, $\tilde{E} = E$ and $(R_i(t))_{t\geq 0}$ is topologically transitive for some $i \in \mathbb{N}^0_{m_n-1}$. Then $(R_i(t))_{t\geq 0}$ is hypercyclic and the set of all hypercyclic vectors of $(R_i(t))_{t\geq 0}$, denoted by $\mathrm{HC}(R_i)$, is a dense G_δ-subset of E ([16]). Furthermore, the condition $\rho(A) \neq \emptyset$ combined with the proofs of [19, Lemma 3.1, Theorem 3.2] implies that $\mathrm{HC}(R_i) \cap D_\infty(A)$ is a dense subset of E.

The proof of following theorem follows from Lemma 2 and the argumentation used in the proof of [24, Theorem 2.3].

Theorem 1. *Suppose $i \in \mathbb{N}^0_{m_n-1}$, Ω is an open connected subset of $\mathbb{C}$, $\Omega \cap (-\infty, 0] = \emptyset$ and $P_\Omega := \{P_\lambda : \lambda \in \Omega\} \subseteq \sigma_p(A)$. Let $f : P_\Omega \to E$ be an analytic mapping such that $f(P_\lambda) \in Kern(P_\lambda - A) \setminus \{0\}$, $\lambda \in \Omega$ and let $\tilde{E} := \overline{span\{f(P_\lambda) : \lambda \in \Omega\}}$. Suppose Ω_+ and Ω_- are two open connected subsets of Ω, and each of them admits a cluster point in Ω. If*

$$\lim_{t\to+\infty} \big|F_i(\lambda, t)\big| = +\infty, \ \lambda \in \Omega_+ \ \textit{and} \ \lim_{t\to+\infty} F_i(\lambda, t) = 0, \ \lambda \in \Omega_-, \tag{8}$$

then $(R_i(t))_{t\geq 0}$ is $\tilde{E}$-topologically mixing.

Remark 1. 1. Assume that $\langle x^*, f(P_\lambda)\rangle = 0$, $\lambda \in \Omega$ for some $x^* \in E^*$ implies $x^* = 0$. Then $\tilde{E} = E$.
2. It is not clear how one can prove an extension of [14, Theorem 2.1] for the most simplest time-fractional evolution equations of the form (1).
3. The previous theorem can be slightly improved in the following manner. Suppose $l \in \mathbb{N}$, $\Omega_1, \cdots, \Omega_l$ are open connected subsets $\Omega_1, \cdots \Omega_l$ of $\mathbb{C}$, as well as $\Omega_{j,+}$ and $\Omega_{j,-}$ are open connected subsets of Ω_j which admits a cluster point in Ω_j, and satisfy (8) with Ω_+ and Ω_- replaced respectively by $\Omega_{j,+}$ and $\Omega_{j,-}$ $(1 \leq j \leq l)$. Assume, additionally, that $f_j : P_{\Omega_j} \to E$ is an analytic mapping, $\Omega_j \cap (-\infty, 0] = \emptyset$, $P_{\Omega_j} \subseteq \sigma_p(A)$, and $f_j(P_\lambda) \in \mathrm{Kern}(A - P_\lambda) \setminus \{0\}$, $\lambda \in \Omega_j$ $(1 \leq j \leq l)$. Set $\tilde{E} := \overline{span\{f_j(P_\lambda) : \lambda \in \Omega_j, \ 1 \leq j \leq n\}}$ and assume that Ω'_j is an open connected subset of Ω_j which admits a cluster point in Ω_j for $1 \leq j \leq l$. Then
$$\tilde{E} = \overline{span\big\{f_j\big(P_\lambda\big) : \lambda \in \Omega'_j, \ 1 \leq j \leq l\big\}}$$
and one can repeat literally the proof of Theorem 1 in order to see that $(R_i(t))_{t\geq 0}$ is $\tilde{E}$-topologically mixing (cf. also [7]).
4. Let $Cf(P_\lambda) \in \tilde{E}$, $\lambda \in \Omega$. Then $A_{|\tilde{E}}$ is the densely defined integral generator of the $C_{|\tilde{E}}$-resolvent propagation family $((R_0(t)_{|\tilde{E}})_{t\geq 0}, \cdots, (R_{m_n-1}(t)_{|\tilde{E}})_{t\geq 0})$ in the Banach space $\tilde{E}$, $C^{-1}_{|\tilde{E}} A_{|\tilde{E}} C_{|\tilde{E}} = A_{|\tilde{E}}$ and the proof of Theorem 1 implies that $((R_0(t)_{|\tilde{E}})_{t\geq 0}, \cdots, (R_{m_n-1}(t)_{|\tilde{E}})_{t\geq 0})$ is topologically mixing in $\tilde{E}$. The additional assumption $\overline{C(\tilde{E})} = \tilde{E}$ implies that $((R_0(t)_{|\tilde{E}})_{t\geq 0}, \cdots, (R_{m_n-1}(t)_{|\tilde{E}})_{t\geq 0})$ is hypercyclic and that the set of all hypercyclic vectors of $((R_0(t)_{|\tilde{E}})_{t\geq 0}, \cdots, (R_{m_n-1}(t)_{|\tilde{E}})_{t\geq 0})$ is dense in $\tilde{E}$.

5. The assumptions of Theorem 1 hold provided that $n=2$, $c_1=0$, $\alpha_2>0$, $\alpha=0$, $i=0$ and $\Omega \cap i\mathbb{R} \neq \emptyset$ ([24]). In this case, $F_0(\lambda,t)=E_{\alpha_2}(\lambda^{\alpha_2}t^{\alpha_2})$, $t\geq 0$, and there exist $\lambda_0 \in \Omega$ and $\delta>0$ such that (8) holds with $\Omega_+ = \{\lambda \in \Omega : |\lambda-\lambda_0|<\delta,\ \arg(\lambda)\in(\frac{\pi}{2}-\delta,\frac{\pi}{2})\}$ and $\Omega_- = \{\lambda \in \Omega : |\lambda-\lambda_0|<\delta,\ \arg(\lambda)\in(\frac{\pi}{2},\frac{\pi}{2}+\delta)\}$.
6. It is worth noting that the condition (8) of Theorem 1 does not hold in general. In order to illustrate this, we shall present two simple counterexamples. Consider first the case: $n=2$, $\alpha_2-\alpha=1$, $\alpha_1-\alpha=-1$, $c_1>0$, $i=0$ and $D_0=\{1\}$ (notice that in the final part of Example 1(1), given below, one has $D_0=\emptyset$). Then, for every $t\geq 0$,

$$F_0(\lambda,t)=\Bigl(1+\frac{c_1}{\lambda^2-c_1}\Bigr)\Bigl(1+\frac{1}{\lambda^2}\Bigr)e^{\lambda t}-\frac{c_1}{\lambda^2-c_1}\Bigl(1+\frac{\lambda^2}{c_1^2}\Bigr)e^{c_1t/\lambda}+\frac{1}{c_1},$$

which shows that there does not exist an open connected subset Ω_- of $\mathbb{C}$ such that $\lim_{t\to+\infty}F_0(\lambda,t)=0$, $\lambda\in\Omega_-$. Suppose now $n=4$, $\alpha_j=j-1$, $j\in\mathbb{N}_4$, $\alpha=1$, $i=2$ and $c_1\in\mathbb{C}\setminus\{0\}$. Then $D_2=\emptyset$ and, for every $t\geq 0$,

$$F_2(\lambda,t)=\frac{e^{\lambda t}}{(\lambda-\lambda_1)(\lambda-\lambda_2)}+\frac{e^{\lambda_1 t}}{(\lambda_1-\lambda)(\lambda_1-\lambda_2)}+\frac{e^{\lambda_2 t}}{(\lambda_2-\lambda)(\lambda_2-\lambda_1)},$$

where $\lambda_{1,2}:=(-\lambda^2\pm\sqrt{\lambda^4+4c_1\lambda})/(2\lambda)$. It is not difficult to prove that, for every $\lambda\in\mathbb{C}\setminus\{0\}$, the following relation holds: $\Re\lambda\neq\Re\lambda_1$. This implies that, for every $\lambda\in\mathbb{C}$ with $\Re\lambda>0$, one has $\lim_{t\to+\infty}|F_2(\lambda,t)|=+\infty$. Regrettably, there does not exist an open connected subset Ω_- of $\mathbb{C}$ such that $\lim_{t\to+\infty}F_2(\lambda,t)=0$, $\lambda\in\Omega_-$.
7. As far as we know, in the handbooks containing tables of Laplace transforms, the explicit forms of functions like $F_i(\lambda,t)$ have not been presented as known images, except for some very special cases of the coefficients α_j, c_j. In this place, we would like to point out the following fact. Suppose $\alpha_n-\alpha_j\in\mathbb{Q}$, $j\in\mathbb{N}^0_{n-1}$. By the well-known formula [5, (1.26)], we obtain that there exists a number $\zeta\in(0,1)$, independent of λ, such that the function $F_i(\lambda,t)$ can be represented as the finite convolution products of functions like $E_{1,\zeta}(p_\lambda t)$.

We recommend for the reader the reference [25] for the basic hypercyclic and chaotic properties of fractionally integrated C-cosine functions. Notice that, in general, the notion of chaoticity makes no sense for the equations of the form (1).

We shall omit the proof of the following extension of [24, Theorem 2.4].

Theorem 2. *Suppose $R(C)$ is dense in E and there exists $i\in\mathbb{N}^0_{m_n-1}$ such that $(R_i(t))_{t\geq 0}$ is hypercyclic. Then $\sigma_p(A^*)=\emptyset$.*

We close the paper by giving some illustrative examples (for some other applications, the reader may consult the references [2-4, 10, 15, 33, 36-37]).

Example 1. 1. ([13], [12], [25], [24]) Let $a,\ b,\ c>0$, $\zeta\in(0,2)$, $c<\frac{b^2}{2a}<1$ and

$$\varLambda:=\Bigl\{\lambda\in\mathbb{C}:\Bigl|\lambda-\Bigl(c-\frac{b^2}{4a}\Bigr)\Bigr|\leq\frac{b^2}{4a},\ \Im(\lambda)\neq 0 \text{ if } \Re(\lambda)\leq c-\frac{b^2}{4a}\Bigr\}.$$

Consider the following abstract time-fractional equation:

$$\begin{cases}\mathbf{D}_t^{\alpha}u(t)=au_{xx}+bu_x+cu:=-Au,\\ u(0,t)=0,\ t\geq 0,\\ u(x,0)=u_0(x),\ x\geq 0,\ \text{and } u_t(x,0)=0,\ \text{if } \alpha\in(1,2).\end{cases}$$

As it is known, the operator $-A$ with domain $D(-A)=\{f\in W^{2,2}([0,\infty)):f(0)=0\}$, generates an analytic strongly continuous semigroup of angle $\frac{\pi}{2}$ in the space $E=L^2([0,\infty))$; the same assertion holds in the case that the operator $-A$ acts on $E=L^1([0,\infty))$ with domain $D(-A)=\{f\in W^{2,1}([0,\infty)):f(0)=0\}$. Assume first $\zeta\in[1,2)$, $\theta\in(\zeta\frac{\pi}{2}-\pi,\pi-\zeta\frac{\pi}{2})$ and $P(z)=\sum_{j=0}^n a_jz^j$ is a non-constant complex polynomial such that $a_n>0$ and

$$-e^{i\theta}P(-\varLambda)\ \cap\ \bigl\{te^{\pm i\zeta\frac{\pi}{2}}:t\geq 0\bigr\}\ \neq\emptyset. \tag{9}$$

Then it is not difficult to prove that $-e^{i\theta}P(A)$ generates an analytic C_0-semigroup of angle $\frac{\pi}{2}-|\theta|$. Taking into account [23, Theorem 2.17], one gets that the operator $-e^{i\theta}P(A)$ is the integral generator of an exponentially bounded, analytic ζ-times regularized resolvent family $(R_{\zeta,\theta,P}(t))_{t\geq 0}$ of angle $\frac{\pi-|\theta|}{\zeta}-\frac{\pi}{2}$. It is not difficult to show that the conditions of Theorem 1 are satisfied with $\tilde{E}=E$, which implies that $(R_{\zeta,\theta,P}(t))_{t\geq 0}$ is topologically mixing. Suppose now $\zeta\in(0,1)$, $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $P(z)=\sum_{j=0}^n a_jz^j$ is a non-constant complex polynomial such that $a_n>0$ and (9) holds. Then $-e^{i\theta}P(A)$ is the integral generator of an exponentially bounded, analytic ζ-times regularized resolvent family $(R_{\zeta,\theta,P}(t))_{t\geq 0}$ of angle $\min((\frac{1}{\zeta}-1)\frac{\pi}{2},\frac{\pi}{2})$. Using the above arguments, we easily infer that $(R_{\zeta,\theta,P}(t))_{t\geq 0}$ is topologically mixing. Notice that (9) holds if $c<\frac{b^2}{4a}$; in the case $c\geq\frac{b^2}{4a}$, one can prove that (9) holds provided $a_0=0$ or $P(z)=\sum_{j=0}^n a_j(z+d)^j$, $z\in\mathbb{C}$, where $d\in\mathbb{C}$ and $0\in\operatorname{int}(d-\varLambda)$. Consider now the equation (1) with $n=2$, $\alpha_2=2$, $\alpha_1=0$, $\alpha=1$, $c_1>0$, and A replaced by $-e^{i\theta}P(A)$ therein. Using Lemma 1(1), one gets that $-e^{i\theta}P(A)$ is the integral generator of an exponentially bounded, analytic resolvent propagation family $((R_0(t))_{t\geq 0},(R_1(t))_{t\geq 0})$ of angle $\frac{\pi}{2}-|\theta|$. Moreover, $F_0(\lambda,t)=(1+c_1(\lambda^2-c_1)^{-1})e^{\lambda t}-c_1(\lambda^2-c_1)^{-1}e^{c_1t/\lambda}$, $t\geq 0$. By Theorem 1, we easily infer that the condition

$$e^{i\theta}P(-\varLambda)\ \cap\ i\mathbb{R}\neq\emptyset \tag{10}$$

implies that $(R_0(t))_{t\geq 0}$ is topologically mixing. Finally, suppose that $n=2$, $\alpha_2-\alpha=1$, $\alpha_1-\alpha=-1$, $i=1$, $c_1>0$ and $2<\alpha_2\leq 3$. Then $m_2=3$,

$D_1=\emptyset$ and $F_1(\lambda,t)=\lambda^{-1}(1+c_1(\lambda^2-c_1)^{-1})e^{\lambda t}-\lambda(\lambda^2-c_1)^{-1}e^{c_1t/\lambda}$, $t\geq 0$. By Lemma 1(1), we get that $-e^{i\theta}P(A)$ is the integral generator of an exponentially bounded, analytic resolvent propagation family $((R_0(t))_{t\geq 0},(R_1(t))_{t\geq 0},(R_2(t))_{t\geq 0})$ of angle $\frac{\pi}{2}-|\theta|$. If the condition (10) is satisfied, then one can apply Theorem 1 in order to see that $(R_1(t))_{t\geq 0}$ is topologically mixing.

2. ([18], [24]) Theorem 1 can be applied in the analysis of (subspace) topologically mixing properties of time-fractional wave equation and time-fractional heat equation on symmetric spaces of non-compact type (cf. [18, Theorem 3.1(a), Theorem 3.2, Corollary 3.3]); here we shall also provide some applications to the abstract Cauchy problem (1). Consider, for example, the situation of [18, Theorem 3.1(a)]. Let X be a symmetric space of non-compact type and rank one, let $p>2$, let the parabolic domain P_p and the positive real number c_p possess the same meaning as in [18] and let $P(z)=\sum_{j=0}^n a_jz^j$, $z\in\mathbb{C}$ be a non-constant complex polynomial with $a_n>0$. Assume first $\zeta\in(1,2)$, $\pi-n\arctan\frac{|p-2|}{2\sqrt{p-1}}-\zeta\frac{\pi}{2}>0$ and $\theta\in(n\arctan\frac{|p-2|}{2\sqrt{p-1}}+\zeta\frac{\pi}{2}-\pi,\pi-n\arctan\frac{|p-2|}{2\sqrt{p-1}}-\zeta\frac{\pi}{2})$. Then it is obvious that $-e^{i\theta}P(\Delta^{\natural}_{X,p})$ is the integral generator of an exponentially bounded, analytic ζ-times regularized resolvent family $(R_{\zeta,\theta,P}(t))_{t\geq 0}$ of angle $\frac{1}{\zeta}(\pi-n\arctan\frac{|p-2|}{2\sqrt{p-1}}-\zeta\frac{\pi}{2}-|\theta|)$. Keeping in mind that $\operatorname{int}(P_p)\subseteq\sigma_p(\Delta^{\natural}_{X,p})$, the condition

$$-e^{i\theta}P\bigl(\operatorname{int}(P_p)\bigr)\ \cap\ \bigl\{te^{\pm i\zeta\frac{\pi}{2}}:t\geq 0\bigr\}\ \neq\emptyset \tag{11}$$

implies that $(R_{\zeta,\theta,P}(t))_{t\geq 0}$ is topologically mixing. Suppose now $n=2$, $0<a<2$, $\alpha_2=2a$, $\alpha_1=0$, $\alpha=a$, $c_1>0$, $i=0$ and $|\theta|<\min(\frac{\pi}{2}-n\arctan\frac{|p-2|}{2\sqrt{p-1}},\frac{\pi}{2}-n\arctan\frac{|p-2|}{2\sqrt{p-1}}-\frac{\pi}{2}a)$. Then $D_0=\emptyset$ and, by Lemma 1(1), $-e^{i\theta}P(\Delta^{\natural}_{X,p})$ is the integral generator of an exponentially bounded, analytic resolvent propagation family $((R_{\theta,P,0}(t))_{t\geq 0},\cdots,(R_{\theta,P,\lceil 2a\rceil-1}(t))_{t\geq 0})$ of angle $\min(\frac{\pi-n\arctan\frac{|p-2|}{2\sqrt{p-1}}-|\theta|}{a}-\frac{\pi}{2},\frac{\pi}{2})$. Furthermore, the equality [5, (1.26)] can serve one to simply verify that:

$$\begin{aligned}F_0(\lambda,t)=&\frac{\lambda^a t^{-a}}{\lambda^{2a}-c_1}\Bigl(E_{a,2-a}\bigl(\lambda^a t^a\bigr)-E_{a,2-a}\bigl(c_1\lambda^{-a}t^a\bigr)\Bigr)\\&+\frac{\lambda^a}{\lambda^{2a}-c_1}\Bigl[\lambda^a E_a\bigl(\lambda^a t^a\bigr)+(a-1)\lambda^a E_{a,2}\bigl(\lambda^a t^a\bigr)\\&-c_1\lambda^{-a}E_a\bigl(c_1\lambda^{-a}t^a\bigr)-(a-1)c_1\lambda^{-a}E_{a,2}\bigl(c_1\lambda^{-a}t^a\bigr)\Bigr],\quad t>0.\end{aligned}$$

Invoking the asymptotic expansion formulae (5)-(7) and the above expression, it can be shown without any substantial difficulties that the condition

$$-e^{i\theta}P\bigl(\operatorname{int}(P_p)\bigr)\ \cap\ \Bigl\{\bigl(it\bigr)^a+c_1\bigl(it\bigr)^{-a}:t\in\mathbb{R}\setminus\{0\}\Bigr\}\ \neq\emptyset$$

implies that $(R_{\theta,P,0}(t))_{t\geq 0}$ is topologically mixing. Finally, let $\zeta\in(0,1)$ and let

$$\theta\in\Bigl(n\arctan\frac{|p-2|}{2\sqrt{p-1}}-\frac{\pi}{2},\frac{\pi}{2}-n\arctan\frac{|p-2|}{2\sqrt{p-1}}\Bigr).$$

Then the validity of (11) provides that $-e^{i\theta}P(\Delta^{\natural}_{X,p})$ is the integral generator of a topologically mixing ζ-times regularized resolvent family $(R_{\zeta,\theta,P}(t))_{t\geq 0}$ of angle $\min((\frac{1}{\zeta}-1)\frac{\pi}{2},\frac{\pi}{2})$. It is clear that (11) holds if $P(z)$ is of the form $P(z)=\sum_{j=0}^n a_j(z-c)^j$, $z\in\mathbb{C}$, where $c>c_p$.

3. ([7], [34], [24]) Suppose $\zeta\in(0,1)$, $E:=L^2(\mathbb{R})$, $c>\frac{b}{2}>0$, $\Omega:=\{\lambda\in\mathbb{C}:\Re\lambda<c-\frac{b}{2}\}$, $\phi\in E^*=E$ and $\mathcal{A}_cu:=u''+2bxu'+cu$ is the bounded perturbation of the one-dimensional Ornstein-Uhlenbeck operator acting with domain $D(\mathcal{A}_c):=\{u\in L^2(\mathbb{R})\ \cap\ W^{2,2}_{loc}(\mathbb{R}):\mathcal{A}_cu\in L^2(\mathbb{R})\}$. Then $\mathcal{A}_c$ is the integral generator of a topologically mixing ζ-times regularized resolvent family $(R_\zeta(t))_{t\geq 0}$ which cannot be hypercyclic provided $b<0$ or $c\leq\frac{b}{2}$ ([7], [24]). Notice also that the above assertions continue to hold in the case of ζ-times regularized resolvent families generated by bounded perturbations of multi-dimensional Ornstein-Uhlenbeck operators [7, Proposition 4.1, Theorem 4.2]; for the sake of simplicity, in the sequel of this example we shall consider only the hypercyclic and topologically mixing properties of resolvent propagation families generated by the operator $\mathcal{A}_c$ defined above. Suppose $\alpha_n-\alpha<1$. Then an application of Lemma 1(2) shows that $\mathcal{A}_c$ is the integral generator of an exponentially bounded, analytic resolvent propagation family $((R_0(t))_{t\geq 0},\cdots,(R_{m_n-1}(t))_{t\geq 0})$ of angle $\min(\frac{\pi}{2(\alpha_n-\alpha)}-\frac{\pi}{2},\frac{\pi}{2})$. If $b<0$, then $\sigma_p(\mathcal{A}_c^*)\neq\emptyset$ (cf. [7]) and, by Theorem 2, there does not exist $i\in\mathbb{N}^0_{m_n-1}$ such that $(R_i(t))_{t\geq 0}$ is hypercyclic (the case $c\leq\frac{b}{2}$ is more complicated in the newly arisen situation since it is not clear how one can prove the boundedness of $(R_i(t))_{t\geq 0}$, in general). Consider now the following case: $n=3$, $\frac{1}{3}<a<\frac{1}{2}$, $\alpha_3=3a$, $\alpha_2=2a$, $\alpha_1=0$, $\alpha=a$, $c_1<0$, $c_2>0$ and $i=1$. Then $D_1=\emptyset$ and

$$\mathcal{L}\bigl(F_1\bigl(\lambda,t\bigr)\bigr)(z)=\frac{z^{3a-2}}{z^{3a}+c_2z^{2a}-z^a\bigl(\lambda^{2a}+c_1\lambda^{-a}+c_2\lambda^a\bigr)+c_1}.$$

Set $\lambda_{1,2}:=\frac{-c_2-\lambda^a\pm\sqrt{(c_2+\lambda^a)^2+4c_1\lambda^{-a}}}{2}$. Then one can simply prove that the set $\Upsilon=\{\lambda\in\mathbb{C}:(\lambda^a-\lambda_1)(\lambda^a-\lambda_2)(\lambda_1-\lambda_2)\neq 0\}$ is finite and that, for every $z\in\mathbb{C}\setminus\{0\}$ and $\lambda\in\mathbb{C}\setminus\Upsilon$,

$$z^{3a}+c_2z^{2a}-z^a\bigl(\lambda^{2a}+c_1\lambda^{-a}+c_2\lambda^a\bigr)+c_1=\bigl(z^a-\lambda^a\bigr)\bigl(z^a-\lambda_1\bigr)\bigl(z^a-\lambda_2\bigr).$$

Using the equality [5, (1.26)], we get that, for every $\lambda\in\mathbb{C}\setminus\Upsilon$,

$$F_1(\lambda,t)=\frac{t^{1-2a}E_{a,2-2a}(\lambda^a t^a)}{(\lambda^a-\lambda_1)(\lambda^a-\lambda_2)}+\frac{t^{1-2a}E_{a,2-2a}(\lambda_1 t^a)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda^a)}+\frac{t^{1-2a}E_{a,2-2a}(\lambda_2 t^a)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda^a)}. \tag{12}$$

Clearly, $P_\lambda=\lambda^{2a}+c_2\lambda^a+c_1\lambda^{-a}$, $\lambda\in\mathbb{C}\setminus\{0\}$, $\lim_{\lambda\to0}(\lambda_1-(-\frac{c_2}{2}+\sqrt{c_1\lambda^{-a}}))=0$ and $\lim_{\lambda\to0}(\lambda_2-(-\frac{c_2}{2}-\sqrt{c_1\lambda^{-a}}))=0$. This implies that there exists a sufficiently small number $\epsilon_1>0$ such that, for every $\lambda\in\mathbb{C}$ with $\Re\lambda>0$ and $|\lambda|\le\epsilon_1$, the following holds: $\Re\lambda_2\le-\frac{c_2}{4}$ and

$$\operatorname{dist}\Big(\lambda_1,\Big\{z\in\mathbb{C}:\arg\Big(z+\frac{c_2}{2}\Big)\in\Big[\frac{\pi}{2}-\frac{\pi a}{4},\frac{\pi}{2}\Big]\Big\}\Big)<\min\Big(\frac{c_2}{4},\frac{c_2}{2}\cot\frac{\pi a}{4}\Big). \tag{13}$$

Arguing similarly, we obtain that there exists a sufficiently small number $\epsilon_2>0$ such that, for every $\lambda\in\mathbb{C}$ with $\arg(\lambda)\in(\frac{\pi}{2},\frac{\pi}{2a})$ and $|\lambda|\le\epsilon_2$, the following holds: $\Re\lambda_2\le-\frac{c_2}{4}$ and

$$\operatorname{dist}\Big(\lambda_1,\Big\{z\in\mathbb{C}:\arg\Big(z+\frac{c_2}{2}\Big)\in\Big[\frac{\pi}{4},\frac{\pi}{2}-\frac{\pi a}{4}\Big]\Big\}\Big)<\frac{c_2}{4}. \tag{14}$$

Furthermore, our assumption $c_1<0$ implies that there exists a sufficiently small number $\epsilon_3>0$ such that, for every $\lambda\in\mathbb{C}\setminus\{0\}$ with $|\arg(\lambda)|\le\frac{\pi}{2a}$ and $|\lambda|\le\epsilon_3$, we have $\Re(P_\lambda)=\Re(\lambda^{2a}+c_2\lambda^a+c_1\lambda^{-a})\le\epsilon_3^{2a}+|c_2|\epsilon_3^a<c-\frac{b}{2}$. Let $\epsilon_4>0$ satisfy that, for every $\lambda\in\mathbb{C}\setminus\{0\}$ and $|\lambda|\le\epsilon_4$, one has $\lambda\in\Upsilon$. Put $\epsilon:=\min(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)$, $\Omega:=\Omega_1:=\Omega_2:=\{z\in\mathbb{C}\setminus\{0\}:|\arg(z)|\le\frac{\pi}{2a},\ |z|<\epsilon\}$, $\Omega_+:=\Omega_{1,+}:=\Omega_{2,+}:=\{z\in\mathbb{C}:\Re z>0,|z|<\epsilon\}$ and $\Omega_-:=\Omega_{1,-}:=\Omega_{2,-}:=\{z\in\mathbb{C}\setminus\{0\}:\arg(z)\in(\frac{\pi}{2},\frac{\pi}{2a}),\ |z|<\epsilon\}$. Then it is obvious that $P_\Omega\subseteq\sigma_p(\mathcal{A}_c)$. Define $f_1:P_\Omega\to E$ and $f_2:P_\Omega\to E$ by $f_1(z):=\mathcal{F}^{-1}(e^{-\frac{\xi^2}{2b}}\xi|\xi|^{-(2+\frac{z-c}{b})})(\cdot)$, $z\in P_\Omega$ and $f_2(z):=\mathcal{F}^{-1}(e^{-\frac{\xi^2}{2b}}|\xi|^{-(1+\frac{z-c}{b})})(\cdot)$, $z\in P_\Omega$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform on the real line and its inverse transform, respectively. Exploiting (12)-(14) and (5)-(7), we easily infer that:

$$\lim_{t\to+\infty}\big|F_1(\lambda,t)\big|=+\infty,\ \lambda\in\Omega_+\ \text{and}\ \lim_{t\to+\infty}F_1(\lambda,t)=0,\ \lambda\in\Omega_-.$$

By Remark 1(3) and the consideration given in [24, Example 2.5(iii)], we reveal that $(R_1(t))_{t\ge0}$ is topologically mixing.

4. The study of qualitative properties of the abstract Basset-Boussinesq-Oseen equation:

$$u'(t)-A\mathbf{D}_t^\alpha u(t)+u(t)=f(t),\quad t\ge0,\ u(0)=0\quad(\alpha\in(0,1)), \tag{15}$$

describing the unsteady motion of a particle accelerating in a viscous fluid under the action of the gravity, has been initiated by C. Lizama and H. Prado in [31]. For further results concerning the C-wellposedness of (15), the references [27] and [28] are of importance. Our intention here is to clarify the most important facts about hypercyclic and topologically mixing properties of once integrated solutions of the equation (15) with $f(t)\equiv0$.

Clearly, $n=2$, $\alpha_2=1$, $\alpha_1=0$, $c_1=1$, $D_0=\emptyset$ and the analysis is quite complicated in the general case since

$$\mathcal{L}\big(F_0(\lambda,t)\big)(z)=\frac{1}{z+1-z^\alpha\big(\lambda^{1-\alpha}+\lambda^{-\alpha}\big)}.$$

The cases $\alpha=\frac12$ and $\alpha=\frac13$ can be considered similarly as in the parts (2) and (3). Suppose now $\alpha=\frac23$, $A\equiv\mathcal{A}_c$ and $c-\frac{b}{2}>2^{1/3}+2^{2/3}$ (cf. (3)). Then $\mathcal{A}_c$ is the integral generator of an exponentially bounded, analytic resolvent propagation family $(R_0(t))_{t\ge0}$ of angle $\frac{\pi}{2}$. Put $\lambda_{1,2}:=\frac{-\lambda^{1/3}\pm\sqrt{\lambda^{2/3}+4\lambda^{(-1)/3}}}{2}$. Then the sets $\Upsilon_1:=\{\lambda\in\mathbb{C}\setminus\{0,-4\}:(\lambda-\lambda_1)(\lambda-\lambda_2)=0\}$ and $\Upsilon_2:=\{\lambda\in\mathbb{C}\setminus\{0\}:\Re\lambda=\Re(\lambda_1^3)\}$ are finite. Furthermore, for every $\lambda\in\mathbb{C}\setminus((-\infty,0]\ \cup\ \Upsilon_1)$, one has:

$$F_0(\lambda,t)=\frac{E_{1/3,1/3}\big(\lambda^{1/3}t^{1/3}\big)}{\big(\lambda^{1/3}-\lambda_1\big)\big(\lambda^{1/3}-\lambda_2\big)}+\frac{E_{1/3,1/3}\big(\lambda_1 t^{1/3}\big)}{\big(\lambda_1-\lambda^{1/3}\big)\big(\lambda_1-\lambda_2\big)}+\frac{E_{1/3,1/3}\big(\lambda_2 t^{1/3}\big)}{\big(\lambda_2-\lambda^{1/3}\big)\big(\lambda_2-\lambda_1\big)}.$$

Since the function $s\mapsto s^{1/3}+s^{(-2)/3}$, $s>0$ attaines its global minimum $2^{1/3}+2^{2/3}$ for $s=2$, we obtain that there exist positive real numbers ε_1 and ε_2 such that $\varepsilon_1<2<\varepsilon_2$ and $\Re(P_\lambda)=\Re(\lambda^{1/3}+\lambda^{(-2)/3})<c-\frac{b}{2}$, provided $\varepsilon_1<|\lambda|<\varepsilon_2$. Set $\Omega:=\Omega_1=:=\Omega_2:=\{\lambda\in\mathbb{C}:\varepsilon_1<|\lambda|<\varepsilon_2\}$ and $\Omega_+:=\Omega_{1,+}=:=\Omega_{2,+}:=\{\lambda\in\mathbb{C}:\Re\lambda>0,\ \varepsilon_1<|\lambda|<\varepsilon_2,\ \lambda\notin\Upsilon_2\}$. It is clear that $\Re\lambda_2<0$ for $\lambda\in\mathbb{C}\setminus\{0\}$, and that $\lim_{\lambda\to-2,\Im\lambda>0}\lambda_1=\lim_{\lambda\to-2,\Im\lambda>0}\frac{-\lambda^{1/3}+\sqrt{\lambda^{2/3}+4\lambda^{(-1)/3}}}{2}=\frac{-(-2)^{1/3}+\sqrt{(-2)^{2/3}+4(-2)^{(-1)/3}}}{2}$. Direct calculation shows that the argument of the last written number belongs to the set $(-\frac{2\pi}{3},-\frac{\pi}{6})$, which implies that there exists a sufficiently small number $\epsilon>0$ such that the set $\Omega_-:=\Omega_{1,-}:=\Omega_{2,-}:=\{\lambda\in\mathbb{C}:\Im\lambda>0,\ |\lambda+2|<\epsilon\}$ is a subset of Ω, and that $\arg(\lambda)\in(-\frac{2\pi}{3},-\frac{\pi}{6})$ for $\lambda\in\Omega_-$. Using Remark 1(3) and (5)-(7), we obtain that $(R_0(t))_{t\ge0}$ is topologically mixing.

5. ([11], [24]) Let B, ω_1, ω_2, V_{ω_2,ω_1}, E, a and b possess the same meaning as in [11, Section 5] and let $Q(z)$ be a non-constant complex polynomial of degree n. Assume $0<\zeta<2$, $N\in\mathbb{N}$, $N>\frac{n}{2\zeta}$ and

$$R_\zeta(t)=\Big(E_\zeta\big(t^\zeta Q(z)\big)e^{-(-z^2)^N}\Big)(B),\ t\ge0, \tag{16}$$

where the right hand side of (16) is defined by means of the $H_{a,b}$ functional calculus developed in [11]. Then $R((e^{-(-z^2)^N})(B))$ is dense in E, and $(R_\zeta(t))_{t\ge0}$ is a ζ-times $(e^{-(-z^2)^N})(B)$-regularized resolvent family generated by $Q(B)$. Moreover, the condition

$$Q\big(\operatorname{int}\big(V_{\omega_2,\omega_1}\big)\big)\ \cap\ \big\{te^{\pm i\zeta\frac{\pi}{2}}:t\ge0\big\}\ \neq\emptyset$$

implies that $(R_\zeta(t))_{t\ge0}$ is both topologically mixing and hypercyclic (cf. also [25, Example 36(ii)] for the case $\zeta=2$).

We leave to the interested reader the problem of finding some other applications of functional calculi in the analysis of hypercyclic and topologically mixing properties of the abstract Cauchy problem (1).

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