L.A.S. and negative Schwarzian derivative do not imply G.A.S. in Clark's equation

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If an orbit (x_n) of (DE) converges to $u \in I$, then u is a fixed point.

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Our more specific aim: to study whether L.A.S. may imply G.A.S. for (DE).

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In what follows we assume that the interval map $h : I \to I$ satisfies the following properties:

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$$Sh(x) = \frac{h'''(x)}{h'(x)} - \frac{3}{2}\left(\frac{h''(x)}{h'(x)}\right)^2 .$$

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If additionally, we have

(S3) $Sh(x) < 0$ for any $x \in I$ (except possibly at its turning point),

then we say that h *belongs to the class S*.

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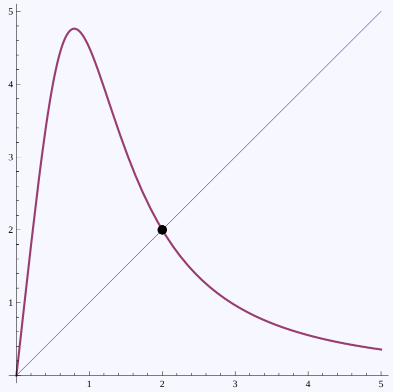

Figure 1: Shepherd's function $h(x) = px/(1 + x^q)$ with $p = 9$, $q = 3$, $u = 2$, belongs to the class S.

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What about "L.A.S. $\Leftrightarrow$ G.A.S" for (DE) if it is "dominated", in some sense by a one-dimensional map h?

The higher order case

Below, $\alpha : I^{k+1} \to (0,1)$:

$$x_{n+1} = \alpha(x_n, \dots, x_{n-k})x_{n-k} + (1 - \alpha(x_n, \dots, x_{n-k}))h(x_n) \qquad \text{(E1)}$$

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Relevant example (Tilman and Wedin 1991):

$$x_{n+1} = px_n + (q + rx_{n-1})e^{-x_n}, \quad I = (0,\infty),\ 0 < p, r < 1,\ q > 0;$$

here

$$\alpha(x) = 1 - re^{-x},$$

$$h(x) = \frac{px + qe^{-x}}{1 - re^{-x}}.$$

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Relevant example (Clark 1976):

$$x_{n+1} = \alpha x_n + (1-\alpha)h(x_{n-k}), \quad 0 < \alpha < 1 \qquad \text{(CE)}$$

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Typical maps h for Clark's equation ($I = (0, \infty)$, $p, q > 0$):

- Shepherd's function $h(x) = px/(1 + x^q)$ (Mackey and Glass, 1977)
- Ricker's function $h(x) = pxe^{-qx}$ (Gurney, Blythe and Nisbet, 1980)

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An important fact: v is a fixed point for (E1) and (E2) $\Leftrightarrow$ v is a fixed point of h.
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Theorem (Fisher 1984 and many more...)

G.A.S. (respectively, L.A.S.) for h implies G.A.S (respectively, L.A.S) for (E1) and (E2). In particular, if h belongs to the class S and $|h'(u)| \leq 1$, then u is a global attractor both for (E1) and (E2).

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DeVault et al. 1995, El-Morshedy and J.L. 2008

Indeed if k is odd, then

G.A.S (respectively L.A.S.) for $h \Leftrightarrow$ G.A.S (respectively L.A.S.) for (E1)

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In what follows we always assume $h'(u) < -1$.

L.A.S. for Clark's equation

Let $(r_k(\Theta), \alpha_k(\Theta))$ be given by

$$r_k(\Theta) = \frac{\sin(\Theta/(k+1))}{\sin(\Theta) - \sin(k\Theta/(k+1))},$$
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This curve can also be seen as the graph of a decreasing function $\alpha = a_k(r)$, $r \in (-\infty, 1)$, with

$$\lim_{r\to-\infty} a_k(r) = 1, \quad \lim_{r\to-1} a_k(r) = 0;$$

in particular,

$$a_1(r) = 1 + \frac{1}{r}.$$

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Theorem (Kuruklis 1994)

Let $r = h'(u)$. Then u is locally attracting (respectively, unstable) for (CE) if $\alpha > a_k(r)$ (respectively, $\alpha < a_k(r)$).

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Theorem (Tkachenko and Trofimchuk 2005)

Assume that g belongs to the class S and let $r = h'(u)$. Then u is globally attracting for (CE) if

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In the case $k = 1$ they improve the above condition as follows:

$$\text{either} \quad \alpha^2 \geq \frac{r+1}{r-1} \text{ and } \alpha \leq 0.88, \quad \text{or} \quad \alpha \geq \max\left\{0.88, \frac{r+0.88}{r}\right\}.$$

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It has been conjectured that if h belongs to the class S, then $\alpha > a_k(r)$ is actually enough to get global attraction for (CE), that is, L.A.S. implies G.A.S. (Györi and Trofimchuk 2000, El-Morshedy and Liz 2005). Some numerical estimates support it (Wang and Wei 2008, Liz 2009).

L.A.S. and G.A.S for Clark's equation

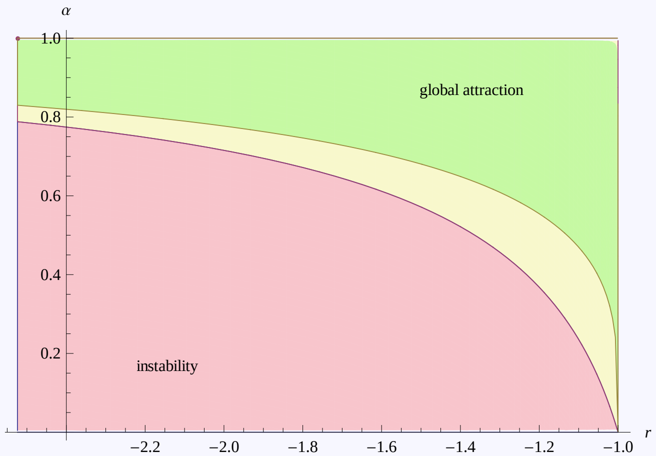

Figure 2: Instability and global attraction for Clark's equation ($k = 3$).

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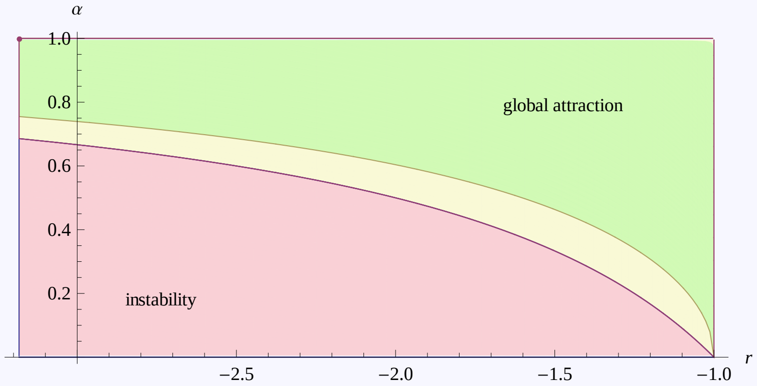

Figure 3: Instability and global attraction for Clark's equation ($k = 1$).

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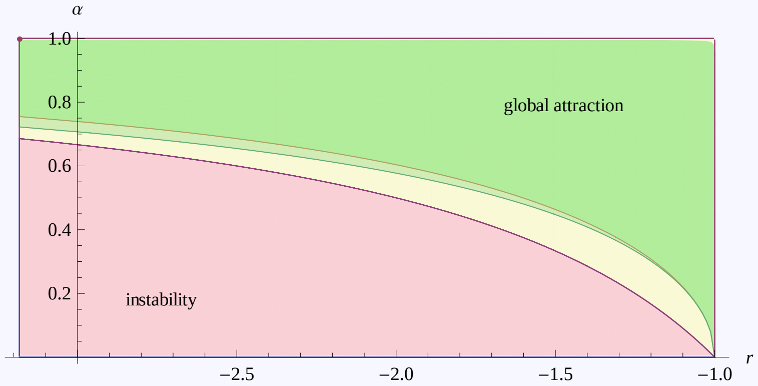

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The Neimark-Sacker bifurcation for Clark's equation

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- if $a_k(r) - \epsilon < \alpha < a_k(r)$, then there is an invariant (attracting) curve near u; if $a_k(r) \leq \alpha < a_k(r) + \epsilon$, then there is no invariant curve near u (*supercritical N-S bifurcation*).

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- if $a_k(r) - \epsilon < \alpha \leq a_k(r)$, then there is no invariant curve near u; if $a_k(r) < \alpha < a_k(r) + \epsilon$, then there is an (unstable) invariant curve near u (*subcritical N-S bifurcation*).

In the supercritical case the conjecture is reinforced; in the subcritical case the conjecture is disproved!

The Neimark-Sacker bifurcation for Clark's equation

$$N_k(\Theta) = \frac{1}{\sin\left(\frac{k\Theta}{k+1}\right)\cos\Theta - k\sin\left(\frac{\Theta}{k+1}\right)} \mathrm{Re}\left(\frac{\sin\left(\frac{k\Theta}{k+1}\right)e^{-i\Theta} - k\sin\left(\frac{\Theta}{k+1}\right)}{1 - e^{i\Theta} + i\sin\Theta\, e^{2i\Theta}\frac{e^{-\frac{3i\Theta}{2(k+1)}}}{\cos\left(\frac{\Theta}{2(k+1)}\right)}}\right)$$

$$+ \frac{\cos\left(\frac{\Theta}{2(k+1)}\right)}{\sin\left(\frac{\Theta}{2}\right)\sin\left(\frac{k\Theta}{2(k+1)}\right)},$$

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The case $k = 1$:

$$N_1(\Theta) = \frac{3 - 2\cos(\Theta/2)}{2 - 2\cos(\Theta/2)}$$

The Neimark-Sacker bifurcation for Clark's equation

$$N_k(\Theta) = \frac{1}{\sin\left(\frac{k\Theta}{k+1}\right)\cos\Theta - k\sin\left(\frac{\Theta}{k+1}\right)} \operatorname{Re}\left(\frac{\sin\left(\frac{k\Theta}{k+1}\right)e^{-i\Theta} - k\sin\left(\frac{\Theta}{k+1}\right)}{1 - e^{i\Theta} + i\sin\Theta e^{2i\Theta}\frac{e^{-\frac{3i\Theta}{2(k+1)}}}{\cos\left(\frac{\Theta}{2(k+1)}\right)}}\right)$$
$$+ \frac{\cos\left(\frac{\Theta}{2(k+1)}\right)}{\sin\left(\frac{\Theta}{2}\right)\sin\left(\frac{k\Theta}{2(k+1)}\right)},$$

$\Theta \in ((k+1)\pi/(2k+1), \pi)$.

$$\begin{aligned} N_\infty(\Theta) &= \lim_{k\to\infty} N_k(\Theta) \\ &= \frac{2}{\sin\Theta\cos\Theta - \Theta}\operatorname{Re}\left(\frac{\sin\Theta e^{-i\Theta} - \Theta}{(e^{-i\Theta} - 1)^2(e^{i\Theta} + 2)}\right) + \frac{1}{\sin^2(\Theta/2)}, \end{aligned}$$

$\Theta \in (\pi/2, \pi)$.

The Neimark-Sacker bifurcation for Clark's equation

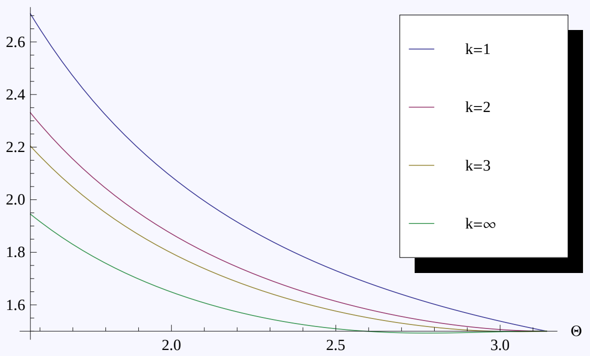

Figure 4: Graphs of maps $N_k(\Theta)$, $k = 1, 2, 3, \infty$.

The Neimark-Sacker bifurcation for Clark's equation

The important things about these maps:

$$N_k(\pi) = 3/2$$
$$N_k'(\pi) = \frac{1}{4\sin(\pi/(k+1))}\left(1 - \cos\left(\frac{\pi}{k+1}\right)\right)\left(2\cos\left(\frac{\pi}{k+1}\right) - 1\right)$$

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In particular, $N_k'(\pi) > 0$ for any $k \geq 3$.

The Neimark-Sacker bifurcation for Clark's equation

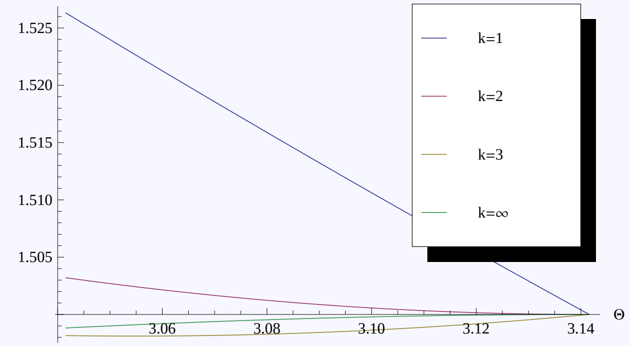

Figure 5: Graphs of maps $N_k(\Theta)$, $k = 1, 2, 3, \infty$ (detail).

The Neimark-Sacker bifurcation for Clark's equation

$$\Sigma h(u) = \frac{h'''(u)h'(u)}{(h''(u))^2}$$

If $h''(u) = 0$:

- if $h'''(u) \leq 0$, then $\Sigma h(u) = \infty$
- if $h'''(u) > 0$, then $\Sigma h(u) = -\infty$

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$$\Sigma h(u) < 3/2 \Leftrightarrow Sh(u) < 0$$

L.A.S. and negative Schwarzian derivative *should* imply G.A.S.!

Preliminaries More about Clark's equation Main results

L.A.S. and negative Schwarzian derivative *should* imply G.A.S.!

Theorem 1

Let $\Theta \in ((k+1)\pi/(2k+1), \pi)$ be such that $h'(u) = r_k(\Theta)$. Then (CE) exhibits a supercritical (respectively, a subcritical) Neimark-Sacker bifurcation at $\alpha = \alpha_k(\Theta)$ if $\Sigma h(u) < N_k(\Theta)$ (respectively, if $N_k(\Theta) < \Sigma h(u)$).

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Corollary

Assume that and one of the following conditions holds:

(a) $k \leq 2$ and $Sh(u) < 0$;

(b) $h'(u) < -1.18$ and $Sh(u) < 0$;

(c) $\Sigma h(u) < 1.49$.

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Remark (Wang and Wei 2008)

If $h(x) = pxe^{-qx}$ is Ricker's function, then $\Sigma h(u) < 1$: the bifurcation is supercritical.

L.A.S. and negative Schwarzian derivative need not imply G.A.S.!

Theorem 2

Let h_ϵ, $0 < \epsilon < \epsilon_0$, be C^4 maps. Assume that for any ϵ there is $u_\epsilon \in I$ such that the following conditions are satisfied:

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(i) $h_\epsilon(u_\epsilon) = u_\epsilon$;

(ii) the map $D(\epsilon) := h'_\epsilon(u_\epsilon)$ is differentiable and $\lim_{\epsilon\to 0} D(\epsilon) = -1$, $\lim_{\epsilon\to 0} D'(\epsilon) = d < 0$;

(iii) the map $T(\epsilon) := \Sigma h_\epsilon(u_\epsilon)$ is differentiable and $\lim_{\epsilon\to 0} T(\epsilon) = 3/2$, $\lim_{\epsilon\to 0} T'(\epsilon) = 0$.

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Then, if $k \geq 3$, $\epsilon > 0$ is small enough and we put $h = h_\epsilon$, $u = u_\epsilon$, (CE) exhibits a subcritical Neimark-Sacker bifurcation at $\alpha = \alpha_k(\Theta)$ with $h'(u) = r_k(\Theta)$.

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A simple example belonging to the class S

$$h_\epsilon(x) = \frac{1}{(1-2\epsilon)(\epsilon + (1-\epsilon)x) + 2\epsilon(\epsilon + (1-\epsilon)x)^2},$$

with $\epsilon > 0$ small enough.

L.A.S. and negative Schwarzian derivative need not imply G.A.S.!

For the map h_ϵ we take:

- $k = 3$,
- $\epsilon = 0.00167086$,
- $\alpha = 0.00573994$,

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- $k = 3$,
- $\epsilon = 0.00167086$,
- $\alpha = 0.00573994$,

and depict pairs (x_{n+1}, x_n) for orbits $(x_n)_{n=-3}^{\infty}$ starting at the following initial conditions:

- $(1.898919, 1.570831, 0.995705, 0.638023)$ (blue),
- $(1.8, 1.570831, 0.995705, 0.638023)$ (magenta),
- $(2, 1.570831, 0.995705, 0.638023)$ (gold),
- $(1.219971, 0.0768226, 0.00488285, 0.0308514)$ (green).

L.A.S. and negative Schwarzian derivative need not imply G.A.S.!

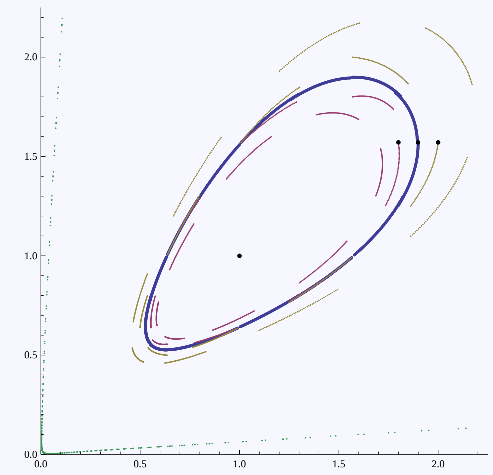

Figure 6: A local, but not global attractor, for Clark's equation.

Final remarks

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$$h(x) = \frac{419 + 722x + 6859x^2}{(1 + 19x)^3}.$$

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$$h(x) = \frac{419 + 722x + 6859x^2}{(1 + 19x)^3}.$$

- If $k = 1$, then numerical experiments suggest that if h is a decreasing map belonging to the class S, then (CE) has exactly one metric attractor (a periodic orbit or an invariant curve). This is not true if $k = 2$ (El-Morshedy, Liz and J.L. 2008).

Final remarks

THANK YOU VERY MUCH
FOR YOUR KIND ATTENTION!