Homoclinic trajectories of non-autonomous maps

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Non-autonomous dynamical systems

P. E. Kloeden and M. Rasmussen.
Nonautonomous dynamical systems, volume 176 of *Mathematical Surveys and Monographs*.
American Mathematical Society, Providence, RI, 2011.

C. Pötzsche.
Geometric theory of discrete nonautonomous dynamical systems, volume 2002 of *Lecture Notes in Mathematics*.
Springer, Berlin, 2010.

M. Rasmussen.
Attractivity and bifurcation for nonautonomous dynamical systems, volume 1907 of *Lecture Notes in Mathematics*.
Springer, Berlin, 2007.

Non-autonomous dynamical systems

- Spectral theory
- Invariant fiber bundles
- Hyperbolicity
- Asymptotic behavior
- Homoclinic trajectories
- Non-autonomous bifurcations
- Pullback attractors
- Skew product flow

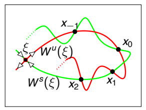
Outline

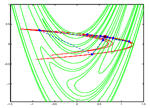
Homoclinic orbits in autonomous systems

Homoclinic orbits in autonomous systems

Let

- $f : \mathbb{R}^k \to \mathbb{R}^k$ be a smooth diffeomorphism,
- ξ be a hyperbolic fixed point, i.e. $\sigma(Df(\xi)) \cap \{x \in \mathbb{C} : |x| = 1\} = \emptyset$.
- Assume that stable and unstable manifold of ξ intersect transversally.





Definition

Homoclinic orbit: $x_{\mathbb{Z}} = (x_n)_{n\in\mathbb{Z}}$:

$$x_{n+1} = f(x_n), \quad n \in \mathbb{Z}, \quad \lim_{n\to\pm\infty} x_n = \xi.$$

Approximation of homoclinic orbits in autonomous systems

Compute a finite orbit segment

$$x_{n_-}, \dots, x_{n_+}$$

by solving a

boundary value problem.
Simplest case: periodic boundary conditions:

$$x = \begin{pmatrix} x_{n_-} \\ \vdots \\ x_{n_+} \end{pmatrix}, \qquad \Gamma(x) = \begin{pmatrix} x_{n+1} - f(x_n), \quad n = n_-, \dots, n_+ - 1 \\ x_{n_-} - x_{n_+} \end{pmatrix}.$$

Often successful: Rough initial guess for Newton's method:

$$u_0 = (\xi, \dots, \xi, g, \xi, \dots, \xi)^T, \quad \text{e.g. } g = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ in the Hénon example.}$$

Homoclinic orbits in autonomous systems

Remark

The dynamics near transversal homoclinic orbits is chaotic, cf. the celebrated Smale-Šil'nikov-Birkhoff Theorem.

L. P. Šil'nikov.
Existence of a countable set of periodic motions in a neighborhood of a homoclinic curve.
Dokl. Akad. Nauk SSSR, 172:298–301, 1967.
Soviet Math. Dokl. 8 (1967), 102–106.

S. Smale.
Differentiable dynamical systems.
Bull. Amer. Math. Soc., 73:747–817, 1967.

K. J. Palmer.
Shadowing in dynamical systems, volume 501 of *Mathematics and its Applications*.
Kluwer Academic Publishers, Dordrecht, 2000.

Approximation of homoclinic orbits in autonomous systems

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Alternative: Compute initial guess via approximations of stable and unstable manifolds.

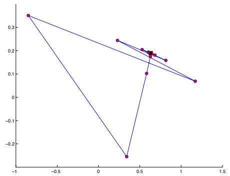
R. K. Ghaziani, W. Govaerts, Y. A. Kuznetsov, and H. G. E. Meijer.
Numerical continuation of connecting orbits of maps in MATLAB.
J. Difference Equ. Appl., 15(8-9):849–875, 2009.

Approximation of homoclinic orbits in autonomous systems

Example: **Hénon's map:**

$$H\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1+y-ax^2\\bx\end{pmatrix}, \quad \text{typical parameters: } a = 1.4,\ b = 0.3.$$

$$n_- = -6, \quad n_+ = 6$$



Approximation of homoclinic orbits in autonomous systems

- W.-J. Beyn, *The numerical computation of connecting orbits in dynamical systems. IMA J. Numer. Anal.*, 10,379–405, 1990.
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- J.-M. Kleinkauf, *Numerische Analyse tangentialer homokliner Orbits. PhD thesis, Universität Bielefeld*, Shaker Verlag, Aachen, 1998.
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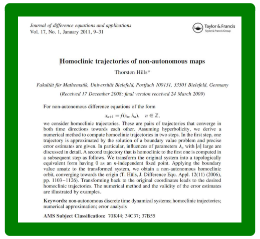
- W.-J. Beyn and Th. Hüls. *Error estimates for approximating non-hyperbolic heteroclinic orbits of maps. Numer. Math.*, 99(2):289–323, 2004.
- Th. Hüls. *Bifurcation of connecting orbits with one nonhyperbolic fixed point for maps. SIAM J. Appl. Dyn. Syst.*, 4(4):985–1007, 2005.
- W.-J. Beyn, Th. Hüls, J.-M. Kleinkauf, and Y. Zou, *Numerical analysis of degenerate connecting orbits for maps. Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 14, 3385–3407, 2004.

Outline

Non-autonomous analog of homoclinic orbits



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Homoclinic trajectories of non-autonomous maps

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For non-autonomous difference equations of the form

$$x_{n+1} = f(x_n, \lambda_n), \quad n \in \mathbb{Z},$$

we consider homoclinic trajectories. These are pairs of trajectories that converge in both time directions towards each other. Assuming hyperbolicity, we derive a numerical method to compute homoclinic trajectories in two steps. In the first step, one trajectory is approximated by the solution of a boundary value problem and precise error estimates are given. In particular, influences of parameters λ_n with $|n|$ large are discussed in detail. A second trajectory that is homoclinic to the first one is computed in a subsequent step as follows. We transform the original system into a topologically equivalent form having 0 as an n-independent fixed point. Applying the boundary value ansatz to the transformed system, we obtain a non-autonomous homoclinic orbit, converging towards the origin (T. Hüls, J. Difference Equ. Appl. 12(11) (2006), pp. 1103–1126). Transforming back to the original coordinates leads to the desired homoclinic trajectories. The numerical method and the validity of the error estimates are illustrated by examples.

Keywords: non-autonomous discrete time dynamical systems; homoclinic trajectories; numerical approximation; error analysis

AMS Subject Classification: 70K44; 34C37; 37B55

Non-autonomous analog of a fixed point

Non-autonomous discrete time dynamical system

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{Z}, \quad f_n \text{ family of smooth diffeomorphisms}$$

autonomous vs. non-autonomous

fixed point	$\leftrightarrow$	bounded trajectory
ξ		$\xi_{\mathbb{Z}}:\ \xi_{n+1} = f_n(\xi_n), \quad n \in \mathbb{Z}$
		$\|\xi_n\| \le C\ \forall n$

J. A. Langa, J. C. Robinson, and A. Suárez.
Stability, instability, and bifurcation phenomena in non-autonomous differential equations.
Nonlinearity, 15(3):887–903, 2002.

Non-autonomous analog of hyperbolicity

Non-autonomous discrete time dynamical system

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{Z}, \quad f_n \text{ family of smooth diffeomorphisms}$$

Hyperbolicity of

a fixed point ξ	$\leftrightarrow$	a bounded trajectory $\xi_{\mathbb{Z}}$
$Df(\xi)$ *has*		The variational equation
no center eigenvalue		$u_{n+1} = Df_n(\xi_n)u_n, \quad n \in \mathbb{Z}$
		has an **exponential dichotomy** on $\mathbb{Z}$.

A counterexample, showing that eigenvalues are meaningless for analyzing stability in non-autonomous systems was given by Vinograd (1952), cf.

F. Colonius and W. Kliemann.
The dynamics of control.
Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 2000.

Exponential dichotomy

Consider the linear difference equation

$$u_{n+1} = A_n u_n, \quad n \in \mathbb{Z}, \quad u_n \in \mathbb{R}^k,\ A_n \in GL(k;\mathbb{R}). \tag{1}$$

Solution operator of (1): $\Phi(n,m) := \begin{cases} A_{n-1}\dots A_m & \text{for } n > m \\ I & \text{for } n = m \\ A_n^{-1}\dots A_{m-1}^{-1} & \text{for } n < m \end{cases}$

Definition

The linear difference equation (1) has an **exponential dichotomy** with data $(K, \alpha, P_n^s, P_n^u)$ on $J = [n_-, n_+] \cap \mathbb{Z}$ if there exist 2 families of projectors P_n^s, P_n^u, $n \in J$, with $P_n^s + P_n^u = I$ for all $n \in J$ and constants $K,\ \alpha > 0$, such that

$$P_n^\kappa \Phi(n,m) = \Phi(n,m) P_m^\kappa \quad \forall n,m \in J, \quad \kappa \in \{s,u\},$$

$$\begin{aligned} \|\Phi(n,m)P_m^s\| &\le Ke^{-\alpha(n-m)} \\ \|\Phi(m,n)P_n^u\| &\le Ke^{-\alpha(n-m)} \end{aligned} \quad \forall n \ge m,\ n,m \in J.$$

Exponential dichotomy – Some references

W. A. Coppel.
Dichotomies in Stability Theory.
Springer-Verlag, Berlin, 1978.
Lecture Notes in Mathematics, Vol. 629.

D. Henry.
Geometric Theory of Semilinear Parabolic Equations.
Springer-Verlag, Berlin, 1981.

K. J. Palmer.
Exponential dichotomies, the shadowing lemma and transversal homoclinic points.
In *Dynamics reported, Vol. 1*, pages 265–306. Teubner, Stuttgart, 1988.

Non-autonomous analog of a homoclinic orbit

Non-autonomous discrete time dynamical system

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{Z}, \quad f_n \text{ family of smooth diffeomorphisms}$$

autonomous vs. non-autonomous

homoclinic orbit	$\leftrightarrow$	homoclinic trajectories:
An orbit $x_{\mathbb{Z}}$ that satisfies		Two bounded trajectories $x_{\mathbb{Z}}, \xi_{\mathbb{Z}}$ satisfying
$\lim_{n\to\pm\infty} x_n = \xi$		$\lim_{n\to\pm\infty} \|x_n - \xi_n\| = 0$

Note that:

$$x_{\mathbb{Z}} \text{ is homoclinic to } \xi_{\mathbb{Z}} \quad \leftrightarrow \quad \xi_{\mathbb{Z}} \text{ is homoclinic to } x_{\mathbb{Z}}$$

Outline

Step 1

Approximation of a bounded trajectory $\xi_{\mathbb{Z}}$.

Step 2

Approximation of a second trajectory $x_{\mathbb{Z}}$ that is homoclinic to $\xi_{\mathbb{Z}}$.

Setup

$$x_{n+1} = f_n(x_n), \qquad n \in \mathbb{Z}$$

f_n is generated by a parameter-dependent map

$$f_n = f(\cdot, \lambda_n), \qquad \lambda_{\mathbb{Z}} \text{ sequence of parameters.}$$

Assumptions

❶ **Smoothness:** $f \in \mathbb{C}^\infty(\mathbb{R}^k \times \mathbb{R}, \mathbb{R}^k)$, $f(\cdot, \lambda)$ diffeomorphism for all $\lambda \in \mathbb{R}$.

❷ There exists $\bar{\lambda}_{\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ such that

$$\bar{\xi}_{n+1} = f(\bar{\xi}_n, \bar{\lambda}_n), \quad n \in \mathbb{Z}$$

has a **bounded solution** $\bar{\xi}_{\mathbb{Z}}$.

❸ The variational equation

$$u_{n+1} = D_x f(\bar{\xi}_n, \bar{\lambda}_n) u_n, \quad n \in \mathbb{Z}$$

has an **exponential dichotomy** on $\mathbb{Z}$.

Approximation of bounded trajectories

Bounded trajectory $\xi_{\mathbb{Z}}$

Zero of the operator $\Gamma : X_{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}} \to X_{\mathbb{Z}}$, defined as

$$\Gamma(\xi_{\mathbb{Z}}, \lambda_{\mathbb{Z}}) := \left(\xi_{n+1} - f(\xi_n, \lambda_n)\right)_{n \in \mathbb{Z}}.$$

Space of bounded sequences on the discrete interval J

$$X_J := \left\{ u_J = (u_n)_{n \in J} \in (\mathbb{R}^k)^J : \sup_{n \in J} \|u_n\| < \infty \right\}$$

Approximation of bounded trajectories

Lemma

Assume ❶ – ❸.
Then there exist two neighborhoods $U(\bar{\lambda}_{\mathbb{Z}})$ and $V(\bar{\xi}_{\mathbb{Z}})$, such that

$$\Gamma(\xi_{\mathbb{Z}}, \lambda_{\mathbb{Z}}) = 0_{\mathbb{Z}}$$

has for all $\lambda_{\mathbb{Z}} \in U(\bar{\lambda}_{\mathbb{Z}})$ a unique solution $\xi_{\mathbb{Z}} \in V(\bar{\xi}_{\mathbb{Z}})$.

Lemma

Assume ❶ – ❸.
Then there exist two neighborhoods $U(\bar{\lambda}_{\mathbb{Z}})$ and $V(\bar{\xi}_{\mathbb{Z}})$, such that

$$u_{n+1} = D_x f(x_n, \lambda_n) u_n, \quad n \in \mathbb{Z}$$

has an exponential dichotomy on $\mathbb{Z}$ for ***any*** *sequence $x_{\mathbb{Z}} \in V(\bar{\xi}_{\mathbb{Z}})$, $\lambda_{\mathbb{Z}} \in U(\bar{\lambda}_{\mathbb{Z}})$. The dichotomy constants are independent of the specific sequence $x_{\mathbb{Z}}$.*

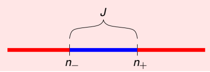
Approximation of bounded trajectories

Aim: Numerical approximations of a bounded solution of

$$\xi_{n+1} = f(\xi_n, \lambda_n) \quad \text{on the **finite interval** } J = [n_-, n_+].$$

Problem

$\xi_J = (\xi_n)_{n \in J}$ depends on λ_n, $n \in J$, **but also on** λ_n, $n \notin J$.



Fortunately

The influence of λ_n, $n \notin J$ **decays exponentially fast towards the middle of the interval** J.
Thus, numerical approximations with high accuracy can be achieved.

Similar observation in autonomous systems

P. Diamond, P. Kloeden, V. Kozyakin, and A. Pokrovskii.
Expansivity of semi-hyperbolic Lipschitz mappings.
Bull. Austral. Math. Soc., 51(2):301–308, 1995.

K. J. Palmer.
Shadowing in dynamical systems, volume 501 of *Mathematics and its Applications*.
Kluwer Academic Publishers, Dordrecht, 2000.

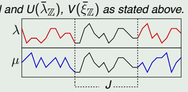
Approximation of bounded trajectories

Theorem

Assume ❶ – ❸.
Let $J = [n_-, n_+]$ be a finite interval and $U(\bar{\lambda}_{\mathbb{Z}})$, $V(\bar{\xi}_{\mathbb{Z}})$ as stated above.

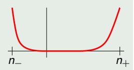
Choose $\lambda_{\mathbb{Z}}, \mu_{\mathbb{Z}} \in U(\bar{\lambda}_{\mathbb{Z}})$

such that $\lambda_n = \mu_n$ for $n \in J$.



Denote by $\xi_{\mathbb{Z}}, \eta_{\mathbb{Z}} \in V(\bar{\xi}_{\mathbb{Z}})$ the bounded solutions w.r.t. $\lambda_{\mathbb{Z}}$ and $\mu_{\mathbb{Z}}$.
Then
there exist constants $C,\ \alpha > 0$ that do not depend on $\lambda_{\mathbb{Z}}$ and $\mu_{\mathbb{Z}}$, such that

$$\|\xi_n - \eta_n\| \le C \left(e^{-\alpha(n - n_-)} + e^{-\alpha(n_+ - n)} \right)$$



holds for all $n \in J$.

Proof ($\|\xi_n - \eta_n\| \le C \left(\mathrm{e}^{-\alpha(n-n_-)} + \mathrm{e}^{-\alpha(n_+-n)} \right)$, where $\lambda_n = \mu_n$, $n \in J = [n_-, n_+]$)

$$\xi_{n+1} = f(\xi_n, \lambda_n), \quad \eta_{n+1} = f(\eta_n, \mu_n), \quad d_{\mathbb{Z}} := \eta_{\mathbb{Z}} - \xi_{\mathbb{Z}}, \quad h_{\mathbb{Z}} := \mu_{\mathbb{Z}} - \lambda_{\mathbb{Z}}.$$

$$\begin{aligned}
d_{n+1} &= f(\xi_n + d_n, \lambda_n + h_n) - f(\xi_n, \lambda_n) \\
&= f(\xi_n + d_n, \lambda_n) + \int_0^1 D_\lambda f(\xi_n + d_n, \lambda_n + \tau h_n) \mathrm{d}\tau \, h_n - f(\xi_n, \lambda_n) \\
&= f(\xi_n, \lambda_n) + \int_0^1 D_x f(\xi_n + \tau d_n, \lambda_n) \mathrm{d}\tau \, d_n \\
&\quad + \int_0^1 D_\lambda f(\xi_n + d_n, \lambda_n + \tau h_n) \mathrm{d}\tau \, h_n - f(\xi_n, \lambda_n) \\
&= \int_0^1 D_x f(\xi_n + \tau d_n, \lambda_n) \mathrm{d}\tau \, d_n + \int_0^1 D_\lambda f(\xi_n + d_n, \lambda_n + \tau h_n) \mathrm{d}\tau \, h_n \\
&= A_n d_n + r_n
\end{aligned}$$

where

$$A_n = \int_0^1 D_x f(\xi_n + \tau d_n, \lambda_n) \mathrm{d}\tau \qquad r_n = \int_0^1 D_\lambda f(\xi_n + d_n, \lambda_n + \tau h_n) \mathrm{d}\tau \, h_n.$$

Proof ($\|\xi_n - \eta_n\| \le C\left(e^{-\alpha(n-n_-)} + e^{-\alpha(n_+-n)}\right)$, where $\lambda_n = \mu_n, n \in J = [n_-, n_+]$)

$$A_n = \int_0^1 D_x f(\xi_n + \tau d_n, \lambda_n)\mathrm{d}\tau \qquad r_n = \int_0^1 D_\lambda f(\xi_n + d_n, \lambda_n + \tau h_n)\mathrm{d}\tau\; h_n$$

By assumption ❸

$$u_{n+1} = D_x f(\bar\xi_n, \bar\lambda_n) u_n, \qquad n \in \mathbb{Z}$$

has an exponential dichotomy on $\mathbb{Z}$.

Due to the Roughness-Theorem and our construction of neighborhoods,

$$u_{n+1} = A_n u_n, \qquad n \in \mathbb{Z}$$

has an exponential dichotomy on $\mathbb{Z}$ with data $(K, \alpha, P_n^s, P_n^u)$.

Solution operator: $\Phi(n,m)$, i.e. $u_n = \Phi(n,m)u_m$

Proof ($\|\xi_n - \eta_n\| \le C\left(e^{-\alpha(n-n_-)} + e^{-\alpha(n_+-n)}\right)$, where $\lambda_n = \mu_n, n \in J = [n_-, n_+]$)

$$A_n = \int_0^1 D_x f(\xi_n + \tau d_n, \lambda_n)\mathrm{d}\tau \qquad r_n = \int_0^1 D_\lambda f(\xi_n + d_n, \lambda_n + \tau h_n)\mathrm{d}\tau\; h_n$$

Unique bounded solution of $u_{n+1} = A_n u_n + r_n$ on $\mathbb{Z}$:

$$u_n = \sum_{m\in\mathbb{Z}} G(n, m+1) r_m,$$

where G is Green's function, defined as

$$G(n,m) = \begin{cases} \Phi(n,m)P_m^s, & n \ge m, \\ -\Phi(n,m)P_m^u, & n < m. \end{cases}$$

Estimates:

$$\|G(n,m)\| = \|\Phi(n,m)P_m^s\| \le Ke^{-\alpha(n-m)}, \quad \text{for } n \ge m,$$
$$\|G(n,m)\| = \|\Phi(n,m)P_m^u\| \le Ke^{-\alpha(m-n)}, \quad \text{for } n < m.$$

Proof ($\|\xi_n - \eta_n\| \le C\left(e^{-\alpha(n-n_-)} + e^{-\alpha(n_+-n)}\right)$, where $\lambda_n = \mu_n, n \in J = [n_-, n_+]$)

$$A_n = \int_0^1 D_x f(\xi_n + \tau d_n, \lambda_n)\mathrm{d}\tau \qquad r_n = \int_0^1 D_\lambda f(\xi_n + d_n, \lambda_n + \tau h_n)\mathrm{d}\tau\; h_n$$

$$u_n = \sum_{m\in\mathbb{Z}} G(n,m+1)r_m, \quad \|G(n,m)\| = \begin{cases} \|\Phi(n,m)P_m^s\| \le Ke^{-\alpha(n-m)}, & n \ge m, \\ \|\Phi(n,m)P_m^u\| \le Ke^{-\alpha(m-n)}, & n < m. \end{cases}$$

$$\begin{aligned} \|u_n\| &\le \sum_{m=-\infty}^{n_--1} \|G(n,m+1)r_m\| + \sum_{m=n_++1}^{\infty} \|G(n,m+1)r_m\| \\ &\le \sum_{m=-\infty}^{n_--1} RKe^{-\alpha(n-m-1)} + \sum_{m=n_++1}^{\infty} RKe^{-\alpha(m+1-n)} \\ &= \frac{RK}{1-e^{-\alpha}}\left(e^{-\alpha(n-n_-)} + e^{-\alpha(n_+-n+2)}\right), \quad n \in J. \end{aligned}$$

Proof ($\|\xi_n - \eta_n\| \le C\left(e^{-\alpha(n-n_-)} + e^{-\alpha(n_+-n)}\right)$, where $\lambda_n = \mu_n, n \in J = [n_-, n_+]$)

$$A_n = \int_0^1 D_x f(\xi_n + \tau d_n, \lambda_n)\mathrm{d}\tau \qquad r_n = \int_0^1 D_\lambda f(\xi_n + d_n, \lambda_n + \tau h_n)\mathrm{d}\tau\; h_n$$

$$u_{n+1} = A_n u_n + r_n, \qquad n \in \mathbb{Z} \qquad (2)$$

$$\|u_n\| \le \frac{RK}{1-e^{-\alpha}}\left(e^{-\alpha(n-n_-)} + e^{-\alpha(n_+-n+2)}\right), \qquad n \in J.$$

$d_{\mathbb{Z}} = \xi_{\mathbb{Z}} - \eta_{\mathbb{Z}}$ is the unique bounded solution of (2), thus

$$\|d_n\| = \|\xi_n - \eta_n\| \le C\left(e^{-\alpha(n-n_-)} + e^{-\alpha(n_+-n)}\right), \qquad n \in J.$$

□

Approximation of bounded trajectories

$\bar\xi_{\mathbb{Z}}$: zero of the operator $\Gamma : X_{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}} \to X_{\mathbb{Z}}$

$$\Gamma(\xi_{\mathbb{Z}}, \lambda_{\mathbb{Z}}) := \left(\xi_{n+1} - f(\xi_n, \lambda_n)\right)_{n\in\mathbb{Z}}.$$

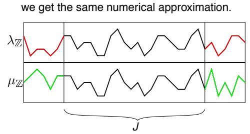
Finite approximation z_J on $J = [n_-, n_+] \cap \mathbb{Z}$

$$\Gamma_J(z_J, \bar\lambda_J) := \left(\left(z_{n+1} - f(z_n, \bar\lambda_n)\right)_{n\in[n_-,n_+-1]}, b(z_{n_-}, z_{n_+})\right) = 0$$

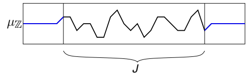
with periodic boundary operator $b(z_{n_-}, z_{n_+}) := z_{n_-} - z_{n_+}$.

Approximation of bounded trajectories

For all parameter sequences that coincide on J, we get the same numerical approximation.



Thus, we choose $\mu_n =$ const for all $n \notin J$.



Approximation of bounded trajectories with constant tails

$$\Gamma_J(z_J, \bar\lambda_J) := \left(\left(z_{n+1} - f(z_n, \bar\lambda_n)\right)_{n\in[n_-,n_+-1]}, b(z_{n_-}, z_{n_+})\right) = 0$$

Assumption ❹

There exist sequence $\bar\mu_{\mathbb{Z}} \in U(\bar\lambda_{\mathbb{Z}})$ with solution $\bar\eta_{\mathbb{Z}} \in V(\bar\xi_{\mathbb{Z}})$ and $\bar\mu \in \mathbb{R}, \bar\eta \in \mathbb{R}^k$ such that

$$\lim_{n\to+\infty} \bar\mu_n = \lim_{n\to-\infty} \bar\mu_n =: \bar\mu \quad \text{and} \quad \lim_{n\to+\infty} \bar\eta_n = \lim_{n\to-\infty} \bar\eta_n =: \bar\eta.$$

Theorem

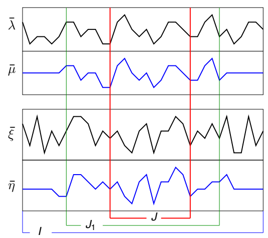
Assume ❶ – ❹.
Then constants $\delta, N, C > 0$ exist, such that $\Gamma_J(z_J, \bar\mu_J) = 0$, with periodic boundary conditions, has a unique solution

$$z_J \in B_\delta(\bar\eta_J) \quad \text{for } J = [n_-, n_+],\ -n_-,\ n_+ \ge N.$$

Approximation error:

$$\|\bar\eta_J - z_J\| \le C\|\bar\eta_{n_-} - \bar\eta_{n_+}\|.$$

Approximation of bounded trajectories with tolerance Δ



$\exists J_1 : \|\bar\xi_J - \bar\eta_J\| \le \frac{\Delta}{2}$

if $\bar\lambda_n = \bar\mu_n$, $n \in J_1$

choose $\bar\mu_n = \bar\mu$ for $n \notin J_1$

$\exists I : \|\bar\eta_I - z_I\| \le \frac{\Delta}{2}$

where $\Gamma_I(z_I) = 0$

For $n \in J$ we get

$$\begin{aligned} &\|\bar\xi_n - z_n\| \\ &\le \|\bar\xi_n - \bar\eta_n\| \\ &\quad + \|\bar\eta_n - z_n\| \\ &\le \frac{\Delta}{2} + \frac{\Delta}{2} = \Delta \end{aligned}$$

Outline

Numerical experiments:

Computation of bounded trajectories

Example: Computation of bounded trajectories

Hénon's map

$$x \mapsto h(x, \lambda, b) = \begin{pmatrix} 1 + x_2 - \lambda x_1^2 \\ b x_1 \end{pmatrix}$$

Fix $b = 0.3$ and choose $\lambda_{\mathbb{Z}} \in [1, 2]^{\mathbb{Z}}$ at random.

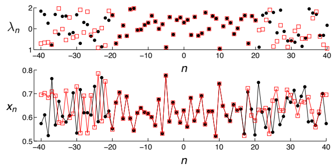
Non-autonomous difference equation

$$x_{n+1} = h(x_n, \lambda_n, b), \qquad n \in \mathbb{Z}.$$

M. Hénon.
A two-dimensional mapping with a strange attractor.
Comm. Math. Phys., 50(1):69–77, 1976.

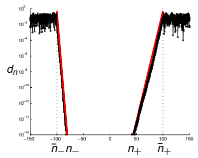
Example: Computation of bounded trajectories

Solutions of $\Gamma_J = 0$ on $J = [-40, 40]$ for two sequences λ_J that coincide on [-20,20].



Example: Computation of bounded trajectories on $J = [-150, 150]$

Given a sequence $\lambda_J \in [1, 2]^J$ with corresponding solution ξ_J.
Let $\mu_J \in [1, 2]^J$ such that $\lambda_n = \mu_n$ for $n \in [-100, 100]$ and solution η_J.



$d_n = \|\xi_n - \eta_n\|$, $n \in J$ for 10 different sequences μ_J.

Interval where $d_n \leq \Delta$

$$n_- = \left\lceil \bar{n}_- + \frac{|\log \Delta|}{\alpha_-} \right\rceil,$$
$$n_+ = \left\lfloor \bar{n}_+ - \frac{|\log \Delta|}{\alpha_+} \right\rfloor,$$

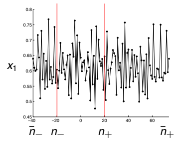
$\alpha_\pm$ dichotomy constants,

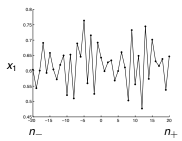
$\Delta = 10^{-16}$.

Example: Computation of bounded trajectories on $J = [-20, 20]$

Choose a buffer interval $[\bar{n}_-, \bar{n}_+]$ such that we get an accurate approximation on $[-20, 20]$:

$$\bar{n}_- = \left\lfloor n_- - \frac{|\log \Delta|}{\alpha_-} \right\rfloor = -40 \quad \text{and} \quad \bar{n}_+ = \left\lceil n_+ + \frac{|\log \Delta|}{\alpha_+} \right\rceil = 74.$$





Outline

Step 1 (done)

Approximation of a bounded trajectory $\xi_{\mathbb{Z}}$.

Step 2

Approximation of a second trajectory $x_{\mathbb{Z}}$ that is homoclinic to $\xi_{\mathbb{Z}}$.

Homoclinic trajectories

Assumptions

❺ Let $\bar{\lambda}_{\mathbb{Z}}$ as in ❷. A solution $\bar{x}_{\mathbb{Z}}$ of

$$x_{n+1} = f(x_n, \bar{\lambda}_n), \quad n \in \mathbb{Z}$$

exists, that is **homoclinic** to $\bar{\xi}_{\mathbb{Z}}$ and non-trivial, i.e. $\bar{x}_{\mathbb{Z}} \neq \bar{\xi}_{\mathbb{Z}}$.

❻ The trajectory $\bar{x}_{\mathbb{Z}}$ is **transversal**, i.e.

$$u_{n+1} = D_x f(\bar{x}_n, \bar{\lambda}_n) u_n, \; n \in \mathbb{Z} \text{ for } u_{\mathbb{Z}} \in X_{\mathbb{Z}} \iff u_{\mathbb{Z}} = 0.$$

Lemma

Assume ❶ – ❻. Then the difference equation

$$u_{n+1} = D_x f(\bar{x}_n, \bar{\lambda}_n) u_n, \quad n \in \mathbb{Z}.$$

has an exponential dichotomy on $\mathbb{Z}$.

Homoclinic trajectories

$\bar{x}_{\mathbb{Z}}$ is homoclinic to $\bar{\xi}_{\mathbb{Z}}$

if and only if

$\bar{y}_{\mathbb{Z}}$ defined as

$$\bar{y}_n = \bar{x}_n - \bar{\xi}_n$$

is a homoclinic orbit of

$$y_{n+1} = g(y_n, \bar{\lambda}_n) := f(y_n + \bar{\xi}_n, \bar{\lambda}_n) - \bar{\xi}_{n+1} \qquad n \in \mathbb{Z}$$

w.r.t. the fixed point 0.

The f and g-system are topologically equivalent due to the kinematic transformation, cf.

B. Aulbach and T. Wanner.
Invariant foliations and decoupling of non-autonomous difference equations.
J. Difference Equ. Appl., 9(5):459–472, 2003.

Approximation of homoclinic trajectories

$$g_n(y) := f(y + \bar{\xi}_n, \bar{\lambda}_n) - \bar{\xi}_{n+1}, \quad y_{n+1} = g_n(y_n), \quad g_n(0) = 0, \quad n \in \mathbb{Z}.$$

Approximation

$$\Gamma_J(y_J) := \left(\left(y_{n+1} - g_n(y_n) \right)_{n \in [n_-, n_+ - 1]}, y_{n_-} - y_{n_+} \right) = 0$$

with periodic boundary conditions.

Assumptions

❼ Denote by P^s_n, P^u_n the dichotomy projectors of

$$u_{n+1} = Df(\bar{\xi}_n, \bar{\lambda}_n) u_n, \quad n \in \mathbb{Z}.$$

Assume for **all** sufficiently large $-n_-, n_+$:

$$\measuredangle(\mathcal{R}(P^s_{n_-}), \mathcal{R}(P^u_{n_+})) > \sigma, \quad \text{for a } 0 < \sigma < \frac{\pi}{2}.$$

Outline

Numerical experiments:

Computation of homoclinic trajectories

(a) for the Hénon system,

(b) for a predator-prey model.

Approximation of homoclinic trajectories

$$g_n(y) := f(y + \bar{\xi}_n, \bar{\lambda}_n) - \bar{\xi}_{n+1}, \quad y_{n+1} = g_n(y_n), \quad g_n(0) = 0, \quad n \in \mathbb{Z}.$$

Theorem

Assume ❶ – ❼.
Then there exist constants δ, N, $C > 0$, such that
$\Gamma_J(y_J) = 0$, with projection boundary conditions, has a unique solution

$$y_J \in B_\delta(\bar{y}_J) \quad \text{for all} \quad J = [n_-, n_+],$$

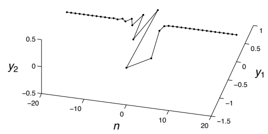
where $-n_-, n_+ \geq N$.
Approximation error: $\|\bar{y}_J - y_J\| \leq C \|\bar{y}_{n_-} - \bar{y}_{n_+}\|$.

Th. Hüls.
Homoclinic orbits of non-autonomous maps and their approximation.
J. Difference Equ. Appl., 12(11):1103–1126, 2006.

Hénon system: Computation of homoclinic trajectories

Homoclinic orbit of the transformed system

$$y_{n+1} = h(y_n + \xi_n, \lambda_n, b) - \xi_{n+1}, \quad n \in J, \qquad h: \text{ Hénon's map}$$

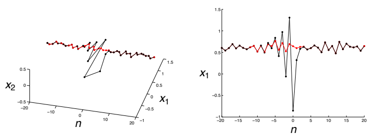


Homoclinic orbit w.r.t. the fixed point 0.

Hénon system: Computation of homoclinic trajectories

Homoclinic trajectories

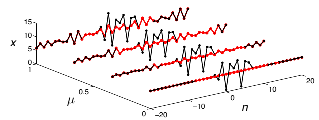
Let $x_n = y_n + \xi_n$, $n \in J$.
Then x_J and ξ_J are two homoclinic trajectories.



Predator-prey model: Computation of homoclinic trajectories

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \exp\left(a\left(1 - \frac{x_n}{K_n}\right) - b y_n\right) \\ c x_n \left(1 - \exp(-b y_n)\right) \end{pmatrix}, \quad n \in \mathbb{Z}.$$

$K_n = 10 + \mu \cdot r_n$, $\quad r_n \in [-\frac{1}{2}, \frac{1}{2}]$ uniformly distributed, $\quad \mu \in [0, 1]$.



Predator-prey model: Computation of homoclinic trajectories

Predator-prey model

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \exp\left(a\left(1 - \frac{x_n}{K_n}\right) - b y_n\right) \\ c x_n \left(1 - \exp(-b y_n)\right) \end{pmatrix}, \quad n \in \mathbb{Z}.$$

x_n	prey at time n,	$a = 7$,
y_n	predator at time n,	$b = 0.2$,
K_n	carrying capacity,	$c = 2$.

J. R. Beddington, C. A. Free, and J. H. Lawton.
Dynamic complexity in predator-prey models framed in difference equations.
Nature, 255(5503):58–60, 1975.

J. D. Murray.
Mathematical biology. I, volume 17 of *Interdisciplinary Applied Mathematics*.
Springer-Verlag, New York, third edition, 2002.

Outline

Some remarks
and
extended results

Remarks: Invariant fiber bundles

Stable and unstable fiber bundles are the non-autonomous generalization of stable and unstable manifolds:

Ψ : solution operator of $x_{n+1} = f_n(x_n)$,

$$S_0^s(\xi_{\mathbb{Z}}) = \left\{x \in \mathbb{R}^k : \lim_{m\to\infty} \|\Psi(m,0)(x) - \xi_m\| = 0\right\},$$
$$S_0^u(\xi_{\mathbb{Z}}) = \left\{x \in \mathbb{R}^k : \lim_{m\to-\infty} \|\Psi(m,0)(x) - \xi_m\| = 0\right\}.$$

Approximation results:

C. Pötzsche and M. Rasmussen.
Taylor approximation of invariant fiber bundles for non-autonomous difference equations.
Nonlinear Anal., 60(7):1303–1330, 2005.

Let $x_{\mathbb{Z}}$ be a homoclinic trajectory w.r.t. $\xi_{\mathbb{Z}}$. Then

$$x_0 \in S_0^s(\xi_{\mathbb{Z}}) \cap S_0^u(\xi_{\mathbb{Z}}).$$

Heteroclinic trajectories

Autonomous world: **Heteroclinic orbit**.
Let ξ^+ and ξ^- two fixed points.
A trajectory $x_{\mathbb{Z}}$ is heteroclinic w.r.t. $\xi^\pm$, if

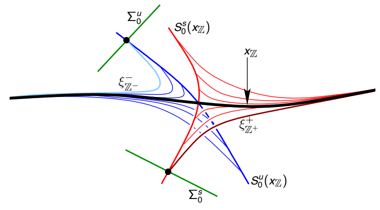
$$\lim_{n\to-\infty} x_n = \xi^-, \qquad \lim_{n\to\infty} x_n = \xi^+.$$

Non-autonomous analog: **Heteroclinic trajectories**.
Let $\xi_{\mathbb{Z}}^-$ be a trajectory that is bounded in backward time, and let $\xi_{\mathbb{Z}}^+$ be a trajectory that is bounded in forward time.
A trajectory $x_{\mathbb{Z}}$ is heteroclinic w.r.t. $\xi_{\mathbb{Z}}^\pm$, if

$$\lim_{n\to-\infty} \|x_n - \xi_n^-\| = 0, \qquad \lim_{n\to\infty} \|x_n - \xi_n^+\| = 0.$$

Th. Hüls and Y. Zou.
On computing heteroclinic trajectories of non-autonomous maps.
Discrete Contin. Dyn. Syst. Ser. B, 17(1):79–99, 2012.

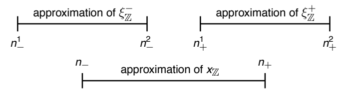
Heteroclinic trajectories



Problem: Families of semi-bounded trajectories exist.
Separate one semi-bounded trajectory by posing an initial condition.

Heteroclinic trajectories

One achieves accurate approximations of semi-bounded and heteroclinic trajectories, by solving *appropriate* boundary value problems.



Boundary operator:

$$b(x_{n_-}, x_{n_+}) = \begin{pmatrix} Y_-^T(x_{n_-} - \xi_{n_-}^-) \\ Y_+^T(x_{n_+} - \xi_{n_+-}^-) \end{pmatrix},$$

Y_-: base of $\mathcal{R}(P_{n_-}^u)^\perp$, $\quad Y_+$: base of $\mathcal{R}(P_{n_+}^s)^\perp$.

Computation of dichotomy projectors

Fix $N \in \mathbb{Z}$ and compute P_N^s as follows:
For each $i = 1, \dots, k$ solve for $n \in \mathbb{Z}$

$$u_{n+1}^i = A_n u_n^i + \delta_{n,N-1} e_i, \quad n \in \mathbb{Z}, \quad A_n = Df_n(\xi_n^\pm)$$

e_i : i-th unit vector, δ : Kronecker symbol.

Unique bounded solution for $n \in \mathbb{Z}$:

$$u_n^i = G(n,N)e_i, \quad G(n,N) = \begin{cases} \Phi(n,N)P_N^s, & n \geq N, \\ -\Phi(n,N)P_N^u, & n < N. \end{cases}$$

Thus

$$u_N^i = G(N,N)e_i = P_N^s e_i.$$

Therefore

$$P_N^s = \begin{pmatrix} u_N^1 & u_N^2 & \dots & u_N^k \end{pmatrix}.$$

Finite approximations can be achieved since **errors decay exponentially fast towards** the midpoint.

Computation of dichotomy projectors

Error estimates for approximate dichotomy projectors:

Th. Hüls.
Numerical computation of dichotomy rates and projectors in discrete time.
Discrete Contin. Dyn. Syst. Ser. B, 12(1):109–131, 2009.

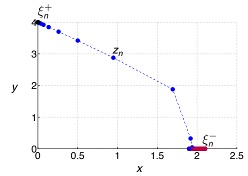
Extended results:

Th. Hüls.
Computing Sacker-Sell spectra in discrete time dynamical systems.
SIAM J. Numer. Anal., 48(6):2043–2064, 2010.

Example: Heteroclinic trajectories

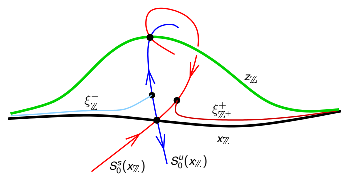
Discrete time model for two competing species x and y:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{s_n x_n}{x_n + y_n} \\ y_n e^{r-(x_n+y_n)} \end{pmatrix}, \quad r = 4, \quad s_n = 2 + \tfrac{1}{5}\sin(\tfrac{1}{5}n)$$



Y. Kang and H. Smith.
Global Dynamics of a Discrete Two-species Lottery-Ricker Competition Model.
Journal of Biological Dynamics 6(2):358-376, 2012.

Homoclinic and heteroclinic trajectories



Heteroclinic trajectories: $\xi_{\mathbb{Z}}^+$, $\xi_{\mathbb{Z}}^-$ and $x_{\mathbb{Z}}$.
If $S_0^s(x_{\mathbb{Z}})$ and $S_0^u(x_{\mathbb{Z}})$ intersect transversally, we find homoclinic trajectories $x_{\mathbb{Z}}$, $z_{\mathbb{Z}}$.

Conclusion

- We derive approximation results for homoclinic (and heteroclinic) trajectories via boundary value problems.
- Justification: **Due to our hyperbolicity assumptions, errors on finite intervals decay exponentially fast toward the midpoint.**
- One can verify these hyperbolicity assumption, using techniques that have been introduced, for example, in:

L. Dieci, C. Elia, and E. Van Vleck.
Exponential dichotomy on the real line: SVD and QR methods.
J. Differential Equations, 248(2):287–308, 2010.

Th. Hüls.
Computing Sacker-Sell spectra in discrete time dynamical systems.
SIAM J. Numer. Anal., 48(6):2043–2064, 2010.

Conclusion

- The exponential decay of error enables accurate computation of covariant vectors:

G. Froyland, Th. Hüls, G.P. Morriss and Th.M. Watson.
Computing covariant vectors, Lyapunov vectors, Oseledets vectors, and dichotomy projectors: a comparative numerical study.
arXiv:1204.0871, 2012.