W-maps and harmonic averages

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Thanks

I am grateful to the organizers for the invitation and giving me a chance to present my results.

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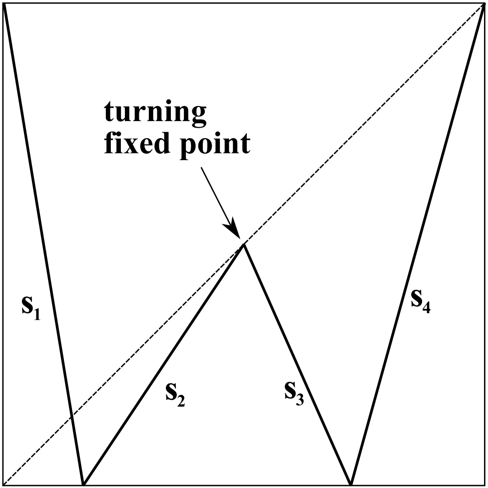
Harmonic mean

$$H(a,b) = \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

W-map



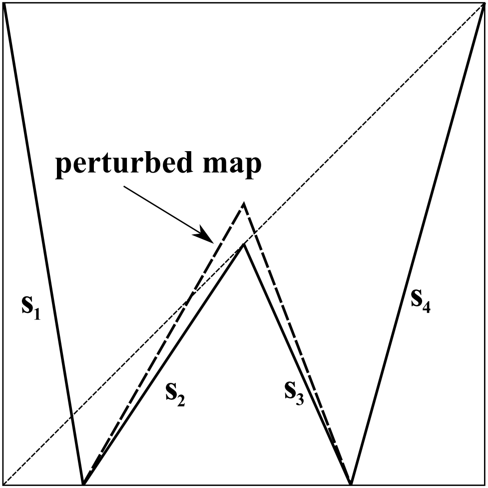
First considered by G. Keller (1994) with $s_1 = s_4 = 4$, $s_2 = s_3 = 2$.

Acim Stability of map τ

acim = absolutely continuous invariant measure

We consider τ_0 with acim μ_0 and a family of perturbations τ_a with acim's μ_a such that $\tau_a \to \tau_0$ as $a \to 0$, say in Skorokhod metric.
We say, τ_0 is acim stable if $\mu_a \to \mu_0$ as $a \to 0$, say in weak$*$ topology.
Keller constructed perturbations such that his W-map was not acim stable under these perturbations.

Our perturbations

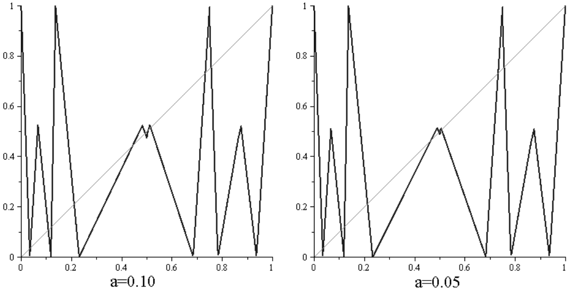


Why is there a problem?

From standard Lasota-Yorke (1973) inequality it follows that τ is acim stable if $|\tau'| \geq \lambda > 2$. Stability of isolated eigenvalues and corresponding eigenfunctions of Frobenius-Perron operator was proved by Keller and Liverani (1999).
Standard method to improve the slope is to consider an iterate of τ. It does not work for perturbations of a map with a turning fixed point.

Iterates of perturbed τ_0 I

Second iterates for $a = 0.10$ and $a = 0.05$:



The results

Three cases:

$\frac{1}{s_2} + \frac{1}{s_3} > 1$: There exists a small invariant subinterval around the turning fixed point x_0 and

$$\mu_a \to \delta_{\{x_0\}} \ ,$$

$*$-weakly.

$\frac{1}{s_2} + \frac{1}{s_3} = 1$: for example $s_2 = s_3 = 2$.
τ_a are exact on $[0,1]$ and

$$\mu_a \to \alpha \delta_{\{x_0\}} + (1-\alpha)\mu_0 \ ,$$

$*$-weakly. To prove this we used the general formulas for acim of piecewise linear eventually expanding maps (Góra, 2009).

The results:

$\frac{1}{s_2} + \frac{1}{s_3} < 1$:
τ_0 is acim stable, i.e.,

$$\mu_a \longrightarrow \mu_0 \ ,$$

not only $*$-weakly but also in L^1. The proof is based on a slightly stronger Lasota-Yorke inequality (Eslami and Góra, to appear).

Stronger Lasota-Yorke inequality:

Theorem

Let $\tau : [0,1] \to [0,1]$ be piecewise expanding with q branches, piecewise C^{1+1} and satisfy

$$\eta = \max_{1 \le i < q} \left(\frac{1}{s_i} + \frac{1}{s_{i+1}} \right) < 1 \,, \tag{1}$$

where $s_i = \min |\tau_i'|$, $i = 1, 2, \ldots, q$.
Then, for every $f \in BV([0,1])$,

$$\bigvee_I P_\tau f \le \eta \bigvee_I f + \gamma \int_I |f| \, dm \,. \tag{2}$$

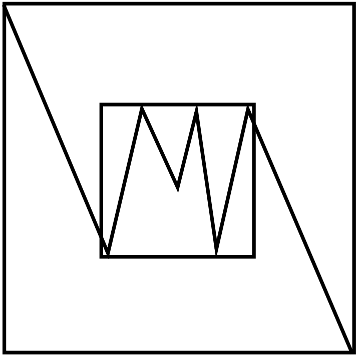
$\gamma = \frac{M}{s^2} + \frac{2}{s \min_{1 \le i \le q} m(I_i)}$, *where $s = \min s_i$, I_i is the domain of branch τ_i and M is the common Lipschitz constant of τ_i', $i = 1, 2, \ldots, q$.*

Stability

Now, the whole stability theory holds under the above slightly weaker assumption.
In particular, Ulam's approximation method works under the assumption (1), (Góra and Boyarsky, to appear in Discrete and Continuous Dynamical System - A).
Ulam's method works also for standard W-map $(s_1 = s_4 = 4,\, s_2 = s_3 = 2)$.

A small detail

The above inequality holds if we assume additionally that $\tau(0), \tau(1) \in \{0, 1\}$. This restriction can be removed considering our system onto a slightly bigger interval and properly extended:



Minimax Problem

The constant

$$\eta = \left(\frac{1}{s_1} + \frac{1}{s_2}\right)$$

shows up in the following minimax problem:
Let $s_1, s_2 > 1$ and $\alpha + \beta = c$, where $\alpha, \beta > 0$. Then,

$$\min_{\alpha,\beta} \max\{s_1\alpha, s_2\beta\} = \frac{1}{\frac{1}{s_1} + \frac{1}{s_2}} c \, .$$

Proof:

Proof: We have

$$\min_{\alpha,\beta} \max\{s_1\alpha, s_2\beta\} = \min_{\alpha} \max\{s_1\alpha, s_2(c-\alpha)\} .$$

The line $f(\alpha) = s_1\alpha$ is increasing while the line $g(\alpha) = s_2(c-\alpha)$ is decreasing. The $\min_\alpha \max\{s_1\alpha, s_2(c-\alpha)\}$ occurs where the lines intersect, i.e., at

$$\alpha = \frac{s_2 c}{s_1 + s_2} ,$$

which gives

$$\min_{\alpha,\beta} \max\{s_1\alpha, s_2\beta\} = \frac{s_1 s_2 c}{s_1 + s_2} = \frac{1}{\frac{1}{s_1} + \frac{1}{s_2}} c .$$

Corollary:

If piecewise expanding τ satisfies

$$\eta = \max_{1 \leq i < q} \left(\frac{1}{s_i} + \frac{1}{s_{i+1}} \right) < 1 \, ,$$

then for arbitrary small interval J the largest connected component of $\tau^n(J)$ grows as $\eta^{-n} m(J)$ until it contains a whole domain I_i of one of the branches of τ.

Lower bound for the densities

For one transformation density is bounded away from 0 (Keller 1978, Kowalski 1979).
It is possible to construct a family of piecewise expanding maps τ_n with slopes $|\tau_n'| > 2$, with acims $\mu_n = f_n m$, converging to the standard W-map such that $\operatorname{supp} f_n = [0,1]$ and $\mu_n \to \delta_{\{1/2\}}$ $*$-weakly. Then, there is no uniform lower bound for densities f_n (Li, preprint).

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