Different approaches to the global periodicity problem

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Talk based on the work made in collaboration with:

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1.The problem. An example

Consider the Lyness recurrence

$$x_{j+2} = \frac{a + x_{j+1}}{x_j}, \quad a \in \mathbb{C},$$

or equivalently, the DDS generated by

$$F_a(x, y) = \left(y, \frac{a+y}{x}\right).$$

QUESTION: For which values of a the map F_a is globally periodic?.
Recall that it is said that F is globally periodic if there exists $m = m(a)$ such that

$$F_a^m = F_a \circ \ldots^{(m)} \ldots \circ F_a = \mathrm{Id}$$

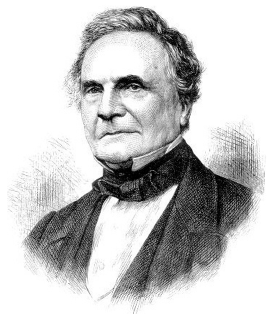
The answer is well known: $a = 0$ and $m = 6$ and $a = 1$ and $m = 5$.

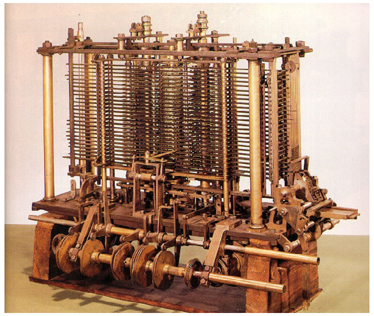
1.The problem

A map $F : \mathcal{U} \to \mathcal{U}$ is said globally m-periodic if $F^m = \mathrm{Id}$ and m is the smallest natural with this property.
The functional equation

$$F^m = \mathrm{Id}$$

is called functional Babbage equation.





Charles Babbage (1791-1871) Babbage's difference engine-2

1. The problem

Given a class of maps F, we want to find and characterize the globally periodic cases.
Recall that the difference equation

$$x_{j+n} = f(x_j, x_{j+1}, \ldots, x_{j+n-1})$$

can be studied considering the DDS given by the map

$$F(x_1, \ldots, x_n) = (x_2, \ldots, x_{n-1}, f(x_1, \ldots, x_n)).$$

The goal of this talk is to present some of the techniques that we are using to approach the above problem and some of our results.

2. Techniques for studying global periodicity

- Find special properties of the dynamical system induced by F.
- The local linearization given by the Montgomery-Bochner Theorem.
- Theory of normal forms.
- Properties of the so called vanishing sums. These are polynomial identities with integer coefficients involving only roots of the unity.

3. Properties of the globally periodic maps-I

First we present some classical results when $\mathcal{U} = \mathbb{R}^n$:
Recall that a map F defined on an open set U, $\mathcal{C}^k$- linearizes if there exists a $\mathcal{C}^k$-homeomorphism, $\psi : U \to \psi(U) \subset \mathbb{R}^n$, for which

$$\psi \circ F \circ \psi^{-1}$$

is the restriction of a linear map to $\psi(U)$. The map ψ is called a linearization of F on U.

- Any continuous globally periodic map on $\mathbb{R}^2$, $\mathcal{C}^0$-linearizes. (Kerékjártó, 1919 and 1920)
- For $n \geq 3$ there are globally periodic continuous maps on $\mathbb{R}^n$ that do not linearize. (Bing, 1952 and 1964)
- For $n \geq 7$ there are globally periodic differentiable maps on $\mathbb{R}^n$ without fixed points. (Kister, 1969)

3. Properties of the globally periodic maps-II

MONTGOMERY-BOCHNER THEOREM

- If x is a fixed of a $\mathcal{C}^k$ ($k \geq 1$) m-periodic map F, then F $\mathcal{C}^k$-linearizes in a neighborhood of x. The linearization ψ is explicitly given by

$$\psi = \frac{1}{m} \sum_{i=0}^{m-1} ((DF)_x)^{-i} F^i = \mathrm{Id} + \cdots$$

Montgomery and L. Zippin, 1955.
IDEA OF THE PROOF: It is easy to see that

$$\psi \circ F = L \circ \psi$$

where L is the linear map $L(y) = (DF)_x y$, because $F^m = L^m = \mathrm{Id}$.

3. Properties of the globally periodic maps-III

We use the following well-known properties for globally periodic maps:

- If a map F is m-periodic then F^k, for any integer k is also periodic.
- If a map $F : \mathcal{U} \to \mathcal{U}$ is periodic then it has to be bijective in $\mathcal{U}$.
- If a rational map is periodic of period m in open subset of $\mathbb{R}^n$ then it has to be periodic –also of period m– in the whole real or complex space, except at the points where F or its iterates are not well defined.
- If $F : \mathcal{U} \to \mathcal{U}$ is a periodic map of period m and $\mathbf{x} \in \mathcal{U}$ is a fix point of F then $((DF)_x)^m = \text{Id}$.
- The eigenvalues of $(DF)_x$ have to be m roots of the unity.
- The matrix $(DF)_x$ diagonalices.
- The fixed points of a periodic map can not be neither attractor nor repeller.
- They have zero algebraic entropy.

3. Properties of the globally periodic maps-IV

An easy, but useful, consequence of the Mongomery-Bochner Theorem is:

Proposition

Let F be a differentiable map having a fixed point x_0. Assume that F is m-periodic and let k be the minimum positive k such that $((DF)_{x_0}))^k = \text{Id}$. Then $k = m$.

For instance a simple corollary is:

Corollary

If a smooth map $F(x, y) = (-x + ..., -y + ...)$ is m-periodic then it is an involution $(m = 2)$.

3. Properties of the globally periodic maps-V

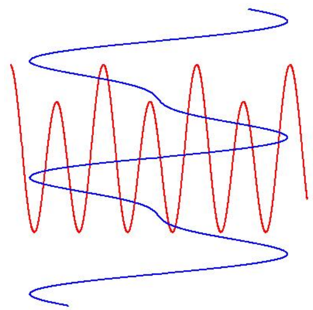
We also have proved the following "more dynamical properties" when the map F is m-periodic:

- If it is $\mathcal{C}^1$ then the DDS generated by F has an absolutely continuous invariant measure ν, that is there exists an integrable map g such that
$$\nu(A) = \int_A g(x)\, dx, \quad \text{and} \quad \nu(F^{-1}(A)) = \nu(A).$$
- The map DDS generated by F, defined on a subset of $\mathbb{R}^n$, has n functionally independent first integrals. Moreover if the map is bijective this property "essentially" characterizes the globally periodic maps.

Recall that a non-constant function H is called a first integral or invariant of the DDS generated by F if
$$H(F(x)) = H(x).$$
Notice that the level sets of H are invariant by F

4. "Equivalence" between global periodicity and existence of n first integrals

When and why a planar map with two functionally independent first integrals is periodic?



The set of intersection points is invariant by F.
We need that the number of intersections is bounded.

4. "Equivalence" between global periodicity and existence of n first integrals-II

Example. The map

$$F(x, y) = (x + 2\pi, y)$$

has the two functionally independent first integrals

$$H_1(x, y) = y - \sin(x), \qquad H_2(x, y) = y.$$

and is clearly non periodic.

4. "Equivalence" between global periodicity and existence of n first integrals-III

For instance the Lyness map

$$F(x, y) = (y, \frac{a + y}{x})$$

has for all $a \in \mathbb{C}$ the first integral

$$H_1(x, y) = \frac{(x + 1)(y + 1)(x + y + a)}{xy}$$

and when $a = 0$ it has also the first integral

$$H_2(x, y) = \frac{x^4y^2 + x^2y^4 + y^4 + y^2 + x^2 + x^4}{x^2y^2}$$

and when $a = 1$,

$$H_2(x, y) = \frac{y^4x + (x^3 + x^2 + 2x + 1)y^3 + (x^3 + 5x^2 + 3x + 2)y^2 + (x^4 + 2x^3 + 3x^2 + 3x + 1)y + x^3 + 2x^2 + x}{x^2y^2}.$$

4. "Equivalence" between global periodicity and existence of n first integrals-IV

How to construct the n independent first integrals for any globally periodic map?

Proposition

Let $F : \mathcal{U} \subset \mathbb{R}^n \to \mathcal{U}$ be a globally m–periodic map on U. Let

$$\Phi : \mathcal{U}^m = \overbrace{\mathcal{U} \times \mathcal{U} \times \cdots \times \mathcal{U}}^{m} \longrightarrow \mathbb{K}$$

be a symmetric function. Then, whenever it is not a constant function,

$$H_\phi(x) = \Phi(x, F(x), \ldots, F^{m-1}(x))$$

is a first integral of the DDS generated by F.

4. "Equivalence" between global periodicity and existence of n first integrals-V

Example. Consider the map

$$F(x, y) = (y, \frac{c}{xy}).$$

corresponding to the difference equation $x_{n+2} = \dfrac{c}{x_n x_{n+1}}$.

It holds

$$(x, y) \to (y, \frac{c}{xy}) \to (\frac{c}{xy}, x) \to (x, y),$$

so it is 3-periodic. So $n = 2$ and $m = 3$.

The first integrals are constructed using two suitable symmetric functions of the orbit.

4. "Equivalence" between global periodicity and existence of n first integrals-VI

Recall that

$$(x,y) \to (y, \frac{c}{xy}) \to (\frac{c}{xy}, x) \to (x,y),$$

Taking $\sigma_1(a,b,c,d,e,f) = a+b+c+d+e+f$ we get:

$$H_1(x,y) = 2\left(x+y+\frac{c}{xy}\right).$$

Taking $\sigma_2(a,b,c,d,e,f) = a^2+b^2+c^2+d^2+e^2+f^2$,

$$H_2(x,y) = 2\left(x^2+y^2+\frac{c^2}{x^2y^2}\right).$$

Note that the symmetric function $\sigma_3(a,b,c,d,e,f) = abcdef$ gives a constant function and is not a first integral.

5. Applications of the given tools

Consider the $n-$th order rational difference equation

$$x_{j+n} = \frac{A_1x_j + A_2x_{j+1} + \cdots + A_nx_{j+n-1} + A_0}{B_1x_j + B_2x_{j+1} + \cdots + B_nj_{j+n-1} + B_0},$$

with initial condition $(x_1, x_2, \ldots, x_n) \in (0,\infty)^n$, and $\sum_{i=0}^n A_i > 0$, $\sum_{i=0}^n B_i > 0$, $A_i \geq 0$, $B_i \geq 0$, and $A_1^2 + B_1^2 \neq 0$.

The following globally periodic difference equations are well known:

$$x_{j+2} = \frac{x_{j+1}}{x_j}, \quad x_{j+2} = \frac{x_{j+1}+1}{x_j}, \quad x_{j+3} = \frac{x_{j+1}+x_{j+2}+1}{x_j}$$

$$x_{j+1} = x_j, \quad x_{j+1} = \frac{1}{x_j} \quad \text{(trivial cases)}.$$

5. Applications of the given tools-II

Each $m-$periodic k-th order difference equation produces in a natural way periodic difference equation of higher order. For instance the ones of order 2 given in the previous slide

$$x_{j+2} = \frac{x_{j+1}}{x_j}, \quad x_{j+2} = \frac{x_{j+1}+1}{x_j}, \tag{1}$$

produce the following ones

$$x_{j+2\ell} = \frac{x_{j+\ell}}{x_j}, \qquad x_{j+2\ell} = \frac{x_{j+\ell}+1}{x_j},$$

for any positive integer ℓ, which are also periodic. Moreover taking $x_n = \alpha y_n, \alpha \neq 0$ they can be written as

$$y_{j+2\ell} = \frac{\alpha y_{j+\ell}}{y_j}, \quad y_{j+2\ell} = \frac{\alpha y_{j+\ell} + \alpha^2}{y_j}.$$

We will say that they are equivalent to (1).

5. Applications of the given tools-III

Theorem

Consider the $n-$th order rational difference equation

$$x_{j+n} = \frac{A_1x_j + A_2x_{j+1} + \cdots + A_nx_{j+n-1} + A_0}{B_1x_j + B_2x_{j+1} + \cdots + B_nj_{j+n-1} + B_0},$$

with the above hypotheses. Then for

$$n \in \{1,2,3,4,5,7,9,11\}$$

all the globally periodic cases are equivalent the 5 ones given in the previous slides (3 of the Lyness type and the 2 trivial ones).

Open Question

Is the above result true for any n?

Remark

When A_j and B_j are no more non-negative there are globally periodic maps of all periods: the Möbius maps.

5. Applications of the given tools-IV

Theorem

Any $(n+1)$-periodic recurrence of class $\mathcal{C}^k$ defined in an open connected subset of $\mathbb{R}^n$ can be $\mathcal{C}^k$-linearized.

The proof consists in showing that the Montgomery-Bochner linearization is in this case globally invertible.

For other globally m-periodic difference equations the same idea works.

Unfortunately, recently we have proved that there are simple explicit involutions $(m = 2)$ (not coming from a difference equation) for which the linearization given by the Montgomery-Bochner Theorem is not globally invertible.

5. Applications of the given tools-V

Proposition

Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by

$$F(x, y) = \left(x + 4xy + f(x, y), -y + 2(x^2 + y^2) - f(x, y)\right), \tag{2}$$

where $f(x, y) = 4(x + y)^2(y - x) - 4(x + y)^4$. Then F is an involution, has $(0, 0)$ as a fixed point and its associated Montgomery-Bochner linearization $\psi = \frac{1}{2}\left(\mathrm{Id} + (DF)^{-1}_{(0,0)} \circ F\right)$ is not a global diffeomorphism.

Open Question

Is the Montgomery-Bochner linearization associated to a globally periodic $\mathcal{C}^1$ difference equation with a fixed point globally invertible?

Open Question

Is any $\mathcal{C}^1$ globally periodic map with a fixed point globally linearizable?

6. A concrete case: the Coxeter recurrence

For any $n \geq 1$, the map

$$F_n(x_1, \ldots, x_n) = \left(x_2, \ldots, x_n, 1 - \cfrac{x_n}{1 - \cfrac{x_{n-1}}{1 - \cfrac{x_{n-2}}{1 - \cdots \cfrac{x_2}{1 - x_1}}}}\right)$$

is associated to the Coxeter recurrence which is $(n + 3)$-periodic.
It holds:

Theorem

The Coxeter maps have exactly $\left[\frac{n+2}{2}\right]$ fixed points, all them with positive coordinates. Moreover at each of these fixed points F_n is locally conjugated to a different linear map.

We do not know how to prove that these conjugations are global (in the corresponding open sets).

6. A warning!

For many globally periodic real difference equations, including the linear ones,

$$x_{j+n} = f(x_j, x_{j+1}, \ldots, x_{j+n-1}),$$

given its associated map

$$F(x_1, \ldots, x_n) = (x_2, \ldots, x_n, f(x_1, \ldots, x_n)),$$

the following properties hold:

- **P1:** For each fixed $y \in \mathbb{R}$ and $w \in \mathbb{R}$ the map

  $$f_y(w) := f(w, y, \ldots, y)$$

  is an involution.
- **P2:** The map $\sigma \circ F$ is an involution, where $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is $\sigma(x_1, x_2, \ldots, x_{n-1}, x_n) = (x_n, x_{n-1}, \ldots, x_2, x_1)$.

It can be seen that for $n = 2$ both properties are equivalent.

6. A warning!-II

For instance for the globally 8-periodic Lyness recurrence, $f(x,y,z)=\frac{1+y+z}{x}$, both **P1** and **P2** hold:

- $f_{\mathbf{y}}(w):=f(w,y,y)=\dfrac{1+2y}{w}$ is an involution.
- The map
$$\sigma\circ F(x,y,z)=\left(\frac{1+y+z}{x},z,y\right)$$
is also an involution.

We prove that this is not always the case.

6. A warning!-III

Proposition

None of the properties **P1** and **P2** is a necessary condition for global periodicity of real difference equations.

Proof: Consider $F=\Phi\circ L\circ\Phi^{-1}$, where

$$\Phi(x,y)=(x+g(y),y+g(-x-y))$$

with

$$g(z)=-z-z^2-z^3 \qquad \text{and} \qquad L(x,y)=(y,-x-y).$$

It can be seen that F writes as $F(x,y)=(y,f(x,y))$, for a suitable f.

6. A warning!-IV

Nevertheless for the map $F=\Phi\circ L\circ\Phi^{-1}$ given in the previous slide if we consider $\widetilde{\sigma}:=\Phi\circ\sigma\circ\Phi^{-1}$ it holds that

- $\widetilde{\sigma}$ is an involution
- The map $\widetilde{\sigma}\circ F$ is also an involution

Open Question

Is property **P2** always true, changing σ by a suitable involution?

7. Normal form theory

Given a ordinary differential equation $\dot{x}=F(x)$ or a DDS $x_{j+1}=F(x_j)$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ the Normal Form Theory provides a method to remove the unessential parameters in F. This theory goes back to Poincaré and Lyapunov.
For instance given a planar analytic differential equation

$$(\dot{x},\dot{y})=(-y+f_2(x,y)+f_3(x,y)+\cdots,x+g_2(x,y)+g_3(x,y)+\cdots),$$

is well known that the origin is either a center or a focus. In this later case its stability is given by the sign of the first non-null number of a list of polynomial expressions, called the Lyapunov quantities $V_3,V_5,\ldots$ obtained from the coefficients of f_j and g_j. In particular the first one

$$V_3=V_3(f_2,f_3,g_2,g_3)$$

7. Normal form theory-II

It turns out that the normal form of the differential equation

$$(\dot{x},\dot{y}) = (-y + f_2(x,y) + f_3(x,y) + \cdots, x + g_2(x,y) + g_3(x,y) + \cdots),$$

is

$$(\dot{x},\dot{y}) = (-y-(x^2+y^2)(V_3y+T_2x)+\cdots, x+(x^2+y^2)(V_3x-T_2y)+\cdots).$$

So only V_3 (that gives the stability) and T_2 (that gives information about the "period" of the orbits) remain.
Following the same idea, since by the Montgomery-Bochner Theorem says that globally periodic maps with a fixed point locally linearize we can apply the normal form theory for DDS to obtain the obstructions to be linearized.
As far as we know this approach is new. We will call these obstructions global periodicity conditions although they are also linerizability conditions.

7. Normal form theory-III

Theorem

Consider a smooth complex map of the form

$$F(x,y) = \left(\alpha x + \sum_{i+j\geq 2} f_{i,j}x^iy^j, \frac{1}{\alpha}y + \sum_{i+j\geq 2} g_{i,j}x^iy^j\right), \qquad (3)$$

where α is a primitive m-root of unity, $m \geq 5$. Then the conditions $\mathcal{P}_1(F) = \mathcal{P}_2(F) = 0$ are necessary for F to be m-periodic, where

$$\begin{aligned}\mathcal{P}_1(F) :=& (f_{2,1} + f_{1,1}g_{1,1})\,\alpha^4 - f_{1,1}\,(2f_{2,0} - g_{1,1})\,\alpha^3\\ &+ (2g_{2,0}f_{0,2} - f_{1,1}f_{2,0} + f_{1,1}g_{1,1})\,\alpha^2\\ &- (f_{2,1} + f_{1,1}f_{2,0})\,\alpha + f_{1,1}f_{2,0},\\ \mathcal{P}_2(F) :=& g_{0,2}g_{1,1}\alpha^4 - (g_{1,2} + g_{0,2}g_{1,1})\,\alpha^3\\ &+ (f_{1,1}g_{1,1} + 2g_{2,0}f_{0,2} - g_{0,2}g_{1,1})\,\alpha^2\\ &+ g_{1,1}\,(-2g_{0,2} + f_{1,1})\,\alpha + f_{1,1}g_{1,1} + g_{1,2}.\end{aligned}$$

7. Normal form theory-IV

As an application of the theorem applications we study the global periodicity for the 2-periodic Lyness recurrence

$$x_{j+2} = \frac{a_j + x_{j+1}}{x_j}, \quad \text{where} \quad a_j = \begin{cases} a & \text{for } j = 2\ell+1,\\ b & \text{for } j = 2\ell,\end{cases} \qquad (4)$$

and $a, b \in \mathbb{C}$.

Theorem

The only globally periodic recurrences in (4) are:

(I) The cases $a = b = 0$ (6-periodic) and $a = b = 1$ (5-periodic).
(II) The cases $a = (-1 \pm i\sqrt{3})/2$ and $b = \bar{a} = 1/a$, 10-periodic.

7. Normal form theory-V

Main steps of the proof:

I) The sequence $\{x_j\}$ can be reobtained as

$$(x_1,x_2) \xrightarrow{G_a} (x_2,x_3) \xrightarrow{G_b} (x_3,x_4) \xrightarrow{G_a} (x_4,x_5) \xrightarrow{G_b} (x_5,x_6) \xrightarrow{G_a} \cdots$$

where $G_\alpha(x,y) = (y,(\alpha+y)/x)$, with $\alpha \in \{a,b\}$. So the behavior of (4) is given by the dynamical system generated by the map:

$$G_{b,a}(x,y) := G_b \circ G_a(x,y) = \left(\frac{a+y}{x}, \frac{a+bx+y}{xy}\right)$$

7. Normal form theory-VI

II) Given the family of maps

$$G_{b,a}(x,y) := G_b \circ G_a(x,y) = \left(\frac{a+y}{x}, \frac{a+bx+y}{xy}\right),$$

if we introduce the new parameters

$$\begin{aligned} a &= \frac{B^3\,(\lambda^2+1)+\lambda\,(2\,B^3-1)}{B\,(\lambda+1)^2}, \\ b &= -B+(B^2-a)^2, \quad \text{with} \quad B(\lambda+1)\neq 0 \quad \text{and} \quad \lambda\neq 0, \end{aligned} \tag{5}$$

it holds:

- We cover all the values of a and b in $\mathbb{C}$.
- The fixed point is (B, B^2-a), where a is given above.
- The eigenvalues of $G_{b,a}$ at this point are λ and $1/\lambda$.

7. Normal form theory-VII

III) The cases $m = 1, 2, 3$ and 4 are trivial.
For $m \geq 5$, translating the fixed point to the origin and making a linear change we can apply our Normal form Theorem to the new map $F_{B,\lambda}$. All the coefficients of this map are rational functions of B and λ.
From both conditions $\mathcal{P}_i(F_{B,\lambda}) = 0, i = 1, 2$, we obtain the same periodicity condition $C_1(B,\lambda) = 0$, where $C_1(B,\lambda)$ is

$$\begin{aligned} &B^6\lambda^{10}+9B^6\lambda^9+35B^6\lambda^8+80B^6\lambda^7+124B^6\lambda^6+2B^3\lambda^9+142B^6\lambda^5+8B^3\lambda^8 \\ &+124B^6\lambda^4+18B^3\lambda^7+80B^6\lambda^3+32B^3\lambda^6+35B^6\lambda^2+40B^3\lambda^5+9B^6\lambda \\ &+32B^3\lambda^4+\lambda^7+B^6+18B^3\lambda^3+3\lambda^6+8B^3\lambda^2+2\lambda^5+2B^3\lambda+3\lambda^4+\lambda^3. \end{aligned}$$

7. Normal form theory-VIII

Computing similarly the condition $\mathcal{P}_3(F_{B,\lambda}) = 0$, we obtain a new periodicity condition $C_2(B,\lambda) = 0$, where $C_2(B,\lambda)$ is huge polynomial of degree 37 in B and λ.
To study the solutions of

$$C_1(B,\lambda) = 0, \quad C_2(B,\lambda) = 0,$$

we compute

$$R(\lambda) := \mathrm{Res}(C_1(B,\lambda), C_2(B,\lambda); B)$$

where Res is the resultant of both polynomials.
We get

$$R(\lambda) = \lambda^{36}\,(\lambda-1)^{24}\,(\lambda+1)^{72}\,\left(\lambda^2+1\right)^6\left(\lambda^2+\lambda+1\right)^{24}S^6(\lambda)\,T^6(\lambda),$$

where

$$S(\lambda) = \lambda^4+\lambda^3+\lambda^2+\lambda+1$$

and

$$T(\lambda) = 3\lambda^4+15\lambda^3+20\lambda^2+15\lambda+3$$

7. Normal form theory-IX

Then a necessary condition for for $F_{B,\lambda}$ to be m-periodic, for $m \geq 5$, is

$$\lambda^{36}\,(\lambda-1)^{24}\,(\lambda+1)^{72}\,\left(\lambda^2+1\right)^6\left(\lambda^2+\lambda+1\right)^{24}S^6(\lambda)\,T^6(\lambda) = 0,$$

where

$$S(\lambda) = \lambda^4+\lambda^3+\lambda^2+\lambda+1$$

and

$$T(\lambda) = 3\lambda^4+15\lambda^3+20\lambda^2+15\lambda+3,$$

where λ is also a primitive m-th of the unity is that either $S(\lambda) = 0$ or $T(\lambda) = 0$.
It can be seen that $T(\lambda)$ has no roots which are roots of the unity.
On the other hand the roots of $S(\lambda)$ are 5-th roots of the unity and give rise to a $F_{B,\lambda}$ which is globally 5-periodic. They correspond to the 5 and 10 periodic Lyness recurrences.

8. Vanishing sums-I

This is an ongoing project, also in collaboration with V. Mañosas(UPC) and X. Xarles(UAB).

We want to apply the properties of the so called vanishing sums to detect globally periodic cases.
To fix the ideas here we will center our attention to a family of maps already studied by Bedford

$$F(x,y) = (y, \frac{a+y}{b+x}), \quad a,b \in \mathbb{C}$$

It is known that the only periods appearing are $m \in \{5,6,8,12,18,30\}$.

We will approach to this problem with this new tool.

8. Vanishing sums-II

As in the study of the 2-periodic Lyness difference equation we introduce a rational change of parameters:

$$a = U(\lambda, \mu),$$
$$b = V(\lambda, \mu).$$

The map has generically two fixed points. With these new parameters the eigenvalues of DF at one of the fixed points are λ, μ.

The eigenvalues s at the other fixed point satisfy the equation

$$(\lambda\mu - \lambda - \mu)s^2 + (\lambda + \mu)s + 1 - \lambda - \mu = 0.$$

Since all λ, μ and s have to be roots of the unity we have got what is called a vanishing sum.They are subject of classical interest and classified by Conway and Jones (1967).

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