SYMPLECTIC DIFFERENCE SYSTEMS WITH PERIODIC COEFFICIENTS

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Symplectic difference systems

- Symplectic difference system:

$$(SDS) \qquad z_{k+1} = \mathcal{S}_k z_k,$$

where $z \in \mathbb{R}^{2n}$, $\mathcal{S} \in \mathbb{R}^{2n \times 2n}$ is symplectic, i.e.

$$\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

- (SDS) in entries:

$$z = \begin{pmatrix} x \\ u \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$$

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k$$

$x, u \in \mathbb{R}^n$, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathbb{R}^{n \times n}$.

- Symplecticity in terms of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$:

$$\begin{aligned} \mathcal{A}^T\mathcal{C} - \mathcal{C}^T\mathcal{A} &= 0, \\ \mathcal{B}^T\mathcal{D} - \mathcal{D}^T\mathcal{B} &= 0, \\ \mathcal{A}^T\mathcal{D} - \mathcal{C}^T\mathcal{B} &= I, \end{aligned}$$

equivalently ($S^T\mathcal{J}S = \mathcal{J}$ iff $S\mathcal{J}S^T = \mathcal{J}$)

$$\begin{aligned} \mathcal{A}\mathcal{B}^T - \mathcal{B}\mathcal{A}^T &= 0, \\ \mathcal{C}\mathcal{D}^T - \mathcal{D}\mathcal{C}^T &= 0, \\ \mathcal{A}\mathcal{D}^T - \mathcal{B}\mathcal{C}^T &= I. \end{aligned}$$

Particular cases of (SDS)

- Sturm-Liouville difference equation ($r_k \neq 0$):

$$(SL) \qquad \Delta(r_k \Delta x_k) + p_k x_{k+1} = 0.$$

Substitution $u = r\Delta x$:

$$\Delta x_k = \frac{1}{r_k} u_k, \quad \Delta u_k = -p_k x_{k+1}$$

i.e.,

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{r_k} \\ -p_k & 1 - \frac{p_k}{r_k} \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}.$$

- Linear Hamiltonian difference system:

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k,$$

where $A, B, C \in \mathbb{R}^{n\times n}$, $I - A$ invertible, $B^T = B$, $C^T = C$.

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} (I-A)^{-1} & (I-A)^{-1}B \\ C(I-A)^{-1} & C(I-A)^{-1}B + I - A^T \end{pmatrix}_k \begin{pmatrix} x_k \\ u_k \end{pmatrix}$$

and the matrix is the last system *is symplectic*.

Linear Hamiltonian differential systems

Linear Hamiltonian *differential* system:

$$(LHdS) \qquad z' = \lambda \mathcal{J} \mathcal{H}(t) z$$

$z = \binom{x}{u} \in \mathbb{C}^{2n}$, $\mathcal{H} \in \mathbb{C}^{2n\times 2n}$ is Hermitean and periodic, i.e.,

$$\mathcal{H}^*(t) = \mathcal{H}(t), \quad \mathcal{H}(t+T) = \mathcal{H}(t).$$

- M. I. Krein, Stablility zones... 1955, “Traffic rules” for eigenvalues of the monodromy matrix of (LHdS).
- λ_0 is the point of strong stability of (LHdS) if there exists $\delta > 0$ such that (LHdS) is *stable*, i.e., all solutions are bounded on $\mathbb{R}$, for $|\lambda - \lambda_0| < \delta$.
- The set of strong stability points of (LHdS) is *open*, i.e., it consists of (finite or infinite) system of disjoint open intervals.

- System of positive type:

$$\mathcal{H}(t) \geq 0, \ t \in [0, T], \quad \int_0^T \mathcal{H}(t)\, dt > 0.$$

Here > 0 resp. ≥ 0 means positive (semi) definiteness of a given Hermitean matrix.

- Let $Z \in C^{2n \times 2n}$ be the fundamental matrix of (LHdS), $Z(T)$ is called the monodromy matrix of (LHdS).
- ρ the eigenvalue of $Z(T)$ (= the multiplier of (LHdS)), $Z(T)\xi = \rho\xi$, $z(0) = \xi$, then

$$z(t + T) = \rho z(t).$$

$\mathcal{J}$-monotonicity

- We suppose that (LHdS) is of positive type.
- Fundamental formula: Z the fundamental matrix of (LHdS), then

$$Z^*(s)\mathcal{J}Z(s)\big|_t^{t+T} = \underbrace{(\bar{\lambda} - \lambda)}_{-2i\,\mathrm{Im}\,\lambda} \int_t^{t+T} Z^*(s)\mathcal{H}(s)Z(s)\,ds$$

- $\mathcal{J}$-monotonicity of the fundamental matrix Z:

$$i[Z^*(T)\mathcal{J}Z(T)] - \mathcal{J}] >, \; =, \; < 0$$

depending on whether $\mathrm{Im}\,\lambda > 0, \; = 0, \; < 0$.

Periodic symplectic difference system

(*SDS*) $$z_{k+1} = \mathcal{S}_k(\lambda) z_k$$

where $\mathcal{S}_{k+N}(\lambda) = \mathcal{S}_k(\lambda)$ for $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$.

(H1) There exist Hermitean matrices $\mathcal{A}_k(\lambda) \in C^1$:

$$S_k^*(\lambda)\mathcal{J}\mathcal{S}_k(\lambda) - \mathcal{J} = (\bar{\lambda} - \lambda)\mathcal{A}_k(\lambda), \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

In particular, for $\lambda \in \mathbb{R}$ the matrices $\mathcal{S}_k$ are *$\mathcal{J}$-unitary*, i.e.,

$$\mathcal{S}_k^*(\lambda)\mathcal{J}\mathcal{S}_k(\lambda) = \mathcal{J}$$

and for $\mathcal{S}(\lambda) \in \mathbb{R}^{2n\times 2n}$ *symplectic*.

(H2) $\mathcal{S}_k(0) = I$, $S_k(\lambda)$ are differentiable, and $S_k^{[1]} := \mathcal{S}'(0)$ satisfy

$$(S_k^{[1]})^*\mathcal{J} + \mathcal{J}S_k^{[1]} = 0.$$

Second order matrix difference system

$$\Delta^2 x_{k-1} + \lambda^2 P_k x_k = 0, \quad P_k^* = P_k, \quad P_{k+N} = P_k.$$

- A. Halanay, V. Rasvan, Dynam Systems Appl. 1999.

The substitution $u_k = \frac{1}{\lambda}\Delta x_k$, $z = \binom{x}{u}$,

$$z_{k+1} = \underbrace{\left[I + \lambda \underbrace{\begin{pmatrix} 0 & I \\ -P_k & 0 \end{pmatrix}}_{S^{[1]}} + \lambda^2 \begin{pmatrix} -P_k & 0 \\ 0 & 0 \end{pmatrix} \right]}_{\mathcal{S}_k(\lambda)} z_k$$

- Assumptions (H1), (H2) are satisfied.

In particular,

- (H1):

$$\mathcal{S}^*(\lambda)\mathcal{J}\mathcal{S}(\lambda) = \mathcal{J} + (\bar{\lambda} - \lambda)\begin{pmatrix} P + |\lambda|^2 P^* P & \bar{\lambda}P^* \\ -\lambda P & I \end{pmatrix}$$

- (H2):

$$\mathcal{S}'(0) = S^{[1]} = \begin{pmatrix} 0 & I \\ -P & 0 \end{pmatrix}$$

and

$$-\mathcal{J}S^{[1]} = \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix},$$

in particular, $-\mathcal{J}S_k^{[1]} \geq 0$ and $\mathcal{J}\sum_{k=0}^{N-1} S_k^{[1]} > 0$ if and only if

$$P_k^{[1]} \geq 0,\ k = 0, \dots, N-1 \quad \sum_{k=0}^{N-1} P_k > 0.$$

Hamiltonian difference system

$$\Delta \begin{pmatrix} x_k \\ u_k \end{pmatrix} = \lambda \mathcal{J} \underbrace{\begin{pmatrix} -C_k & A_k^* \\ A_k & B_k \end{pmatrix}}_{\mathcal{H}_k} \begin{pmatrix} x_{k+1} \\ u_k \end{pmatrix}$$

with symmetric matrices B, C

- V. Rasvan, Arch. Math. (Brno), 2000

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \underbrace{\begin{pmatrix} (I-\lambda A)^{-1} & \lambda (I-\lambda A)^{-1} B \\ \lambda C (I-\lambda A)^{-1} & \lambda^2 C (I-\lambda A)^{-1} B + I - \lambda A^* \end{pmatrix}_k}_{\mathcal{S}_k(\lambda)} \begin{pmatrix} x_k \\ u_k \end{pmatrix}$$

We have

$$\mathcal{S}(\lambda) = I + \lambda \underbrace{\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}}_{\mathcal{J}\mathcal{H}} + S^{[2]}(\lambda)$$

with

$$\mathcal{S}_k^{[2]}(\lambda) = \begin{bmatrix} (I-\lambda A)^{-1} - I - \lambda A & \lambda\left[(I-\lambda A)^{-1}B - B\right] \\ \lambda\left[C(I-\lambda A)^{-1} - C\right] & \lambda^2 C(I-\lambda A)^{-1}B \end{bmatrix} = o(\lambda)$$

as $\lambda \to 0$ and

$$\mathcal{S}^*(\lambda)\mathcal{J}\mathcal{S}(\lambda) = \mathcal{J} + (\bar\lambda - \lambda)D^*(\lambda) \underbrace{\begin{pmatrix} -C & A^* \\ A & B \end{pmatrix}}_{\mathcal{H}} D(\lambda),$$

where

$$D(\lambda) = \begin{pmatrix} (I - \lambda A)^{-1} & \lambda (I - \lambda A)^{-1} B \\ 0 & I \end{pmatrix}.$$

In particular, for solutions of (LHS) we have

$$z_k^* \mathcal{J} z_k \Big|_{k=0}^{N} = (\bar{\lambda} - \lambda) \sum_{k=0}^{N-1} \begin{pmatrix} x_{k+1} \\ u_k \end{pmatrix}^* \mathcal{H}_k \begin{pmatrix} x_{k+1} \\ u_k \end{pmatrix}.$$

“Exponential” case

The case $\mathcal{S}_k(\lambda) = S_k^\lambda = e^{\lambda \log S_k}$ with *symplectic* matrices S_k, $S_{k+N} = S_k$. Denote $R_k := \log S_k$. Then $R_k^* \mathcal{J} + \mathcal{J} R_k = 0$ and

$$\mathcal{S}_k(\lambda) = \sum_{j=0}^{\infty} R_k^j \frac{\lambda^j}{j!}.$$

Then (suppressing the index k)

$$\mathcal{S}^*(\lambda) \mathcal{J} \mathcal{S}(\lambda) = \mathcal{J} + (\bar{\lambda} - \lambda) \mathcal{A}(\lambda),$$

where

$$\mathcal{A}(\lambda) = \sum_{j=0}^{\infty} \frac{(\bar{\lambda} - \lambda)^{2j}}{(2j+1)!} (R^*)^j (-\mathcal{J} R) R^j + \sum_{j=1}^{\infty} (-1)^j \frac{(\bar{\lambda} - \lambda)^{2j-1}}{(2j)!} (R^*)^j \mathcal{J} R^j \geq 0$$

if and only if $-\mathcal{J} R = -\mathcal{J} \mathcal{S}'(0) \geq 0$.

Central stability zone

We consider our symplectic system in the form

$$(SDS) \qquad z_{k+1} = \underbrace{\left[I + \lambda S_k^{[1]} + S^{[2]}(\lambda)\right]}_{\mathcal{S}_k(\lambda)} z_k,$$

with (H1) and (H2), in particular

$$(S_k^{[1]})^* \mathcal{J} = \mathcal{J} S_k^{[1]} = -\mathcal{J}^* S_k^{[1]}, \quad k = 0, \dots, N-1,$$

where $S^{[2]}(\lambda) = o(\lambda)$ as $\lambda \to 0$ and $\mathcal{S}_{k+N}(\lambda) = \mathcal{S}_k(\lambda)$. Then we have for the monodromy matrix

$$\mathcal{U}_N(\lambda) = \mathcal{S}_{N-1}(\lambda) \cdots \mathcal{S}_0(\lambda) = I + \lambda \left(\sum_{k=0}^{N-1} S_k^{[1]} \right) + o(\lambda)$$

as $\lambda \to 0$.

Central stability zone

We denote

$$\mathcal{S}^{[1]} = \sum_{k=0}^{N-1} S_k^{[1]}.$$

Theorem. Let

$$-\mathcal{J}\mathcal{S}^{[1]} > 0$$

and suppose that the eigenvalues s_j of $\mathcal{S}^{[1]}$ are distinct. Then there exists $l > 0$ such that solutions of (SDS) are bounded for $|\lambda| < l$, i.e., the interval $(-l, l)$ is contained in the *central stability zone* of (SDS).

- The theorem requires *distinct* eigenvalues of the matrix $\mathcal{S}^{[1]}$ and its proof *does not need* any assumption on $\mathcal{J}$ monotonicity of the monodromy matrix.

Positive type system

Next, we don't suppose that the eigenvalues of $\mathcal{S}^{[1]}$ are distinct, we suppose that (SDS) is of *positive type*:

$$-\mathcal{J} S_k^{[1]} \geq 0,\ k = 0, \dots, N-1, \quad -\mathcal{J} \left(\sum_{k=0}^{N-1} \mathcal{S}_k^{[1]} \right) > 0.$$

and, moreover (compare (H1))

$$\mathcal{S}_k^*(\lambda) \mathcal{J} \mathcal{S}_k(\lambda) = \mathcal{J} + (\bar{\lambda} - \lambda) \underbrace{\left[-\mathcal{J} S_k^{[1]} + \mathcal{B}_k(\lambda) \right]}_{\mathcal{A}_k(\lambda)}$$

with

$$(B) \qquad z_k^* \mathcal{B}_k(\lambda) z_k \geq 0, \quad k = 0, \dots, N-1,$$

for any solution of (SDS).

Krein's traffic rules

- $|\rho| = 1$ the eigenvalue of the monodromy matrix $\mathcal{U}_N$, $\mathcal{L}$ is the corresponding eigenspace.
- If $iu^*\mathcal{J}u > 0$ (< 0) for $\forall u \in \mathcal{L}$, then the multiplier ρ is called of the 1-st (=positive) kind (2-th kind (negative) kind)
- If $\exists 0 \neq u \in \mathcal{L}$: $u^*\mathcal{J}u = 0$, ρ is the multiplier of *indefinite* (=mixed) type.
- If (SDS) is of positive type and (B) holds, there are only multipliers of definite type.
- $\lambda = 0$ is the stability point of (SDS), $\mathcal{U}_N(0) = I$. Multipliers of the positive type (there is n of them) move clockwise and of negative type move counterclockwise when λ increases and *stay on the unit circle*.

Traffic rules cont.

- A multiplier $\rho(\lambda)$ my exit the unit circle only when the multipliers of different kind meet on the unit circle, i.e., at least of them comes through the point $[-1, 0]$, which is the same as that the antiperiodic BVP

$$z_{k+1} = \mathcal{S}_k(\lambda) z_k, \quad z_N + z_0 = 0$$

has a solution, i.e. λ is a solution of

$$(U) \qquad \det \left[\mathcal{U}_N(\lambda) + I\right] = 0$$

- Estimate of the length of the central stability zone: Let Λ_+ be the minimal positive root of (U) and Λ_- the maximal negative root of (U). Then the interval (Λ_-, Λ_+) is contained in the central stability zone of (SDS).

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