# Snap-back repellers in rational difference equations

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## Definition

*Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a (continuous) function. An order one difference equation (DE) is:*

$$x_{k+1} = f(x_k) \qquad k \geq 0 \tag{1}$$

A solution of the DE is a sequence $(x_k)_k \subseteq \mathbb{R}^n$ obtained from (1) taking $x_0 \in \mathbb{R}^n$.

The equation is rational (RDE) if $f$ is a rational function.

If the sequence $(x_k)_k$ has a finite number of elements, we say that $x_0$ is an element of the **forbidden set** of (1).

- The forbidden set is empty when Dom $f = \mathbb{R}^n$.
- If $f$ is rational, the forbidden set includes the poles of $f$.

A general open problem in RDE is to determine the forbidden set.

## Generalized Li-Yorke chaos or Marotto chaos

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a **continuous** function. $x_{k+1} = f(x_k)$ is Li-Yorke chaotic in a generalized sense or Marotto chaotic if:

- LY1 There exists $N \in \mathbb{N}$ such that $\forall p \geq N$ there are (prime) periodic sequences with period $p$.
- LY2 There exists an uncountable set $S \subset \mathbb{R}^n$ holding $f(S) \subset S$ and non containing periodic points, such that for every $x \neq y \in S$,

$$\overline{\lim}_{k\to\infty} ||f^k(x) - f^k(y)|| > 0$$

and for any period $p$ of $f$

$$\overline{\lim}_{k\to\infty} ||f^k(x) - f^k(p)|| > 0$$

- LY3 There is an uncountable subset $S_0 \subset S$, such that for every $x_0 \neq y_0 \in S_0$ is

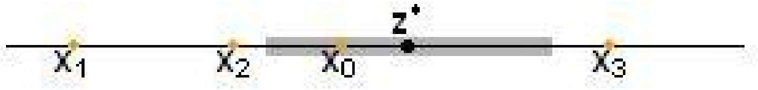
$$\underline{\lim}_{k\to\infty} ||f^k(x_0) - f^k(y_0)|| = 0$$

A **Li-Yorke pair** is $(x, y)$ verifying [LY2] and [LY3].

## Notion of snap-back repeller (SBR)

Let $f \in C^1$ in a neighborhood of a fixed point $z^*$ of $f$. We say that $z^*$ is a SBR if:

- SBR1 All the eigenvalues of $Df(z^*)$ have modulus greater than 1 ($z^*$ is a repeller)
- SBR2 There exist a finite sequence $x_0, x_1, \ldots, x_M$ such that $x_{k+1} = f(x_k)$, $x_M = z^*$ and $x_0 \neq z^*$ belongs to a repelling neighborhood of $z^*$. Moreover, $|Df(x_k)| \neq 0$ for $0 \leq k \leq M - 1$.




Remark that a SBR is a special kind of **homoclinic point** in the sense of Louis Block (1978).

### Theorem (Marotto, 1978 and 2004)

*The existence of a SBR implies Marotto chaos.*

A similar result, without derivatives and only continuity was proved by Peter Kloeden in 1981. In the proof, he used the fact that continuous one-to-one mappings have continuous inverses on compact spaces and the Brouwer fixed point theorem. In this case, saddle points can be considered as well as snap-back repellers when Marotto's result can not be applied.

It is apparent that the existence of a snap-back repeller for a one-dimensional map $f$ is equivalent to the existence of a point of period 3 for the map $f^n$ for some positive integer $n$.

# Open questions

1) Does Marotto's theorem remain true for a RDE?

2) How estimate the forbidden set of a RDE?

## Motivation I

We consider systems of difference equations

$$x_{n+1} = \frac{P_1(x_n, y_n)}{Q_1(x_n, y_n)}$$

$$y_{n+1} = \frac{P_2(x_n, y_n)}{Q_2(x_n, y_n)}$$

where $P_1, Q_1, P_2, Q_2$ are quadratic polynomials in $x_n, y_n$ and look for conditions for the existence of chaotic behavior. For some particular values of the parameters, Mazrooei and Sebdani have recently given an example of the former system, where the origin $(0, 0)$ is a snap-back repeller. So there will be generalized Li-Yorke chaos if Marotto's rule can be applied to this new frame.

## Motivation II

Models containing RDE of this kind are founded in population dynamics and also in the application of the Newton's method to polynomial equations.
Easy examples are:

- inverse parabolas, $x_{k+1} = \frac{1}{x_k^2 - r}$
- inverse logistic equation, $x_{k+1} = \frac{1}{rx_k(1-x_k)}$

# Marotto chaos in RDE: LY1 property

In a RDE, we define SBR in the same way that in the continuous case.

Remark that none of the elements of the finite sequence $x_0$, $x_1$,... , $x_M = z^*$ belongs to the forbidden set.

## Theorem (LY1)

*In a RDE, the existence of a SBR implies the existence of infinite periods (LY1).*

## Sketch of proof

The key idea is fix a special compact neighborhood $B_\epsilon(z^*)$ and to apply Brouwer's fixed point theorem to each function of the form

$$f^{-k} \circ g$$

where

- $g$ is the local inverse of $f^M$ in a neighborhood of $z^*$.
- $f^{-k}$ is the $k$-times composition of a local inverse of $f$ near $z^*$.

## Sketch of proof II

These concepts appear in Li-Yorke and Marotto's works. In the RDE setting, they remain valid because of the local continuity and differentiability of the iteration function.

An important remark is that the repelling character of $z^*$ assures that $f^{-1}(A) \subseteq A$ where $A$ is a convenient neighborhood of $z^*$, and therefore $f^{-k}(x)$ is well defined for all $x \in A$ and $k > 0$.

## Second part of Marotto's theorem

We wonder if it is true or not the existence of an uncountable set of Li-Yorke pairs when the RDE has a SBR.

We claim that the answer is not, but we have not yet been able to find a counterexample.

Our claim is based on the necessity of an additional topological condition if we want to obtain similar results to those of Marotto. We have called it *the compact preimage property (CPP)*.

## Definition (Itinerary)

*Let $f : \mathbb{R}^n \to \mathbb{R}^n$ a function. A sequence of compact subsets of $\mathbb{R}^n$, $\{I_k\}_{k=0}^{+\infty}$, is an itinerary of $f$ when $f(I_k) \supseteq I_{k+1}$.*

Itineraries are the tool that allow us to locate special solutions of a difference equation behaving in a particular way. Indeed, they are used several times in the proof of the second part of Marotto's theorem, where we find the result:

## Lemma

*If $f$ is continuous, and $\{I_k\}_{k=0}^{+\infty}$ is an itinerary, there exist $x_0 \in \mathbb{R}^n$ such that $f^k(x_0) \in I_k$, $\forall k \geq 0$.*

Itineraries' lemma in the real line is an application of the intermediate value property.

In the $n$th-dimensional case the lemma was proved by Phil Diamond in 1976.

In the RDE setting the problem is more complicated because we have in addition the presence of the the forbidden set and the non continuity in the maps.

## Definition (CPP property)

*We say that a function $f$ has the compact pre-image property (CPP) if for every compact subset $K$ in the range of $f$, there exist a compact $L$ such that $f(L) = K$.*

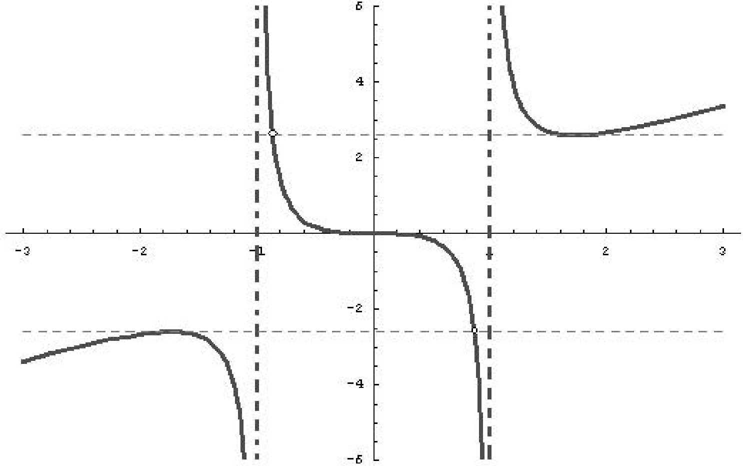
- If $f$ has CPP, this is also the case for $f^k$.
- If $f : \mathbb{R} \to \mathbb{R}$ is continuous, the Intermediate Value Property proves that $f$ is CPP.
- If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, the CPP property is also true (Diamond).

## Lemma

*If $f$ is CPP, and $\{I_k\}_{k=0}^{+\infty}$ is an itinerary, there exist $x_0 \in \mathbb{R}^n$ such that $f^k(x_0) \in I_k, \forall k \geq 0$.*

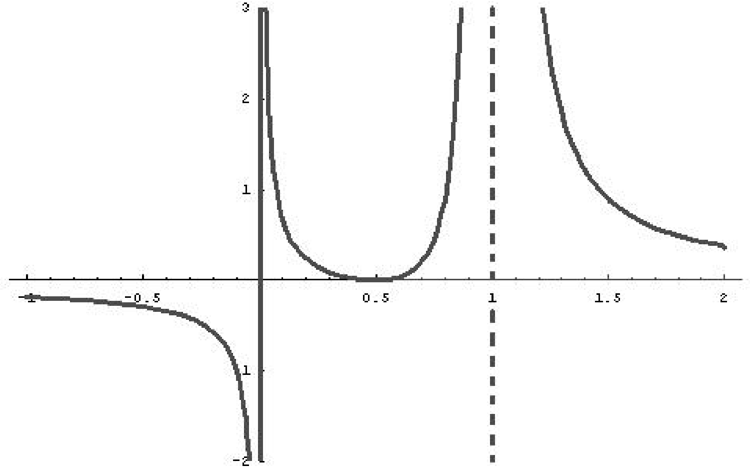
# Example of non-CPP function: reducible case

$$f(x) = \frac{4x^5 - 3x^3}{4x^4 - 7x^2 + 3}$$



## Another non-CPP function

$$f(x) = \frac{1}{3} \cdot \frac{(x - 1/2)^2}{x(x-1)^2}$$



# Main result

**Theorem**

*A RDE with the CPP property having a SBR is chaotic in the sense of Marotto.*

## LY2 and LY3: Sketch of proof.

The definition of SBR allows to construct two compact sets $U$, $V$ of $\mathbb{R}^n$ and $N \in \mathbb{N}$ such that:

- $U \cap V = \emptyset$
- $V \subset f^N(U)$
- $U, V \subset f^N(V)$

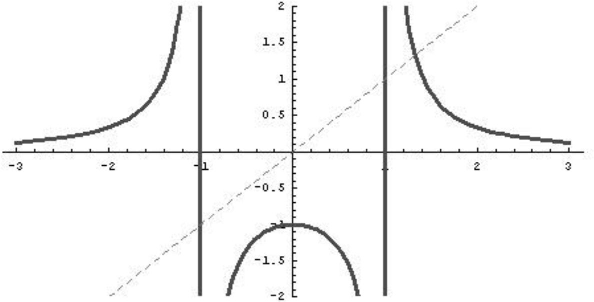
Therefore, any sequence $\{I_k\}_{k=0}^{+\infty}$ with $I_k = U, V$ and with the restriction that $I_k = U$ implies $I_{k+1} = V$, is an itinerary of $f^N$. The set of all these itineraries define an uncountable subset of $\mathbb{R}^n$ which can be refined to obtain the scrambled set of Li-Yorke pairs.

# Numerical examples: inverse parabolas

Consider the RDE:

$$x_{k+1} = \frac{1}{x_k^2 - 1} \tag{2}$$

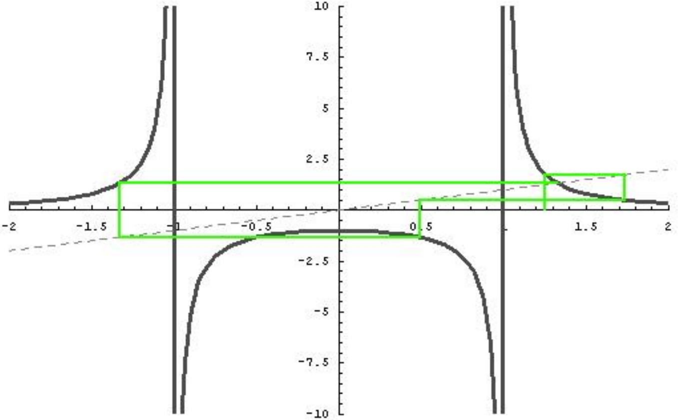
We apply the former theorem to it. This is possible because function $f(x) = \frac{1}{x^2-1}$ is CPP, for each compact interval in the image of $f$ belongs to one of the three branches of the graph, and we have continuity over them.



Moreover, RDE (2) has a SBR in

$$z^* = \frac{\sqrt[3]{9-\sqrt{69}} + \sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{18}} \approx 1.32472$$

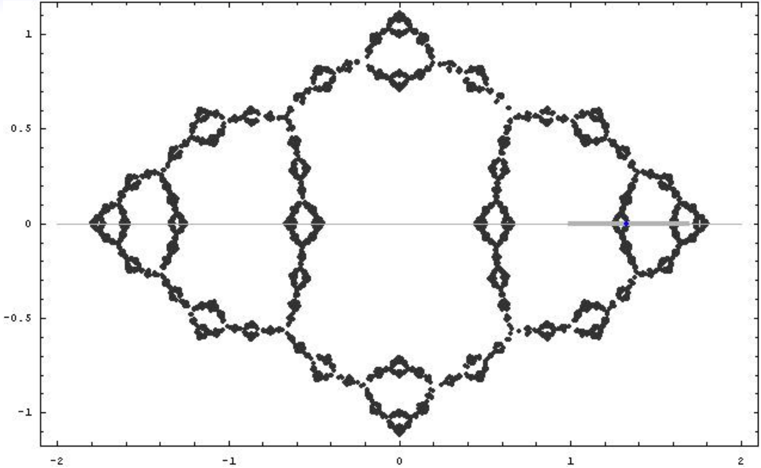
This is illustrated in the following cobweb diagram.



There is another way to locate the SBR of RDE (2). We consider the inverse difference equation:

$$x_{k+1} = \pm\sqrt{1 + \frac{1}{x_k}} \tag{3}$$

Now, we compute the *solution* of (3) corresponding to $x_0 = z^*$. So we must work in the complex domain, but it is easy to verify if some of the branches of this multi-valuate solution belong to a repeller neighborhood of $z^*$.

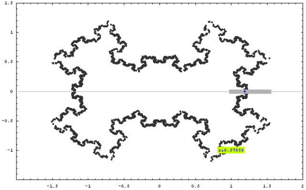



If we start the inverse iteration with the poles of (2), we obtain some similar type of fractal structure. Therefore, it is an estimation of the forbidden set.

Now, we consider the family

$$x_{k+1} = \frac{1}{x_k^2 - r} \qquad r \in \mathbb{R} \tag{4}$$

These RDEs are CPP, so it will be possible to apply Marotto's theorem if we can locate SBRs. A fast way to do that is to draw the fractals corresponding to each RDE of the family.



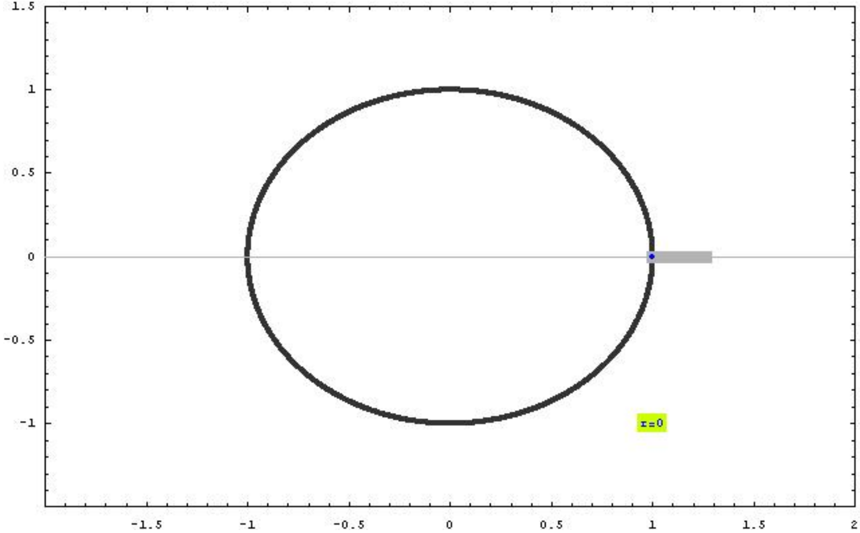
*Web link*

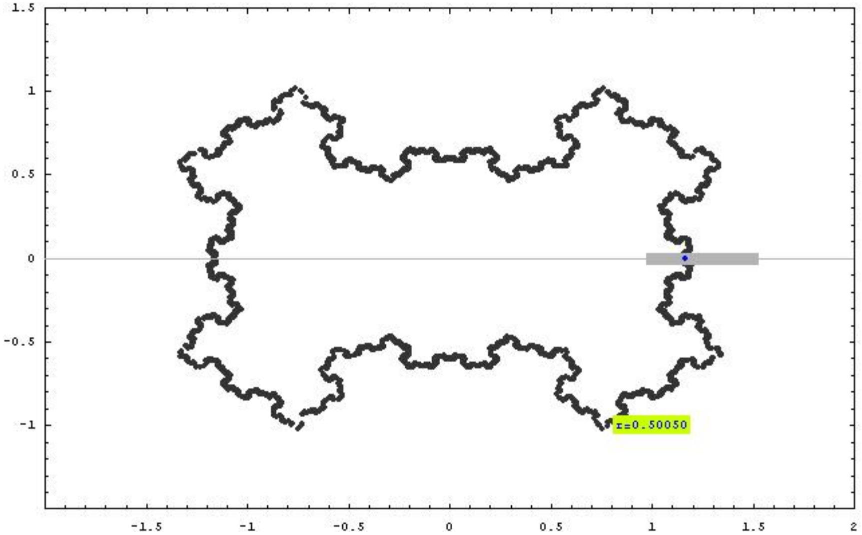
*http://youtu.be/bigcVJYdGbU*

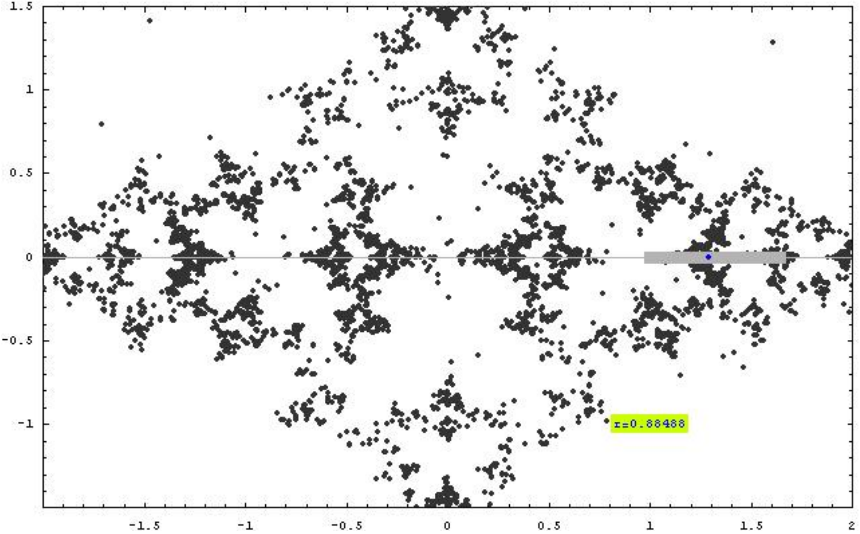
We have estimated that there is a bifurcation point in $r_0 = \frac{1}{\sqrt[3]{2}} \approx 0.79$.

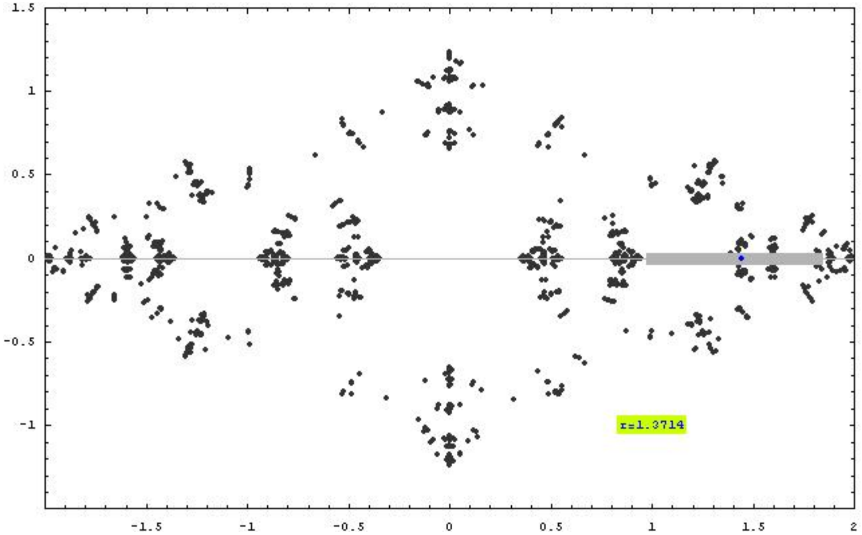
We conjecture that if $r \geq r_0$, then RDE (4) has a SBR and therefore is chaotic.

## Some movie frames...




r=0.50050
1.5
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5
2


1.5
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5
2
r=0.88488


1.5
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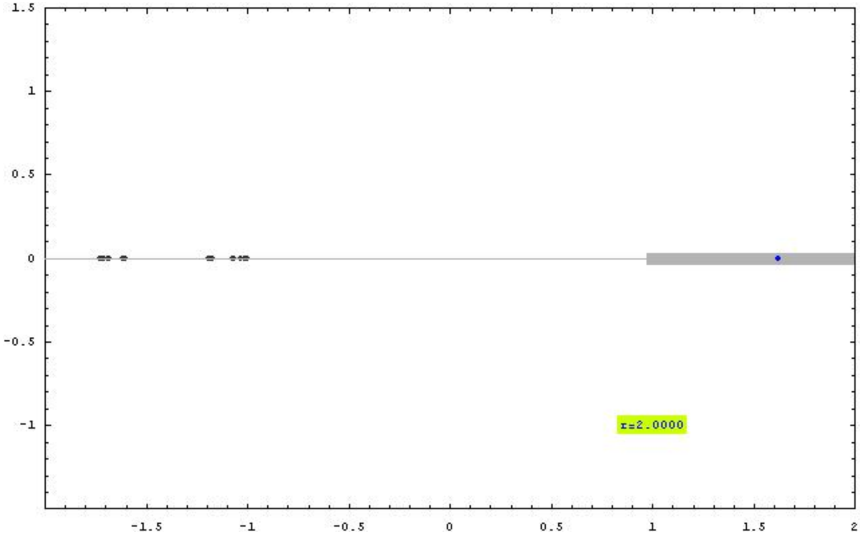
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5
2
r=1.3714


1.5
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5
2
r=2.0000

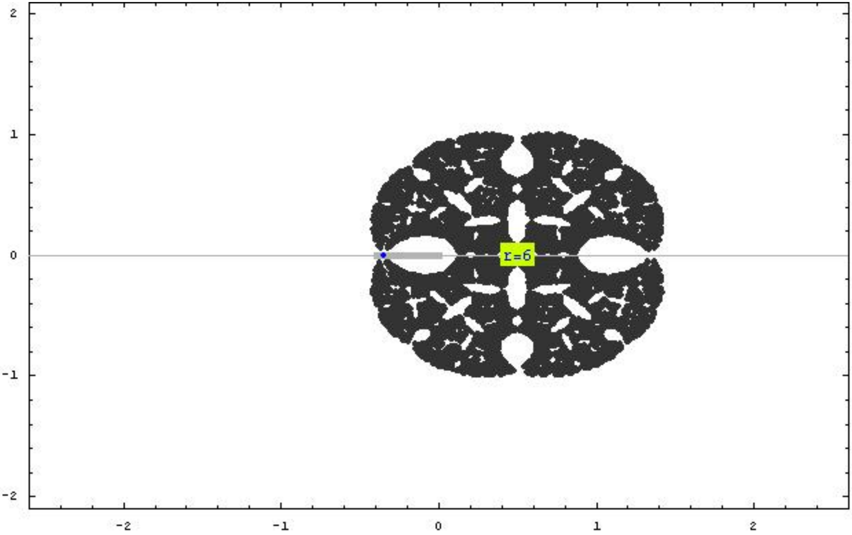
# Inverse logistics

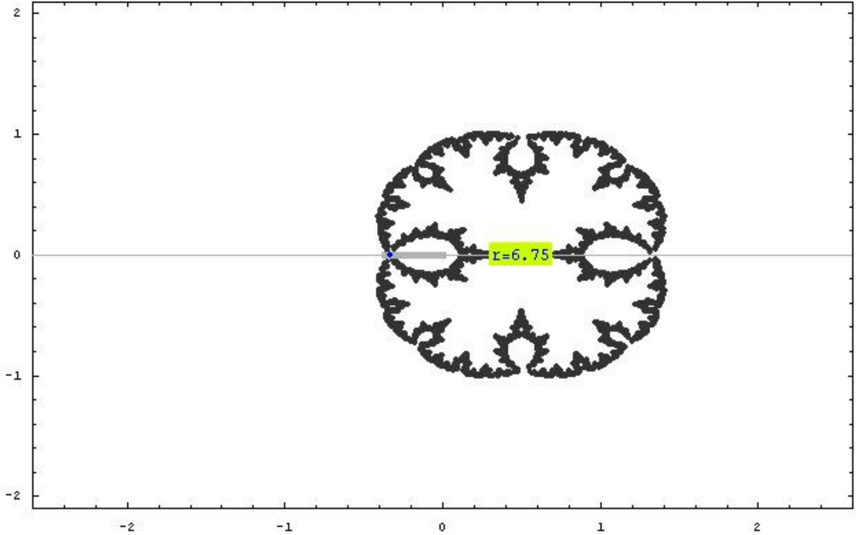
Let $r$ be a nonzero real number and consider the RDE:

$$x_{k+1} = \frac{1}{r x_k (1 - x_k)} \tag{5}$$

- (5) is also CPP, so Marotto's rule still applies.
- Open question: to estimate bifurcation points (if exists)

## Some inverse solutions of the SBR in RDE $x_{k+1} = \frac{1}{rx_k(1-x_k)}$

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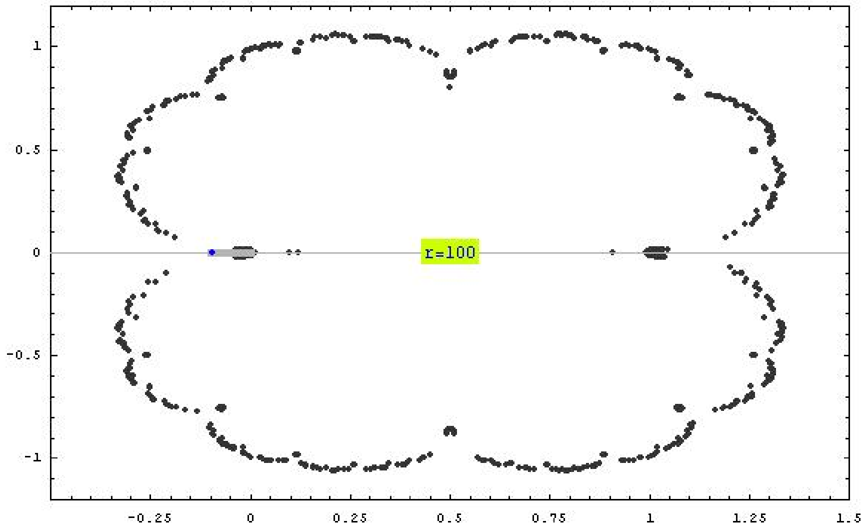
-1
-2
-2
-1
0
1
2
r=6.75


1
0.5
0
-0.5
-1
-0.25
0
0.25
0.5
0.75
1
1.25
1.5
r=100

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