Stability of difference equations with an infinite delay

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Joint work with

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Bohl-Perron Type Theorems

Bohl (1913, J.Reine Angew.Math)
Perron (1930):
If the solution of the initial value problem

$$\frac{dX}{dt} = AX + f, X(0) = 0$$

is bounded for any bounded f, then the solution of the homogeneous equation is exponentially stable.
Equations in a Banach space: M. Krein (1948)
Delay equations: Azbelev, Tyshkevich, Berezansky, Simonov, Chistyakov (1970-1993)
Impulsive delay equations: Anokhin, Berezansky, Braverman (1995)

Difference equations

Bohl-Perron type result for a nondelay difference equation:
[1] C.V. Coffman and J.J. Schäffer, *Dichotomies for linear difference equations*, Math. Ann. 172 (1967), pp. 139–166.
[2] B. Aulbach, N. Van Minh, The concept of spectral dichotomy for linear difference equations. II, *J. Differ. Equations Appl.* **2** (1996), 251–262.

Theorem [2]. If a solution of the equation

$$x_{n+1} = A_n x_n + f_n \tag{1}$$

belongs to ℓ^p, $1 \leq p \leq \infty$, for any sequence f_n in the same space ℓ^p, then the solution of the homogeneous equation

$$x_{n+1} = A_n x_n \tag{2}$$

decays exponentially with the growth of n.

The case of different spaces

If for any $f_n \in \ell^1$ the solution is bounded, then the equation is stable (but, generally speaking, not exponentially). Suppose a solution of $x_{n+1} = A_n x_n + f_n$ belongs to ℓ^∞ for any f_n from ℓ^p, $1 < p < \infty$; what kind of stability can be deduced for $x_{n+1} = A_n x_n$?
Quite recently it was proved in

[3] M. Pituk, A criterion for the exponential stability of linear difference equations, *Appl. Math. Let.* **17** (2004), 779–783.

that under the above conditions the solution is exponentially stable.

Some other relevant references

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- S. Elaydi and S. Murakami, *Asymptotic stability versus exponential stability in linear Volterra difference equations of convolution type*, J. Difference Equ. Appl. 2 (1996), pp. 401–410.
- M. Pituk, *Global asymptotic stability in a perturbed higher order linear difference equation*, Comput. Math. Appl. 45 (2003), 1195–1202.
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Outline of Bohl-Perron type methods

- **Application of solution representations.**
  Some proofs are based on the solution representation

$$x(n) = \sum_{k=1}^{n} X(n, k+1) f(k), \tag{3}$$

where $X(n, k)$ satisfies the semigroup equality

$$X(n, k) = X(n, i) X(i, k), \ \ n > i > k. \tag{4}$$

This is relevant for first order difference equations only.

- **Results are applied to study stability properties.**
  (stability $\Leftrightarrow$ a solution belongs to a certain space)

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Solution representation

For the delay difference equation

$$x(n+1) = \sum_{k=-d}^{n} A(n,k)x(k) + f(n), \quad x(n) = \varphi(n), \; n \le 0, \quad (5)$$

with $d = 0$ (no prehistory) the solution representation for (5) is

$$x(n) = X(n,0)x(0) + \sum_{k=0}^{n} X(n,k+1)f(k)$$

(S. Elaydi,1994, S. Elaydi, S. Zhang,1994). Here $X(n,k) = 0, n < k, X(k,k) = I$ (an identity operator). No semigroup equality is valid. For difference equations, there are two possible solutions of the problem.

Difference and inverse operators

First, we can follow the steps of the proofs for delay differential equations.
Introduce **the difference operator** for the zero initial conditions

$$\mathcal{L}\left(\{x(n)\}_{n=1}^{\infty}\right) = \left\{ x(n+1) - \sum_{k=1}^{n} A(n,k)x(k) \right\},$$

$x(0) = 0$, and **the Cauchy operator**

$$\mathcal{C}\left(\{f(n)\}_{n=0}^{\infty}\right) = \left\{ y(n) = \sum_{l=0}^{n-1} X(n,l+1)f(l) \right\}_{n=0}^{\infty}$$

(at this step we do not specify the space of sequences).

Assumptions

We consider an assumption that the sums of the operators $A(n,l)$ are uniformly bounded

(a1) there exists $K > 0$, such that $\sup_{n \geq 0} \sum_{l=-d}^{n} |A(n,l)| \leq K$;

and a stronger restriction (the delay is also bounded)
(a2) there exists $T > 0$ such that $A(n,l) = 0$ whenever $n - l > T$ and $A(n,l)$ are uniformly bounded: $|A(n,l)| \leq M$ for all n, l.

Lemma 1. Suppose (a2) holds. Then the difference operator is a bounded operator in the space ℓ^p, $1 \leq p \leq \infty$.

Boundedness of delay is necessary

Unlike ℓ^∞, where the boundedness of the delay is not necessary for the action of the operator, in ℓ^p it is crucial as the following example shows.

Example 1. For the equation $x(n+1) = x(n) - x(2)$, $n \geq 2$ the operator

$$\mathcal{L}(\{x(n)\}) = \{x(n) - x(2)\}$$

does not act in ℓ^p : for any sequence $\{x(n)\} \in \ell^p$ such that $x(2) \neq 0$ the resulting sequence does not tend to zero.

Stability

Theorem 1. Suppose (a1) holds. Then the uniform estimate $|X(n,k)| \leq C$ holds if and only if for any $\{f(n)\} \in \ell^1$ the solution $\{x(n)\}$ with the zero initial conditions is bounded $\{x(n)\} \in \ell^\infty$.

Corollary 1. If (a1) holds and for any $\{f(n)\} \in \ell^1$ the solution with the zero initial condition is bounded, then the equation is stable.

It is similar to the result by Aulbach, Van Minh for first order equations.

Bohl-Perron Theorem for Delay Difference Equation

Theorem 2. Suppose (a2) holds and for every sequence $\{f(n)\} \in \ell^p,\ 1 \le p \le \infty$, the solution $\{x(n)\}$ with the zero initial condition also belongs to ℓ^p.
Then there exist $N > 0, \lambda > 0$ such that the fundamental function X satisfies

$$|X(n,l)| \le N e^{-\lambda(n-l)}.$$

Corollary 2. Under the conditions of Theorem 2 the equation is exponentially stable.

Boundedness of the delay is necessary

Example 2. Consider the equation with an unbounded delay

$$x(n+1) = \frac{1}{2}x(n) + x(0) + f(n).$$

Then for any right hand side bounded by f ($|f(n)| \leq f$) the solution is bounded by $2(|x(0)| + f)$ (prove by induction!). However solutions of the corresponding homogeneous equation

$$x(n+1) = \frac{1}{2}x(n) + x(0)$$

do not decay exponentially: for example, a solution with $x(0) = 1$ (a scalar case) is increasing and tends to 2.

Illustration for equations with two delays

As an illustration, consider the autonomous equation with 2 delays:

$$x(n+1) - x(n) = -a_0 x(n) - a_1 x(n-h_1) - a_2 x(n-h_2), \quad (6)$$

where $h_1 > 0, h_2 > 0$.

Corollary. Suppose at least one of the following conditions holds:

1) $1 > a_0 > 0, \ |a_1| + |a_2| < a_0$;

2) $0 < a_0 + a_1 + a_2 < 1, \ |a_1|h_1 + |a_2|h_2 < \dfrac{a_0 + a_1 + a_2}{|a_0| + |a_1| + |a_2|}$;

3) $0 < a_0 + a_2 < 1, \ |a_2|h_2 < \dfrac{a_0 + a_2 - |a_1|}{|a_0| + |a_1| + |a_2|}$.

Then Eq. (6) is exponentially stable.

Known stability results - Cooke and Győri

Cooke, Győri (1994):
the equation

$$x(n+1) - x(n) = -\sum_{k=1}^{N} a_k x(n-h_k), \quad a_k \geq 0, h_k \geq 0,$$

is asymptotically stable if $\sum_{k=1}^{N} a_k h_k < 1$.

Known stability results - Elaydi, Kocić and Ladas

Elaydi (1994), Kocić and Ladas (1993):
the equation

$$x(n+1) - x(n) = -a_0(n)x(n) - \sum_{k=1}^{N} a_k(n)x(g_k(n)), \quad g_k(n) \le n,$$

is asymptotically stable if for some $\varepsilon > 0$

$$\sum_{k=1}^{N} |a_k(n)| \le \begin{cases} a_0(n) - \varepsilon, & 0 < a_0(n) < 1, \\ 2 - a_0(n) - \varepsilon & 1 \le a_0(n) < 2. \end{cases}$$

Known stability results - Győri and Pituk

Győri, Pituk (1997):
the equation

$$x(n+1) - x(n) = -a(n)x(g(n)), \quad a(n) \geq 0, g(n) \leq n$$

is exponentially stable if

$$\sum_{n=1}^{\infty} a(n) = \infty, \quad \limsup_{n\to\infty}(n - g_k(n)) < \infty,$$

$$\limsup_{n\to\infty} \sum_{l=\min_k\{g(n)\}}^{n-1} a(l) < 1.$$

Known stability results - Győry, Hartung

Győry, Hartung (2001):
the equation

$$x(n+1) - x(n) = -\sum_{k=1}^{N} a_k x(g_k(n)), \quad a_k \geq 0, g_k(n) \leq n$$

is exponentially stable if

$$\limsup_{n\to\infty}(n - g_k(n)) < \infty, \quad \sum_{k=1}^{N} a_k \limsup_{n\to\infty}(n - g_k(n)) < 1 + \frac{1}{e} - \sum_{k=1}^{N} a_k.$$

Example - comparison to known results

Example 3. Consider the equation

$$x(n+1) - x(n) = -0.5x(n) - 0.2x(n-5) - 0.3x(n-1).$$

Here $a_0 = 0.5, a_1 = 0.2, a_2 = 0.3, h_1 = 5, h_2 = 1$.
$a_1 h_1 + a_2 h_2 = 1.3 > 1 \Rightarrow$ the conditions of Győri and Cooke do not work.
Since $a_1 + a_2 = 0.5 = a_0$ and $a_0 < 1$, the conditions of Elaydi, Kocić and Ladas ($a_1 + a_2 < a_0 - \varepsilon$) are not satisfied.
$a_1 h_1 + a_2 h_2 = 1.3 < 1 + 1/e - a_1 - a_2$ (Győri and Hartung) does not hold as well.
Part 3 of the corollary works:

$$0 < a_0 + a_2 < 1, \ \ a_2 h_2 = 0.3 < \frac{a_0 + a_2 - a_1}{a_0 + a_1 + a_2} = 0.6.$$

Some Questions: Methods and Results

- Is it possible to use the same method to equations with unbounded delays?
- The technique used is similar to delay differential equations. Can we use a different method to obtain the same result?
- The answer to both questions is positive.

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Reduction (a different method is possible)

Consider the non-autonomous difference equation of a constant order

$$x(n+1) = \sum_{k=0}^{r} A(n,k)x(n-k) + f(n), \;\; n \geq 0. \tag{7}$$

If $Y(n)$, Y_0, $F(n)$ and $D(n)$ are defined as

$$Y(n) = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_{r+1} \end{bmatrix} = \begin{bmatrix} x(n) \\ x(n-1) \\ \dots \\ x(n-r) \end{bmatrix}, \; Y_0 = \begin{bmatrix} \varphi(0) \\ \varphi(-1) \\ \dots \\ \varphi(-r) \end{bmatrix}, \; F(n) = \begin{bmatrix} f(n) \\ 0 \\ \dots \\ 0 \end{bmatrix},$$

$$D(n) = \begin{bmatrix} A(n,0) & A(n,1) & \dots & A(n,r-1) & A(n,r) \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ 0 & 0 & \dots & I & 0 \end{bmatrix}, \tag{8}$$

then Eq. (7) with initial conditions becomes

$$Y(n+1) = D(n)Y(n) + F(n), \quad Y(0) = Y_0. \tag{9}$$

Let us note the following.

1. If $\sup_{n \geq 0} \sum_{k=\max\{n-r,0\}}^{n} |A(n,k)| \leq M$ for some $M > 0$, then in the induced norm $|D(n)| \leq M$.
2. $\{Y(n)\} \in \ell^p$ if and only if $\{x(n)\} \in \ell^p$, where ℓ^p is over $\mathbf{B}^{r+1}$ and $\mathbf{B}$, respectively.
3. Exponential decay of $|x(n)|$ is equivalent to the exponential decay of $|Y(n)|$.

Thus all results known for the first order equation can be applied to the delay equation with a bounded delay, in particular, the Bohl-Perron theorem.

Exponential Memory Decay

Consider the linear difference (Volterra) equation

$$x(n+1) = \sum_{k=0}^{n} A(n,k)x(k) + f(n), \;\; n \geq 0, \tag{10}$$

Let us introduce the restriction that the memory decays exponentially:
(a3) there exist $M > 0, \zeta > 0$, such that $|A(n,k)| \leq Me^{-\zeta(n-k)}$.
wh
Example 4. The equation $x(n+1) = \sum_{k=0}^{n} a\lambda^k x(n-k)$, $0 < \lambda < 1$, satisfies (a3) with $M = |a|$, $\zeta = -\ln \lambda$.
Example 5. The equation $x(n+1) - x(n) = a \exp\{-\beta n\} x([\alpha n])$, $0 < \alpha < 1, \beta > 0$, with a "piecewise constant delay" also satisfies (a2). Here $[t]$ is the maximal integer not exceeding t, $M = \max\{1, |a|\}$, $\zeta = \beta$, since $-\beta n \leq -\beta(n - [\alpha n])$ for any $n \geq 1$.

Bohl-Perron Theorem for Equations with Infinite Delay

Theorem 3. Suppose (a3) holds and for every bounded sequence $\{f(n)\} \in \ell^\infty$ the solution $\{x(n)\}$ of (10) with the zero initial condition is also bounded: $\{x(n)\} \in \ell^\infty$.
Then there exist $N > 0, \lambda > 0$, such that the fundamental function X of (10) satisfies the exponential estimate

$$|X(n,l)| \le N e^{-\lambda(n-l)}. \tag{11}$$

The proof uses the same ideas as for delay differential equations, in particular, applies the solution representations and the Uniform Boundedness Principle.

Some conclusions. What is next?

Under (a3) the exponential estimate of the fundamental function implies the exponential stability of the solution.

- The same method which was applied to equations with bounded delays can be applied to unbounded (but finite delays) - under certain conditions (exponential decay of the kernel).
- For equations with finite delays, the reduction technique was justified (with some inaccuracies in the proof of the equivalence) which allows to consider first order equations in Banach spaces.
- Can we apply the reduction technique to equations with unbounded delays?
- Even equations with infinite memory can be considered this way!

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Some conclusions. What is next?

Under (a3) the exponential estimate of the fundamental function implies the exponential stability of the solution.

- The same method which was applied to equations with bounded delays can be applied to unbounded (but finite delays) - under certain conditions (exponential decay of the kernel).
- For equations with finite delays, the reduction technique was justified (with some inaccuracies in the proof of the equivalence) which allows to consider first order equations in Banach spaces.
- Can we apply the reduction technique to equations with unbounded delays?
- Even equations with infinite memory can be considered this way!

We consider systems of linear difference equations with an infinite delay

$$x(n+1) = L(n)x_n + f(n), \quad n \geq 0, \tag{12}$$

which in particular include *Volterra difference systems*

$$x(n+1) = \sum_{k=-\infty}^{n} L(n, n-k)\, x(k) \; + f(n), \quad n \geq 0. \tag{13}$$

It is assumed that $x(\cdot)$ is a discrete function from $\mathbb{Z}$ to a (real or complex) Banach space $\mathcal{X}$, $f(\cdot)$ is a function from $\mathbb{Z}^+(=\mathbb{N}\cup\{0\})$ to $\mathcal{X}$, where $|\cdot|$ stands for the norm in $\mathcal{X}$, x_n is the semi-infinite prehistory sequence $\{x(n), x(n-1), \cdots, x(n+m), \cdots\}$, $m \leq 0$. The sequence $x_0 = \{x(n+m)\}_{m=-\infty}^{0}$ of the initial conditions belongs to an exponentially weighted ℓ^∞-space $\mathcal{B}^\gamma$ (the phase space): for certain $\gamma \in \mathbb{R}$

$$|x_0|_{\mathcal{B}^\gamma} := \sup_{m\leq 0} |x(m)| e^{\gamma m} < \infty$$

$L(n)$, $n \geq 0$ are bounded linear mappings from $\mathcal{B}^\gamma$ to $\mathcal{X}$.

Let us study relations between uniform exponential stability, uniform stability, and ℓ^p-input ℓ^q-state stability (or shorter (ℓ^p, ℓ^q)-stability) of (12). The problem of finding Bohl-Perron type stability criteria for difference systems with infinite delay naturally requires the phase space settings. We comprehensively solve this problem in the exponentially fading phase spaces $\mathcal{B}^\gamma$, $\gamma > 0$. The method is based on the reduction of the difference system with infinite memory (12) to a first order system with states in the phase space. For systems with bounded delay we have already discussed this method. The main difficulty is the fact that the (ℓ^p, ℓ^q)-stability property of (12) is weaker than that of the reduced first order system.

The Perron property and boundedness

Our main objects are the system (12) of nonhomogeneous linear functional difference equations and the associated homogeneous system

$$x(n+1) = L(n)x_n, \quad n \in \mathbb{Z}^+. \tag{14}$$

The nonhomogeneous system (12) is called *ℓ^p-input ℓ^q-state stable ((ℓ^p, ℓ^q)-stable, in short)* if $x(\cdot, 0, 0_{\mathcal{B}}; f) \in \ell^q(\mathcal{X})$ for any $f \in \ell^p(\mathcal{X})$.

Theorem 4. Assume that $1 \le p, q \le \infty$, $\gamma \in \mathbb{R}$, and function $L : \mathbb{Z}^+ \to \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$ defines system (12). If (12) is (ℓ^p, ℓ^q)-stable, then

$$\|x(\cdot, 0, 0_{\mathcal{B}}; f)\|_q \le K_{p,q,L} \|f\|_p \tag{15}$$

for a certain constant $K_{p,q,L} \ge 1$ depending on L.
The proof is also based on the closed graph principle.

The Main Theorem - Infinite Delay

Theorem 5. Let $\gamma > 0$ and let $L : \mathbb{Z}^+ \to \mathcal{L}(\mathcal{B}^\gamma, \mathcal{X})$ define system (12). phase space $\mathcal{B}^\gamma$. Assume that the pair (p, q) is such that

$$1 \le p \le q \le \infty \qquad \text{and} \qquad (p,q) \neq (1, \infty). \tag{16}$$

Then the following statements are equivalent:

(i) System (14) is UES in $\mathcal{X}$ with respect to (w.r.t.) $\mathcal{B}^\gamma$.

(ii) System (14) is UES in $\mathcal{B}^\gamma$.

(iii) System (12) is (ℓ^p, ℓ^q)-stable and there exists $m \in \mathbb{Z}^-$ such that

$$\|L(\cdot) \Pr_{[-\infty, m]} \|_\infty := \sup_{n \in \mathbb{Z}^+} \|L(n) \Pr_{[-\infty, m]} \|_{\mathcal{B}^\gamma \to \mathcal{X}} < \infty \tag{17}$$

Some Comments and Remarks

The proof of this theorem shows that if any of statements (i)-(iii) is fulfilled, then $\sup_{n\in\mathbb{Z}^+} \|L(n)\|_{\mathcal{B}^\gamma\to\mathcal{X}} < \infty$.
Let $\gamma > 0$ and let a function $L : \mathbb{Z}^+ \to \mathcal{L}(\mathcal{B}^\gamma, X)$ define system (12). Assume that

$$\|L(\cdot) \underset{[-\infty,m]}{\Pr} \|_\infty := \sup_{n\in\mathbb{Z}^+} \|L(n) \underset{[-\infty,m]}{\Pr} \|_{\mathcal{B}^\gamma\to\mathcal{X}} < \infty$$

holds. Then (ℓ^p, ℓ^q)-stability of (12) for a certain pair (p, q) satisfying (16) implies the (ℓ^p, ℓ^q)-stability of (12) for all (p, q) satisfying (16).
Since UE-stability does not depend on the choice of p and q in the (ℓ^p, ℓ^q)-stability property we get the following:
Let $\gamma > 0$ and let a function $L : \mathbb{Z}^+ \to \mathcal{L}(\mathcal{B}^\gamma, X)$ define system (12). Assume that (17) holds. Then (ℓ^p, ℓ^q)-stability of (12) for a certain pair (p, q) satisfying $(p, q) \neq (1, \infty)$ implies the (ℓ^p, ℓ^q)-stability of (12) for all (p, q) satisfying $(p, q) \neq (1, \infty)$.

Bounded Solutions for ℓ^1 RHS

All the results can be applied to equations with a bounded delay. What happens with the pair $(p,q)=(1,\infty)$? The result coincides with the relevant theorem obtained by Aulbach, Van Minh (1996).

Theorem 6. Let $\gamma>0$ and let a function $L:\mathbb{Z}^+\to\mathcal{L}(\mathcal{B}^\gamma,\mathcal{X})$ define system (12). Then the following statements are equivalent:

(i) System (14) is uniformly stable in $\mathcal{B}^\gamma$.

(ii) System (12) is (ℓ^1,ℓ^∞)-stable and condition (17) is fulfilled.

Assumptions are Necessary

Exponential decay of the memory is required.
Example 6. Consider

$$x(1) = f(0), \qquad x(n+1) = a(n)x(1) + f(n), \quad n \in \mathbb{N}, \tag{18}$$

then for the solution $x(n) = x(n, 0, 0_{\mathcal{B}}; f)$ with $f \in \ell^p$, we get

$$x(n+1) = a(n)f(0) + f(n), \quad n \in \mathbb{N}.$$

For instance, if $p = \infty$, then any solution is bounded for a bounded $\{f\}$.
However, the relevant homogeneous equation is obviously not UES.
A more sophisticated example shows that the uniform boundedness of the projections *cannot be replaced* by the less restrictive condition

$$\sup_{n \in \mathbb{Z}^+} \|L(n) \underset{[-\infty, m_n]}{\Pr} \|_{\mathcal{B}^\gamma \to \mathcal{X}} < \infty \tag{19}$$

with non-positive m_n such that $\lim_{n \to \infty} m_n = -\infty$.

Assumptions are Necessary

Also, the phase space decay is required.
Example 7. The system

$$x(n+1) = x(n) + a(n)x(0) + f(n). \tag{20}$$

One can see that:

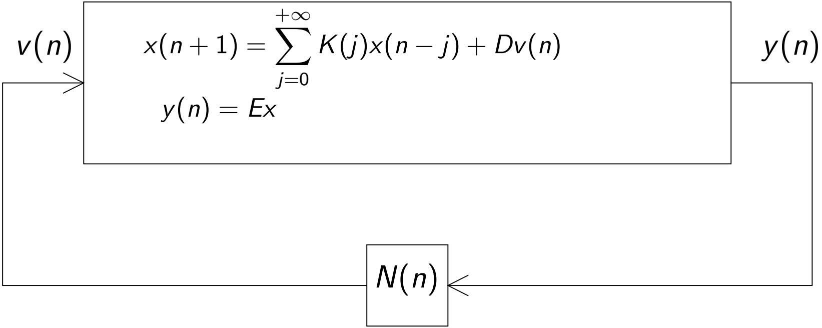
(i) system (20) is (ℓ^1, ℓ^∞)-stable,

(ii) $\| L(\cdot)\, \mathrm{Pr}_{[-\infty,-1]} \|_p < \infty$,

(iii) but the homogeneous system associated with (20) *is not US in $\mathcal{X}$ w.r.t. $\mathcal{B}^0$*.

Stability in the non-decaying phase spaces is still to be studied!

Open Problem: Application to Control



Here $v(n) = N(n)y(n)$, v is an input, y is the output.

Outline

- Let $\gamma > 0$. Assume that either $p \neq 1$ or $q \neq \infty$. Then the homogeneous system is uniformly exponentially stable in $\mathcal{B}^\gamma$ if and only if the system with the right-hand side is (ℓ^p, ℓ^q)-stable and

$$\sup_{n \geq 0} \sum_{k \geq l} e^{k\gamma} \|L(n,k)\|_{\mathcal{X} \to \mathcal{X}} < \infty \text{ for some positive integer } l. \quad (21)$$

  The homogeneous system is uniformly stable in $\mathcal{B}^\gamma$ if and only if the non-homogeneous system is (ℓ^1, ℓ^∞)-stable and (21) holds.

- Under (21), (i) (ℓ^p, ℓ^q)-stability does not depend on p and q (excluding the case $(p, q) = (1, \infty)$), (ii) exponential stability in $\mathcal{B}^\delta$ does not depend on the choice of $\delta \in (0, \gamma]$.

- It is essential that we consider exponentially fading phase spaces $\mathcal{B}^\gamma$, $\gamma > 0$. To some extent, the assumptions of the theorems are necessary.

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Thank you for your attention!

Questions?