P-recursive moment sequences of piecewise D-finite functions and Prony-type algebraic systems

Dmitry Batenkov Gal Binyamini Yosef Yomdin

Weizmann Institute of Science, Israel

18th International Conference on Difference Equations and Applications
July 23-27, 2012, Barcelona

1. Prony-type systems

Linear recurrences with constant coefficients

Definition

The sequence $\{m_k\}_{k=0}^{\infty} \in \mathbb{C}^{\omega}$ is $\mathbb{C}$-**recurrent** if $\exists A_0, \ldots, A_d \in \mathbb{C}$ such that $\forall k \in \mathbb{N}$:

$$A_0 m_k + A_1 m_{k+1} + \cdots + A_d m_{k+d} = 0.$$

Linear recurrences with constant coefficients

Definition

The sequence $\{m_k\}_{k=0}^{\infty} \in \mathbb{C}^{\omega}$ is $\mathbb{C}$-**recurrent** if $\exists A_0, \ldots, A_d \in \mathbb{C}$ such that $\forall k \in \mathbb{N}$:

$$A_0 m_k + A_1 m_{k+1} + \cdots + A_d m_{k+d} = 0.$$

General form of solution

Exponential polynomials (Binet's formula)

$$m_k = \sum_{i=1}^{\mathscr{K}} P_i(k)\, \xi_i^k$$

where $\{\xi_i\}$ are the roots of the **characteristic polynomial** $A_0 + A_1 x + \cdots + A_d x^d$.

Prony system

$$m_k = \sum_{i=1}^{\mathscr{K}} P_i(k)\,\xi_i^k$$

Reconstruction problem

Given few initial terms $m_0, \ldots, m_N$, reconstruct $\{\xi_i, P_i\}$.

Prony system

$$m_k = \sum_{i=1}^{\mathscr{K}} P_i(k)\, \xi_i^k$$

Reconstruction problem

Given few initial terms $m_0, \ldots, m_N$, reconstruct $\{\xi_i, P_i\}$.

Examples

- **Padé approximation:** $\{m_k\}$ are Taylor coefficients of a rational function with poles at $\left\{\xi_i^{-1}\right\}$

Prony system

$$m_k = \sum_{i=1}^{\mathscr{K}} P_i(k)\, \xi_i^k$$

Reconstruction problem

Given few initial terms $m_0, \ldots, m_N$, reconstruct $\{\xi_i, P_i\}$.

Examples

- **Padé approximation:** $\{m_k\}$ are Taylor coefficients of a rational function with poles at $\left\{\xi_i^{-1}\right\}$
- High resolution methods in Signal Processing

Example: finite rate of innovation

- Problem: recovering a signal which has been sampled below Nyquist rate

Example: finite rate of innovation

- Problem: recovering a signal which has been sampled below Nyquist rate
- Assumption: the signal is finite-parametric. For example:

$$x(t) = \sum_{j=0}^{\mathcal{K}} a_j \delta(t - \xi_j)$$

Example: finite rate of innovation

- Problem: recovering a signal which has been sampled below Nyquist rate
- Assumption: the signal is finite-parametric. For example:

$$x(t) = \sum_{j=0}^{\mathscr{K}} a_j \delta(t - \xi_j)$$

- Method: choose a sampling kernel $h(t)$ with certain algebraic properties s.t.

$$y_n = \langle h(t-n), x(t) \rangle = \sum_{j=0}^{\mathscr{K}} a_j \, \mathrm{e}^{-\imath \xi_j n}$$

Example: finite rate of innovation

- Problem: recovering a signal which has been sampled below Nyquist rate
- Assumption: the signal is finite-parametric. For example:

$$x(t) = \sum_{j=0}^{\mathscr{K}} a_j \delta(t - \xi_j)$$

- Method: choose a sampling kernel $h(t)$ with certain algebraic properties s.t.

$$y_n = \langle h(t-n), x(t) \rangle = \sum_{j=0}^{\mathscr{K}} a_j \, \mathrm{e}^{-\imath \xi_j n}$$

- Generalized to piecewise polynomials

Prony solution method

$$m_k = \sum_{i=1}^{\mathscr{K}} P_i(k)\,\xi_i^k; \qquad \sum_{i=1}^{\mathscr{K}} \deg P_i = C$$

Prony solution method

$$m_k = \sum_{i=1}^{\mathscr{K}} P_i(k)\,\xi_i^k; \qquad \sum_{i=1}^{\mathscr{K}} \deg P_i = C$$

1. Solve Hankel-type system

$$\underbrace{\begin{bmatrix} m_0 & m_1 & \cdots & m_{C-1} \\ m_1 & m_2 & \cdots & m_C \\ \vdots & \vdots & \vdots & \vdots \\ m_{C-1} & m_{d+1} & \cdots & m_{2C-1} \end{bmatrix}}_{\overset{\text{def}}{=} M} \times \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{C-1} \end{bmatrix} = - \begin{bmatrix} m_C \\ m_{C+1} \\ \vdots \\ m_{2C} \end{bmatrix}$$

Prony solution method

$$m_k = \sum_{i=1}^{\mathscr{K}} P_i(k)\,\xi_i^k; \qquad \sum_{i=1}^{\mathscr{K}} \deg P_i = C$$

1. Solve Hankel-type system

$$\underbrace{\begin{bmatrix} m_0 & m_1 & \cdots & m_{C-1} \\ m_1 & m_2 & \cdots & m_C \\ \vdots & \vdots & \vdots & \vdots \\ m_{C-1} & m_{d+1} & \cdots & m_{2C-1} \end{bmatrix}}_{\stackrel{\text{def}}{=} M} \times \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{C-1} \end{bmatrix} = - \begin{bmatrix} m_C \\ m_{C+1} \\ \vdots \\ m_{2C} \end{bmatrix}$$

2. $\{\xi_j\}$ are the roots of $x^d + A_{d-1}x^{d-1} + \cdots + A_1 x + A_0 = 0$.

Prony solution method

$$m_k = \sum_{i=1}^{\mathscr{K}} P_i(k)\,\xi_i^k; \qquad \sum_{i=1}^{\mathscr{K}} \deg P_i = C$$

1. Solve Hankel-type system

$$\underbrace{\begin{bmatrix} m_0 & m_1 & \cdots & m_{C-1} \\ m_1 & m_2 & \cdots & m_C \\ \vdots & \vdots & \vdots & \vdots \\ m_{C-1} & m_{d+1} & \cdots & m_{2C-1} \end{bmatrix}}_{\overset{\text{def}}{=} M} \times \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{C-1} \end{bmatrix} = - \begin{bmatrix} m_C \\ m_{C+1} \\ \vdots \\ m_{2C} \end{bmatrix}$$

2. $\{\xi_j\}$ are the roots of $x^d + A_{d-1}x^{d-1} + \cdots + A_1 x + A_0 = 0.$
3. Coefficients of $\{P_i\}$ are found by solving a Vandermonde-type linear system.

Subspace methods

Observations

- $M = V^T B V$, with V-confluent Vandermonde.
- The range of M and V are the same.
- V has the *rotational invariance property*:

$$V^{\uparrow} = V_{\downarrow} J$$

where J is the block Jordan matrix with eigenvalues $\{\xi_j\}$.

Subspace methods

Observations

- $M = V^T BV$, with V-confluent Vandermonde.
- The range of M and V are the same.
- V has the *rotational invariance property*:

$$V^{\uparrow} = V_{\downarrow} J$$

where J is the block Jordan matrix with eigenvalues $\{\xi_j\}$.

ESPRIT method

1. Compute the SVD $M = W \Sigma V^T$.
2. Calculate $\Phi = W_{\downarrow}^{\#} W^{\uparrow}$.
3. Set $\{\xi_i\}$ to be the eigenvalues of Φ with appropriate multiplicities.

Prony systems - solvability

$$m_k = \sum_{j=1}^{\mathscr{K}} \sum_{i=0}^{l_j-1} a_{i,j} \underbrace{k(k-1)\cdot\dots\cdot(k-i+1)}_{\stackrel{\text{def}}{=}(k)_i} \xi_j^{k-i}; \quad \sum_{j=1}^{\mathscr{K}} l_j = C;\ k = 0, 1, \dots, 2C-1$$

Theorem

The Prony system ***has a solution*** *if and only if the sequence* $(m_0, \dots, m_{2C-1})$ *is* $\mathbb{C}$*-recurrent of length at most* C*.*

Prony systems - solvability

$$m_k = \sum_{j=1}^{\mathscr{K}} \sum_{i=0}^{l_j-1} a_{i,j} \underbrace{k(k-1)\cdot\dots\cdot(k-i+1)}_{\stackrel{\text{def}}{=}(k)_i} \xi_j^{k-i}; \quad \sum_{j=1}^{\mathscr{K}} l_j = C;\ k = 0,1,\dots,2C-1$$

Theorem

The Prony system ***has a solution*** *if and only if the sequence $(m_0,\dots,m_{2C-1})$ is $\mathbb{C}$-recurrent of length at most C.*

Theorem

The parameters $\{a_{i,j},\xi_j\}$ can be ***uniquely*** *recovered from the first $2C$ measurements if and only if 1)$\xi_i \neq \xi_j$ for $i \neq j$, and 2)$a_{l_j-1,j} \neq 0$ for all $j = 1,\dots,\mathscr{K}$.*

Prony systems - local stability

Theorem (DB,YY 2010)

Assume that $\max_{k<C}|\Delta m_k| \leq \varepsilon$ *for sufficiently small* ε.
Then there exists a positive constant C_1 *depending only on the nodes* $\xi_1,\ldots,\xi_{\mathscr{K}}$ *and the multiplicities* $l_1,\ldots,l_{\mathscr{K}}$ *such that for all* $i=1,2,\ldots,\mathscr{K}$:

$$|\Delta a_{ij}| \leq \begin{cases} C_1\varepsilon & j=0 \\ C_1\varepsilon\left(1+\frac{|a_{i,j-1}|}{|a_{i,l_i-1}|}\right) & 1\leq j\leq l_i-1 \end{cases}$$

$$|\Delta \xi_i| \leq C_1\varepsilon\frac{1}{|a_{i,l_i-1}|}$$

- This behaviour is observed in experiments

Prony systems - local stability

Theorem (DB,YY 2010)

Assume that $\max_{k<C} |\Delta m_k| \leq \varepsilon$ *for sufficiently small* ε.
Then there exists a positive constant C_1 *depending only on the nodes* $\xi_1, \ldots, \xi_{\mathscr{K}}$ *and the multiplicities* $l_1, \ldots, l_{\mathscr{K}}$ *such that for all* $i = 1, 2, \ldots, \mathscr{K}$:

$$|\Delta a_{ij}| \leq \begin{cases} C_1 \varepsilon & j = 0 \\ C_1 \varepsilon \left(1 + \frac{|a_{i,j-1}|}{|a_{i,l_i-1}|}\right) & 1 \leq j \leq l_i - 1 \end{cases}$$

$$|\Delta \xi_i| \leq C_1 \varepsilon \frac{1}{|a_{i,l_i-1}|}$$

- This behaviour is observed in experiments
- Prony method fails to separate the parameters, worst performance

Prony systems - local stability

Theorem (DB,YY 2010)

Assume that $\max_{k<C}|\Delta m_k| \leq \varepsilon$ *for sufficiently small* ε.
Then there exists a positive constant C_1 *depending only on the nodes* $\xi_1,\dots,\xi_{\mathscr{K}}$ *and the multiplicities* $l_1,\dots,l_{\mathscr{K}}$ *such that for all* $i = 1,2,\dots,\mathscr{K}$*:*

$$|\Delta a_{ij}| \leq \begin{cases} C_1\varepsilon & j=0 \\ C_1\varepsilon\left(1+\frac{|a_{i,j-1}|}{|a_{i,l_i-1}|}\right) & 1 \leq j \leq l_i - 1 \end{cases}$$

$$|\Delta \xi_i| \leq C_1 \varepsilon \frac{1}{|a_{i,l_i-1}|}$$

- This behaviour is observed in experiments
- Prony method fails to separate the parameters, worst performance
- ESPRIT is better, but still not optimal

Algebraic Fourier inversion

Problem

Reconstruct a **piecewise** C^d function f from n Fourier samples

$$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{e}^{-\imath k t}\,\mathrm{d}t.$$

- Approximation accuracy $\sim n^{-1}$ - bad!

Algebraic Fourier inversion

Problem

Reconstruct a **piecewise** C^d function f from n Fourier samples

$$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{e}^{-\imath kt}\,\mathrm{d}t.$$

- Approximation accuracy $\sim n^{-1}$ - bad!

Algebraic approach[Eckhoff(1995)]

- Approximate f by a piecewise polynomial Φ
  - jumps at $\{\xi_i\}$ with magnitudes $\{a_{i,j}\}$.
- Recover Φ from the perturbed Prony-type system

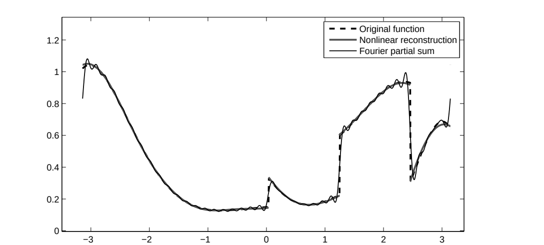
$$c_k(f) = \frac{1}{2\pi}\sum_{j=1}^{\mathscr{K}} \mathrm{e}^{-\imath k\xi_j} \sum_{l=0}^{d} \frac{a_{l,j}}{(\imath k)^{l+1}} + O\left(k^{-d-2}\right)$$

Algebraic Fourier inversion

Theorem (DB,YY 2011)

If f is piecewise-C^{d_1} where $d_1 \geq 2d+1$, then

$$\left|\Delta \xi_j\right| \sim n^{-d-2}$$
$$\left|\Delta a_{l,j}\right| \sim n^{-d-1-l}$$
$$\left|\Delta f\right| \sim n^{-d-1}.$$



2 Piecewise D-finite reconstruction

Piecewise D-finite model

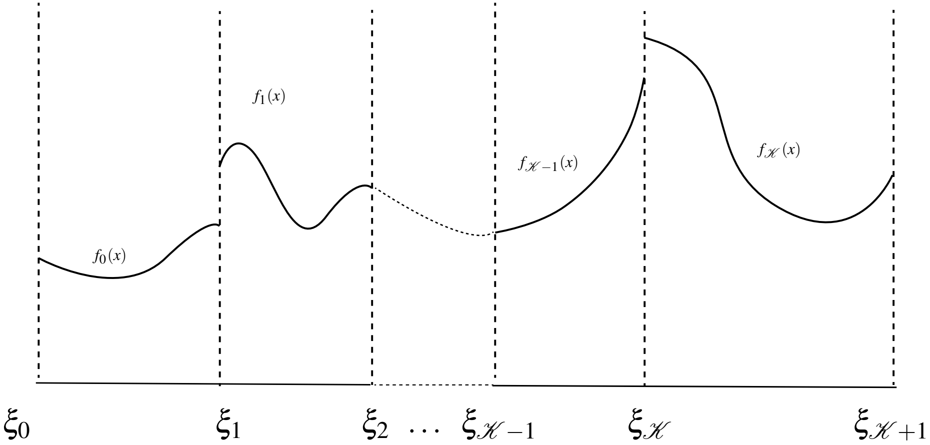


Figure: Piecewise D-finite model

Piecewise D-finite reconstruction

- Every piece satisfies $\mathfrak{D} f_i(x) \equiv 0$, $\mathfrak{D}$ - linear differential operator with polynomial coefficients

$$\mathfrak{D} = \sum_{j=0}^{n} \left(\sum_{i=0}^{d} a_{i,j} x^i \right) \frac{\mathrm{d}^j}{\mathrm{d}x^j} \quad (a_{ij} \in \mathbb{R})$$

Piecewise D-finite reconstruction

- Every piece satisfies $\mathfrak{D} f_i(x) \equiv 0$, $\mathfrak{D}$ - linear differential operator with polynomial coefficients

$$\mathfrak{D} = \sum_{j=0}^{n} \left(\sum_{i=0}^{d} a_{i,j} x^i \right) \frac{\mathrm{d}^j}{\mathrm{d} x^j} \quad (a_{ij} \in \mathbb{R})$$

- Unknown model parameters:

Piecewise D-finite reconstruction

- Every piece satisfies $\mathfrak{D} f_i(x) \equiv 0$, $\mathfrak{D}$ - linear differential operator with polynomial coefficients

$$\mathfrak{D} = \sum_{j=0}^{n} \left(\sum_{i=0}^{d} a_{i,j} x^i \right) \frac{\mathrm{d}^j}{\mathrm{d} x^j} \quad (a_{ij} \in \mathbb{R})$$

- Unknown model parameters:
  - Coefficients of $\mathfrak{D}$, i.e. $\{a_{i,j}\}$,

Piecewise D-finite reconstruction

- Every piece satisfies $\mathfrak{D} f_i(x) \equiv 0$, $\mathfrak{D}$ - linear differential operator with polynomial coefficients

$$\mathfrak{D} = \sum_{j=0}^{n} \left(\sum_{i=0}^{d} a_{i,j} x^i \right) \frac{\mathrm{d}^j}{\mathrm{d}x^j} \quad (a_{ij} \in \mathbb{R})$$

- Unknown model parameters:
  - Coefficients of $\mathfrak{D}$, i.e. $\{a_{i,j}\}$,
  - Jump points $\{\xi_i\}$,

Piecewise D-finite reconstruction

- Every piece satisfies $\mathfrak{D} f_i(x) \equiv 0$, $\mathfrak{D}$ - linear differential operator with polynomial coefficients

$$\mathfrak{D} = \sum_{j=0}^{n} \left(\sum_{i=0}^{d} a_{i,j} x^i \right) \frac{\mathrm{d}^j}{\mathrm{d} x^j} \quad (a_{ij} \in \mathbb{R})$$

- Unknown model parameters:
  - Coefficients of $\mathfrak{D}$, i.e. $\{a_{i,j}\}$,
  - Jump points $\{\xi_i\}$,
  - Initial values of f at $\{\xi_i\}$.

Piecewise D-finite reconstruction

- Every piece satisfies $\mathfrak{D} f_i(x) \equiv 0$, $\mathfrak{D}$ - linear differential operator with polynomial coefficients

$$\mathfrak{D} = \sum_{j=0}^{n} \left(\sum_{i=0}^{d} a_{i,j} x^i \right) \frac{\mathrm{d}^j}{\mathrm{d} x^j} \qquad (a_{ij} \in \mathbb{R})$$

- Unknown model parameters:
  - Coefficients of $\mathfrak{D}$, i.e. $\{a_{i,j}\}$,
  - Jump points $\{\xi_i\}$,
  - Initial values of f at $\{\xi_i\}$.
- Measurements: algebraic moments $m_k(f) = \int_a^b x^k f(x)\,\mathrm{d}x$.

Recurrence relation

$$\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k}}_{\mathfrak{S}\{m_k\}} = \sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}$$

- Idea: integration by parts of the identity $\int_a^b x^k \mathfrak{D} f \equiv 0$.

Recurrence relation

$$\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}\left(-1\right)^{j}\left(i+k\right)_{j} m_{i-j+k}}_{\mathfrak{S}\{m_k\}} = \sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}\left(k\right)_{j}\xi_i^{k-j}$$

- Idea: integration by parts of the identity $\int_a^b x^k \mathfrak{D} f \equiv 0$.
- $c_{i,j}$ - homogeneous bilinear form depending on the values of $\{p_l(x)\}_{l=0}^{n}$ and the "jump function" $f(x^+) - f(x^-)$ with their derivatives up to order $n-1$ at the point $x = \xi_i$.

Recurrence relation

$$\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k}}_{\mathfrak{S}\{m_k\}} = \sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}$$

- Idea: integration by parts of the identity $\int_a^b x^k \mathfrak{D} f \equiv 0$.
- $c_{i,j}$ - homogeneous bilinear form depending on the values of $\{p_l(x)\}_{l=0}^{n}$ and the "jump function" $f(x^+) - f(x^-)$ with their derivatives up to order $n-1$ at the point $x = \xi_i$.
- The RHS is annihilated by constant coefficients difference operator

$$\mathscr{E} = \prod_{i=1}^{\mathscr{K}} (\mathrm{E} - \xi_i \mathfrak{I})^n$$

Recurrence relation

$$\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k}}_{\mathfrak{S}\{m_k\}} = \sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}$$

- Idea: integration by parts of the identity $\int_a^b x^k \mathfrak{D} f \equiv 0$.
- $c_{i,j}$ - homogeneous bilinear form depending on the values of $\{p_l(x)\}_{l=0}^n$ and the "jump function" $f(x^+) - f(x^-)$ with their derivatives up to order $n-1$ at the point $x = \xi_i$.
- The RHS is annihilated by constant coefficients difference operator

$$\mathscr{E} = \prod_{i=1}^{\mathscr{K}} (\mathrm{E} - \xi_i \mathfrak{I})^n$$

- Homogeneous recurrence relation for the moments:

$$\mathscr{E}\,\mathfrak{S}\{m_k\} = 0.$$

Reconstruction procedure

$$\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k} = \sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}$$

$$\mathscr{E}\,\mathfrak{S}\{m_k\} = 0$$

1. Operator $\mathfrak{D}$ is known

Reconstruction procedure

$$\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k} = \sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}$$

$$\mathscr{E}\,\mathfrak{S}\{m_k\} = 0$$

1. Operator $\mathfrak{D}$ is known
   - solve the confluent Prony system directly (LHS is known) for $\{\xi_j, c_{i,j}\}$ and fully recover the function.

Reconstruction procedure

$$\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k} = \sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}$$

$$\mathscr{E}\,\mathfrak{S}\{m_k\} = 0$$

1. Operator $\mathfrak{D}$ is known
   - solve the confluent Prony system directly (LHS is known) for $\{\xi_j, c_{i,j}\}$ and fully recover the function.
2. Operator $\mathfrak{D}$ unknown

Reconstruction procedure

$$\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k} = \sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}$$

$$\mathscr{E}\,\mathfrak{S}\{m_k\} = 0$$

1. Operator $\mathfrak{D}$ is known
   - solve the confluent Prony system directly (LHS is known) for $\{\xi_j, c_{i,j}\}$ and fully recover the function.
2. Operator $\mathfrak{D}$ unknown
   - Solve for coefficients of the difference operator $\mathscr{E}\,\mathfrak{S}$.

Reconstruction procedure

$$\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k} = \sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}$$

$$\mathscr{E}\,\mathfrak{S}\{m_k\} = 0$$

1. Operator $\mathfrak{D}$ is known
   - solve the confluent Prony system directly (LHS is known) for $\{\xi_j, c_{i,j}\}$ and fully recover the function.
2. Operator $\mathfrak{D}$ unknown
   - Solve for coefficients of the difference operator $\mathscr{E}\,\mathfrak{S}$.
   - Factor out the common roots $\{\xi_j\}$ and the remaining factors $\{a_{i,j}\}$.

Reconstruction procedure

$$\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k} = \sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}$$

$$\mathscr{E}\,\mathfrak{S}\{m_k\} = 0$$

1. Operator $\mathfrak{D}$ is known
   - solve the confluent Prony system directly (LHS is known) for $\{\xi_j, c_{i,j}\}$ and fully recover the function.
2. Operator $\mathfrak{D}$ unknown
   - Solve for coefficients of the difference operator $\mathscr{E}\,\mathfrak{S}$.
   - Factor out the common roots $\{\xi_j\}$ and the remaining factors $\{a_{i,j}\}$.
   - Finally solve the linear system for $\{c_{i,j}\}$ and fully recover the function.

Moment uniqueness and vanishing

How many moments are necessary for unique reconstruction?

Moment uniqueness and vanishing

How many moments are necessary for unique reconstruction?

Definition

Given a particular $\mathfrak{D}$ and number of jump points $\mathscr{K}$, the ***moment uniqueness index*** $\tau(\mathfrak{D},\mathscr{K})$ *is the minimal number of moments required for unique reconstruction of any nonzero solution* $\mathfrak{D}f \equiv 0$.

Moment uniqueness and vanishing

How many moments are necessary for unique reconstruction?

Definition

*Given a particular $\mathfrak{D}$ and number of jump points $\mathscr{K}$, the **moment uniqueness index** $\tau(\mathfrak{D},\mathscr{K})$ is the minimal number of moments required for unique reconstruction of any nonzero solution $\mathfrak{D}f \equiv 0$.*

Definition

*Given a particular $\mathfrak{D}$ and number of jump points $\mathscr{K}$, the **moment vanishing index** $\sigma(\mathfrak{D},\mathscr{K})$ is the maximal number of first zero moments of any nonzero solution $\mathfrak{D}f \equiv 0$.*

Moment uniqueness and vanishing

How many moments are necessary for unique reconstruction?

Definition

*Given a particular $\mathfrak{D}$ and number of jump points $\mathscr{K}$, the **moment uniqueness index** $\tau(\mathfrak{D},\mathscr{K})$ is the minimal number of moments required for unique reconstruction of any nonzero solution $\mathfrak{D}f \equiv 0$.*

Definition

*Given a particular $\mathfrak{D}$ and number of jump points $\mathscr{K}$, the **moment vanishing index** $\sigma(\mathfrak{D},\mathscr{K})$ is the maximal number of first zero moments of any nonzero solution $\mathfrak{D}f \equiv 0$.*

Lemma

$\tau(\mathfrak{D},\mathscr{K}) \leq \sigma(\mathfrak{D},2\mathscr{K})$.

Unbouded example

Legendre differential equation

$$\mathfrak{D}_m = \left(1 - x^2\right) \frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x \frac{\mathrm{d}}{\mathrm{d}x} + m(m+1)\mathfrak{I}.$$

- For $m \in \mathbb{N}$ solutions are the Legendre orthogonal polynomials $\{L_m\}$
- First $m-1$ moments of L_m are zero
- Conclusion: $\sigma(\mathfrak{D}_m) = m$
- $\Longrightarrow$ No uniform bound in terms of d,n for generic $\mathfrak{D}$!

Regular operators

Theorem (DB,GB 2012)

Assume that the leading coefficient of the operator $\mathfrak{D}$ *does not vanish on any two consecutive jump points* ξ_j, ξ_{j+1}*. Then*

$$\sigma(\mathfrak{D}) \leq (\mathscr{K}+2)n+d-1.$$

Proof outline

$$\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}\,(-1)^{j}\,(i+k)_{j}\, m_{i-j+k}}_{\mathfrak{S}\{m_k\}} = \underbrace{\sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}\,(k)_{j}\,\xi_{i}^{k-j}}_{\varepsilon_k}$$

1. Some initial $\{m_k\}$ vanish $\Longrightarrow$ sufficient number of initial ε_k vanish.

Proof outline

$$\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k}}_{\mathfrak{S}\{m_k\}} = \underbrace{\sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}}_{\varepsilon_k}$$

1. Some initial $\{m_k\}$ vanish $\implies$ sufficient number of initial ε_k vanish.
2. By Skolem-Mahler-Lech, ε_k can have only finitely many zeros$\implies c_{i,j} = 0$.

Proof outline

$$\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k}}_{\mathfrak{S}\{m_k\}} = \underbrace{\sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}}_{\varepsilon_k}$$

1. Some initial $\{m_k\}$ vanish $\Longrightarrow$ sufficient number of initial ε_k vanish.
2. By Skolem-Mahler-Lech, ε_k can have only finitely many zeros$\Longrightarrow$ $c_{i,j}=0$.
3. $p_n(\xi_j) \neq 0 \Longrightarrow f(\xi_j) = f'(\xi_j) = \cdots = f^{(n-1)}(\xi_j) = 0.$

Resonant Fuchsian operators

Theorem (DB,GB 2012)

*Let $\mathfrak{D}$ be of Fuchsian type, and consider moments in $[0,1]$. If $\mathfrak{D}$ has **at most one negative integer characteristic exponent** at the point $z=0$, then*

$$\sigma(\mathfrak{D},0)=2n+d-1.$$

Proof outline

1. Write functional equation for the Mellin transform $M[f](s)=\int_0^1 t^s f(s)\,\mathrm{d}s$.
2. Check analytic continuation to $\Re s<0$.

Moment generating function

$$\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}\left(-1\right)^{j}\left(i+k\right)_{j} m_{i-j+k}}_{\mu_k} = \underbrace{\sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}\left(k\right)_{j} \xi_i^{k-j}}_{\varepsilon_k}$$

$$I_g\left(z\right) = \sum_{k=0}^{\infty} \frac{m_k}{z^{k+1}} = \int_a^b \frac{f\left(t\right)\mathrm{d}t}{t-z}$$

Theorem

The Cauchy integral I_g satisfies at the neighborhood of ∞ the inhomogeneous ODE

$$\mathfrak{D}\, I_g\left(z\right) = R\left(z\right)$$

where $R\left(z\right)$ is the rational function whose Taylor coefficients are given by ε_k.

General Fuchsian operators

$$\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k}}_{\mu_k=\mathfrak{S}\{m_k\}} = \underbrace{\sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}}_{\varepsilon_k}$$

Lemma

Let $\mathfrak{D}$ be a Fuchsian operator. Then the characteristic polynomial of $\mathfrak{D}$ at the point ∞ coincides with the leading coefficient of the difference operator $\mathfrak{S}$.

General Fuchsian operators

$$\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k}}_{\mu_k=\mathfrak{S}\{m_k\}} = \underbrace{\sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}}_{\varepsilon_k}$$

Lemma

Let $\mathfrak{D}$ be a Fuchsian operator. Then the characteristic polynomial of $\mathfrak{D}$ at the point ∞ coincides with the leading coefficient of the difference operator $\mathfrak{S}$.

Theorem

Let $\mathfrak{D}$ be a Fuchsian operator, and let $\lambda(\mathfrak{D})$ denote the largest positive integer root of its characteristic polynomial at the point ∞. Then

$$\sigma(\mathfrak{D},\mathscr{K}) \le \max\{\lambda(\mathfrak{D}), (\mathscr{K}+2)n+d-1\}.$$

3 D-finite planar domains

2D shapes from complex moments ([Gustafsson et al.(2000)Gustafsson, He, Milanfar, a

- Let $P \subset \mathbb{C}$ be a polygon with vertices $z_1, \dots, z_n$

2D shapes from complex moments ([Gustafsson et al.(2000)Gustafsson, He, Milanfar, a

- Let $P \subset \mathbb{C}$ be a polygon with vertices $z_1, \dots, z_n$
- Measurements: $\mu_k(f) = \iint z^k f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, $z = x + \imath y$ where $f = \chi_P$

2D shapes from complex moments ([Gustafsson et al.(2000)Gustafsson, He, Milanfar, a

- Let $P \subset \mathbb{C}$ be a polygon with vertices $z_1, \ldots, z_n$
- Measurements: $\mu_k(f) = \iint z^k f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, $z = x + \imath y$ where $f = \chi_P$
- Turns out that there exist $c_{1,} \ldots, c_n \in \mathbb{C}$ s.t.

$$k(k-1)\mu_{k-2}(\chi_P) = \sum_{i=1}^{n} c_i z_i^k$$

2D shapes from complex moments ([Gustafsson et al.(2000)Gustafsson, He, Milanfar, a

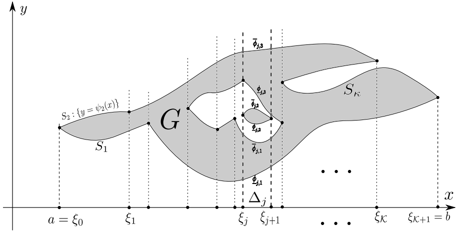
- Let $P \subset \mathbb{C}$ be a polygon with vertices $z_1, \dots, z_n$
- Measurements: $\mu_k(f) = \iint z^k f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, $z = x + \imath y$ where $f = \chi_P$
- Turns out that there exist $c_{1,} \dots, c_n \in \mathbb{C}$ s.t.

$$k(k-1)\mu_{k-2}(\chi_P) = \sum_{i=1}^{n} c_i z_i^k$$

- Special case of *quadrature domains:* any analytic f (in particular $f(z) = z^k$) satisfies

$$\iint_\Omega f(x + \imath y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{i=1}^{n} \sum_{j=0}^{k_j - 1} c_{ij} f^{(j)}(z_i)$$

D-finite domains



$$m_{\alpha,\beta} = \int_a^b x^\alpha \Psi_\beta(x) = \sum_{j=0}^{\mathscr{K}} \int_{\Delta_j} x^\alpha \Psi_{\beta,j}(x)\,\mathrm{d}x$$

$$\Psi_{\beta,j} = \frac{1}{\beta+1} \sum_{l=1}^{s_j} \left\{ \overline{\phi}_{j,l}^{\beta+1}(x) - \underline{\phi}_{j,l}^{\beta+1}(x) \right\}$$

- Ψ_β are piecewise D-finite, are reconstructed via the 1D algorithm.
- $\left\{\phi_{j,l}\right\}$ are reconstructed pointwise via solving Prony-type system.

Bibliography

D. Batenkov.
Moment inversion problem for piecewise D-finite functions.
Inverse Problems, 25(10):105001, October 2009.

D. Batenkov and Y. Yomdin.
Algebraic Fourier reconstruction of piecewise smooth functions.
Mathematics of Computation, 81:277–318, 2012.
doi: 10.1090/S0025-5718-2011-02539-1.
URL http://dx.doi.org/10.1090/S0025-5718-2011-02539-1.

D. Batenkov, V. Golubyatnikov, and Y. Yomdin.
Reconstruction of Planar Domains from Partial Integral Measurements.
In *Proc. Complex Analysis & Dynamical Systems V*, 2011.

K.S. Eckhoff.
Accurate reconstructions of functions of finite regularity from truncated Fourier series expansions.
Mathematics of Computation, 64(210):671–690, 1995.

B. Gustafsson, C. He, P. Milanfar, and M. Putinar.
Reconstructing planar domains from their moments.
Inverse Problems, 16(4):1053–1070, 2000.