P-recursive moment sequences of piecewise
D-finite functions and Prony-type algebraic systems

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Prony-type systems
Linear recurrences with constant coefficients

Definition

The sequence \( \{m_k\}_{k=0}^{\infty} \in \mathbb{C}^\omega \) is \( \mathbb{C} \)-recurrent if \( \exists A_0, \ldots, A_d \in \mathbb{C} \) such that \( \forall k \in \mathbb{N} \):

\[
A_0 m_k + A_1 m_{k+1} + \cdots + A_d m_{k+d} = 0.
\]
Linear recurrences with constant coefficients

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\]

**General form of solution**

Exponential polynomials (Binet’s formula)

\[
m_k = \sum_{i=1}^{\mathcal{K}} P_i(k) \xi_i^k
\]

where \( \{ \xi_i \} \) are the roots of the characteristic polynomial \( A_0 + A_1 x + \cdots + A_d x^d \).
Prony system

\[ m_k = \sum_{i=1}^{\mathcal{H}} P_i(k) \xi_i^k \]

Reconstruction problem

Given few initial terms \(m_0, \ldots, m_N\), reconstruct \(\{\xi_i, P_i\}\).
Prony system

\[ m_k = \sum_{i=1}^{\mathcal{K}} P_i(k) \xi_i^k \]

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Given few initial terms \( m_0, \ldots, m_N \), reconstruct \( \{\xi_i, P_i\} \).

Examples

• Padé approximation: \( \{m_k\} \) are Taylor coefficients of a rational function with poles at \( \{\xi_i^{-1}\} \)
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Examples

- **Padé approximation:** \( \{m_k\} \) are Taylor coefficients of a rational function with poles at \( \{\xi_i^{-1}\} \)
- High resolution methods in Signal Processing
Example: finite rate of innovation

- Problem: recovering a signal which has been sampled below Nyquist rate
Example: finite rate of innovation

- Problem: recovering a signal which has been sampled below Nyquist rate
- Assumption: the signal is finite-parametric. For example:

\[ x(t) = \sum_{j=0}^{\mathcal{K}} a_j \delta(t - \xi_j) \]
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- Method: choose a sampling kernel \( h(t) \) with certain algebraic properties s.t.

\[
y_n = \langle h(t - n), x(t) \rangle = \sum_{j=0}^{\mathcal{K}} a_j e^{-i\xi_j n}
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- Generalized to piecewise polynomials
Prony solution method

\[ m_k = \sum_{i=1}^{\mathcal{K}} P_i(k) \xi_i^k; \quad \sum_{i=1}^{\mathcal{K}} \deg P_i = C \]
Prony solution method

\[ m_k = \sum_{i=1}^{K} P_i(k) \xi_i^k; \quad \sum_{i=1}^{K} \deg P_i = C \]

Solve Hankel-type system

\[
\begin{bmatrix}
  m_0 & m_1 & \cdots & m_{C-1} \\
  m_1 & m_2 & \cdots & m_C \\
  \vdots & \vdots & \ddots & \vdots \\
  m_{C-1} & m_{d+1} & \cdots & m_{2C-1}
\end{bmatrix} \defeq M
\]

\[
\begin{bmatrix}
  A_0 \\
  A_1 \\
  \vdots \\
  A_{C-1}
\end{bmatrix} \times \begin{bmatrix}
  m_C \\
  m_{C+1} \\
  \vdots \\
  m_{2C}
\end{bmatrix} = -
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\]

\[ \text{def} \equiv M \]

2. \( \{ \xi_j \} \) are the roots of \( x^d + A_{d-1}x^{d-1} + \cdots + A_1 x + A_0 = 0. \)
Prony solution method

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2. \( \{\xi_j\} \) are the roots of \( x^d + A_{d-1}x^{d-1} + \cdots + A_1x + A_0 = 0 \).

3. Coefficients of \( \{P_i\} \) are found by solving a Vandermonde-type linear system.
Subspace methods

Observations

- $M = V^T BV$, with $V$-confluent Vandermonde.
- The range of $M$ and $V$ are the same.
- $V$ has the rotational invariance property:

$$V^\uparrow = V_\downarrow J$$

where $J$ is the block Jordan matrix with eigenvalues $\{\xi_j\}$. 
Subspace methods

Observations

- $M = V^T BV$, with $V$-confluent Vandermonde.
- The range of $M$ and $V$ are the same.
- $V$ has the *rotational invariance property*:

$$V^\uparrow = V_\downarrow J$$

where $J$ is the block Jordan matrix with eigenvalues $\{\xi_j\}$.

ESPRIT method

1. Compute the SVD $M = W\Sigma V^T$.
2. Calculate $\Phi = W^\#W^\uparrow$.
3. Set $\{\xi_i\}$ to be the eigenvalues of $\Phi$ with appropriate multiplicities.
The Prony system has a solution if and only if the sequence \((m_0, \ldots, m_{2C-1})\) is \(C\)-recurrent of length at most \(C\).
Prony systems - solvability

\[ m_k = \sum_{j=1}^{\mathcal{K}} \sum_{i=0}^{l_j-1} a_{i,j} k(k-1) \cdots (k-i+1) \xi_j^{k-i}; \quad \sum_{j=1}^{\mathcal{K}} l_j = C; \quad k = 0, 1, \ldots, 2C - 1 \]

**Theorem**

The Prony system has a solution if and only if the sequence \((m_0, \ldots, m_{2C-1})\) is \(\mathbb{C}\)-recurrent of length at most \(C\).

**Theorem**

The parameters \(\{a_{i,j}, \xi_j\}\) can be uniquely recovered from the first \(2C\) measurements if and only if
1) \(\xi_i \neq \xi_j\) for \(i \neq j\), and
2) \(a_{l_j-1,j} \neq 0\) for all \(j = 1, \ldots, \mathcal{K}\).
Theorem (DB, YY 2010)

Assume that \( \max_{k<C} |\Delta m_k| \leq \varepsilon \) for sufficiently small \( \varepsilon \). Then there exists a positive constant \( C_1 \) depending only on the nodes \( \xi_1, \ldots, \xi_K \) and the multiplicities \( l_1, \ldots, l_K \) such that for all \( i = 1, 2, \ldots, K \):

\[
|\Delta a_{ij}| \leq \begin{cases} 
C_1 \varepsilon & j = 0 \\
C_1 \varepsilon \left(1 + \frac{|a_{i,j-1}|}{|a_{i,l_i-1}|}\right) & 1 \leq j \leq l_i - 1 
\end{cases}
\]

\[
|\Delta \xi_i| \leq C_1 \varepsilon \frac{1}{|a_{i,l_i-1}|}
\]

This behaviour is observed in experiments
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- Prony method fails to separate the parameters, worst performance
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This behaviour is observed in experiments

- Prony method fails to separate the parameters, worst performance
- ESPRIT is better, but still not optimal
Algebraic Fourier inversion

Problem

Reconstruct a **piecewise** $C^d$ function $f$ from $n$ Fourier samples

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$ 

- Approximation accuracy $\sim n^{-1}$ - bad!
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Algebraic approach [Eckhoff(1995)]

- Approximate $f$ by a piecewise polynomial $\Phi$
  - jumps at $\{\xi_i\}$ with magnitudes $\{a_{i,j}\}$.
- Recover $\Phi$ from the perturbed Prony-type system

$$c_k(f) = \frac{1}{2\pi} \sum_{j=1}^{\mathcal{K}} e^{-ikt_j} \sum_{l=0}^{d} \frac{a_{l,j}}{(ik)^{l+1}} + O(k^{-d-2})$$
Algebraic Fourier inversion

Theorem (DB, YY 2011)

If \( f \) is piecewise-\( C^{d_1} \) where \( d_1 \geq 2d + 1 \), then

\[
|\Delta \xi_j| \sim n^{-d-2}
\]

\[
|\Delta a_{l,j}| \sim n^{-d-1-l}
\]

\[
|\Delta f| \sim n^{-d-1}.
\]
Piecewise D-finite reconstruction
Figure: Piecewise D-finite model
Piecewise D-finite reconstruction

- Every piece satisfies $\mathcal{D} f_i(x) \equiv 0$, $\mathcal{D}$ - linear differential operator with polynomial coefficients

$$
\mathcal{D} = \sum_{j=0}^{n} \left( \sum_{i=0}^{d} a_{i,j} x^i \right) \frac{d^j}{dx^j} \quad (a_{ij} \in \mathbb{R})
$$
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- Unknown model parameters:
Piecewise D-finite reconstruction

- Every piece satisfies $D f_i(x) \equiv 0$, $D$ - linear differential operator with polynomial coefficients

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- Jump points $\{\xi_i\}$,
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  - Jump points $\{\xi_i\}$,
  - Initial values of $f$ at $\{\xi_i\}$.

- Measurements: algebraic moments $m_k(f) = \int_a^b x^k f(x) \, dx$. 
Recurrence relation

\[ \sum_{j=0}^{n} \sum_{i=0}^{d} a_{i,j} (-1)^j (i + k)_j m_{i-j+k} = \sum_{i=1}^{\aleph} \sum_{j=0}^{n-1} c_{i,j} (k)_j \xi_i^{k-j} \]

\( \mathcal{S}\{m_k\} \)

- Idea: integration by parts of the identity \( \int_a^b x^k \mathcal{D} f \equiv 0 \).
Recurrence relation

\[
\sum_{j=0}^{n} \sum_{i=0}^{d} a_{i,j} (-1)^{j} (i + k)_{j} m_{i-j+k} = \sum_{i=1}^{\mathcal{K}} \sum_{j=0}^{n-1} c_{i,j} (k)_{j} \xi_{i}^{k-j} \\
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- Idea: integration by parts of the identity \( \int_{a}^{b} x^{k} \mathcal{D} f \equiv 0 \).
- \( c_{i,j} \) - homogeneous bilinear form depending on the values of \( \{p_{l}(x)\}_{l=0}^{n} \) and the “jump function” \( f(x^{+}) - f(x^{-}) \) with their derivatives up to order \( n - 1 \) at the point \( x = \xi_{i} \).
Recurrence relation

\[ \sum_{j=0}^{n} \sum_{i=0}^{d} a_{i,j} (-1)^{j} (i + k) j m_{i-j+k} = \sum_{i=1}^{\mathcal{K}} \sum_{j=0}^{n-1} c_{i,j} (k) j \xi^{k-j} \]

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- The RHS is annihilated by constant coefficients difference operator

\[ \mathcal{E} = \prod_{i=1}^{\mathcal{K}} (E - \xi_i \mathcal{I})^n \]
Recurrence relation

\[ \sum_{i=0}^{n} \sum_{j=0}^{d} a_{i,j} (-1)^j (i + k) j m_{i-j+k} = \sum_{i=1}^{\mathcal{K}} \sum_{j=0}^{n-1} c_{i,j} (k) j \xi_i^{k-j} \]

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Homogeneous recurrence relation for the moments:

\[ E \mathcal{S} \{ m_k \} = 0. \]
Reconstruction procedure

\[
\sum_{j=0}^{n} \sum_{i=0}^{d} a_{i,j} (-1)^j (i + k)_j m_{i-j+k} = \sum_{i=1}^{n-1} \sum_{j=0}^{n-1} c_{i,j} (k)_j \xi_i^{k-j}
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Operator \(\mathcal{D}\) is known

- solve the confluent Prony system directly (LHS is known) for \(\{ \xi_{j}, c_{i,j}\}\) and fully recover the function.
Reconstruction procedure

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\sum_{j=0}^{n} \sum_{i=0}^{d} a_{i,j} (-1)^j (i + k) m_{i-j+k} = \sum_{i=1}^{n-1} \sum_{j=0}^{n-1} c_{i,j} (k) \xi_{i}^{k-j} \\
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1. **Operator \( \mathcal{D} \) is known**

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2. **Operator \( \mathcal{D} \) unknown**
Reconstruction procedure

\[ \sum_{j=0}^{n} \sum_{i=0}^{d} a_{i,j} (-1)^j (i + k) \sum_{j=0}^{i+k} c_{i,j} (k) \xi_{i-j} \]  

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1. Operator ∇ is known
   - solve the confluent Prony system directly (LHS is known) for \{ξ_j, c_{i,j}\} and fully recover the function.

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   - Solve for coefficients of the difference operator \mathcal{E} \mathcal{S}.
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   - Solve for coefficients of the difference operator \(\mathcal{E} \mathcal{S}\).
   - Factor out the common roots \(\{\xi_j\}\) and the remaining factors \(\{a_{i,j}\}\).
Reconstruction procedure

\[ \sum_{j=0}^{n} \sum_{i=0}^{d} a_{i,j} (-1)^j (i+k)_j m_{i-j+k} = \sum_{i=1}^{\mathcal{K}} \sum_{j=0}^{n-1} c_{i,j} (k)_j \xi_i^{k-j} \]

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1. **Operator \( \mathcal{D} \) is known**
   - solve the confluent Prony system directly (LHS is known) for \( \{\xi_j, c_{i,j}\} \) and fully recover the function.

2. **Operator \( \mathcal{D} \) unknown**
   - Solve for coefficients of the difference operator \( \mathcal{E} \mathcal{S} \).
   - Factor out the common roots \( \{\xi_j\} \) and the remaining factors \( \{a_{i,j}\} \).
   - Finally solve the linear system for \( \{c_{i,j}\} \) and fully recover the function.
Moment uniqueness and vanishing

How many moments are necessary for unique reconstruction?
Moment uniqueness and vanishing

How many moments are necessary for unique reconstruction?

**Definition**

_Given a particular_ \( \mathcal{D} \) _and number of jump points_ \( \mathcal{K} \), _the moment uniqueness index_ \( \tau(\mathcal{D}, \mathcal{K}) \) _is the minimal number of moments required for unique reconstruction of any nonzero solution_ \( \mathcal{D}f \equiv 0 \).
How many moments are necessary for unique reconstruction?

Definition

Given a particular $\mathcal{D}$ and number of jump points $\mathcal{K}$, the **moment uniqueness index** $\tau(\mathcal{D}, \mathcal{K})$ is the minimal number of moments required for unique reconstruction of any nonzero solution $\mathcal{D} f \equiv 0$.

Definition

Given a particular $\mathcal{D}$ and number of jump points $\mathcal{K}$, the **moment vanishing index** $\sigma(\mathcal{D}, \mathcal{K})$ is the maximal number of first zero moments of any nonzero solution $\mathcal{D} f \equiv 0$. 
Moment uniqueness and vanishing

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Lemma

$\tau(\mathcal{D}, \mathcal{K}) \leq \sigma(\mathcal{D}, 2\mathcal{K})$. 

D.Batenkov, G.Binyamini, Y.Yomdin (WIS) 
Moments of piecewise functions 
ICDEA 2012
Unbounded example

Legendre differential equation

\[ \mathcal{D}_m = \left(1 - x^2\right) \frac{d^2}{dx^2} - 2x \frac{d}{dx} + m (m + 1) \mathcal{I}. \]

- For \( m \in \mathbb{N} \) solutions are the Legendre orthogonal polynomials \( \{L_m\} \)
- First \( m - 1 \) moments of \( L_m \) are zero
- Conclusion: \( \sigma (\mathcal{D}_m) = m \)
- \( \implies \) No uniform bound in terms of \( d, n \) for generic \( \mathcal{D} \)!
Theorem (DB, GB 2012)

Assume that the leading coefficient of the operator $\mathcal{D}$ does not vanish on any two consecutive jump points $\xi_j, \xi_{j+1}$. Then

$$\sigma(\mathcal{D}) \leq (\mathcal{K} + 2)n + d - 1.$$
Some initial \( \{ m_k \} \) vanish \( \Longrightarrow \) sufficient number of initial \( \varepsilon_k \) vanish.
Proof outline

1. Some initial \( \{ m_k \} \) vanish \( \implies \) sufficient number of initial \( \varepsilon_k \) vanish.

2. By Skolem-Mahler-Lech, \( \varepsilon_k \) can have only finitely many zeros \( \implies c_{i,j} = 0 \).
Proof outline

\[ \sum_{j=0}^{n} \sum_{i=0}^{d} a_{i,j} (-1)^j (i+k)_j m_{i-j+k} = \sum_{i=1}^{n-1} \sum_{j=0}^{\mathcal{K}} c_{i,j} (k)_j \xi_i^{k-j} \]

\( \mathcal{S} \{m_k\} \)

\( \mathcal{E}_k \)

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2. By Skolem-Mahler-Lech, \( \varepsilon_k \) can have only finitely many zeros \( \implies c_{i,j} = 0 \).

3. \( p_n(\xi_j) \neq 0 \implies f(\xi_j) = f'(\xi_j) = \cdots = f^{(n-1)}(\xi_j) = 0 \).
Resonant Fuchsian operators

**Theorem (DB, GB 2012)**

Let $\mathcal{D}$ be of Fuchsian type, and consider moments in $[0, 1]$. If $\mathcal{D}$ has at most one negative integer characteristic exponent at the point $z = 0$, then

$$\sigma(\mathcal{D}, 0) = 2n + d - 1.$$  

**Proof outline**

1. Write functional equation for the Mellin transform
   $$M[f](s) = \int_0^1 t^s f(t) \, dt.$$
2. Check analytic continuation to $\Re s < 0$. 

Moment generating function

\[
\sum_{j=0}^{d} \sum_{i=0}^{n} a_{i,j} (-1)^j (i+k)_j m_{i-j+k} = \mu_k \sum_{i=1}^{n-1} \sum_{j=0}^{\mathcal{K}} c_{i,j} (k)_j \xi_i^{k-j} \]

\[
I_g(z) = \sum_{k=0}^{\infty} \frac{m_k}{z^{k+1}} = \int_a^b \frac{f(t)}{t-z} dt
\]

**Theorem**

The Cauchy integral \(I_g\) satisfies at the neighborhood of \(\infty\) the inhomogeneous ODE

\[
\mathcal{D} I_g(z) = R(z)
\]

where \(R(z)\) is the rational function whose Taylor coefficients are given by \(\varepsilon_k\).
General Fuchsian operators

\[ \sum_{j=0}^{n} \sum_{i=0}^{d} a_{i,j} (-1)^{j} (i + k)_{j} m_{i-j+k} = \mu_{k} = S \{m_{k}\} \]

\[ \sum_{i=1}^{n-1} \sum_{j=0}^{n-1} c_{i,j} (k)_{j} \xi_{i}^{k-j} = \varepsilon_{k} \]

**Lemma**

Let \( \mathcal{D} \) be a Fuchsian operator. Then the characteristic polynomial of \( \mathcal{D} \) at the point \( \infty \) coincides with the leading coefficient of the difference operator \( \mathcal{S} \).

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Moments of piecewise functions
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General Fuchsian operators

\[
\sum_{j=0}^{n} \sum_{i=0}^{d} a_{i,j} (-1)^j (i + k)_j m_{i-j+k} = \sum_{i=1}^{\mathcal{K}} \sum_{j=0}^{n-1} c_{i,j} (k)_{j} \xi_{i}^{k-j} \]

\[\mu_k = \mathcal{G}\{m_k\}\]

Lemma

Let $\mathcal{D}$ be a Fuchsian operator. Then the characteristic polynomial of $\mathcal{D}$ at the point $\infty$ coincides with the leading coefficient of the difference operator $\mathcal{G}$.

Theorem

Let $\mathcal{D}$ be a Fuchsian operator, and let $\lambda(\mathcal{D})$ denote the largest positive integer root of its characteristic polynomial at the point $\infty$. Then

\[
\sigma(\mathcal{D}, \mathcal{K}) \leq \max\{\lambda(\mathcal{D}), (\mathcal{K} + 2)n + d - 1\}.
\]
D-finite planar domains
2D shapes from complex moments ([Gustafsson et al.(2000)Gustafsson, He, Milanfar, and]

- Let $P \subset \mathbb{C}$ be a polygon with vertices $z_1, \ldots, z_n$
2D shapes from complex moments
([Gustafsson et al. (2000) Gustafsson, He, Milanfar, and

- Let $P \subset \mathbb{C}$ be a polygon with vertices $z_1, \ldots, z_n$
- Measurements: $\mu_k(f) = \iint z^kf(x,y) \, dx \, dy$, $z = x + iy$ where $f = \chi_P$
Let $P \subset \mathbb{C}$ be a polygon with vertices $z_1, \ldots, z_n$

Measurements: $\mu_k(f) = \int \int z^k f(x, y) \, dx \, dy$, $z = x + iy$ where $f = \chi_P$

Turns out that there exist $c_1, \ldots, c_n \in \mathbb{C}$ s.t.

$$k(k-1)\mu_{k-2}(\chi_P) = \sum_{i=1}^{n} c_i z_i^k$$
2D shapes from complex moments ([Gustafsson et al. (2000)](Gustafsson, He, Milanfar, and)

- Let $P \subset \mathbb{C}$ be a polygon with vertices $z_1, \ldots, z_n$
- Measurements: $\mu_k(f) = \iint z^k f(x, y) \, dx \, dy$, $z = x + iy$ where $f = \chi_P$
- Turns out that there exist $c_1, \ldots, c_n \in \mathbb{C}$ s.t.

\[
k(k - 1) \mu_{k-2}(\chi_P) = \sum_{i=1}^{n} c_i z_i^k
\]

- Special case of *quadrature domains*: any analytic $f$ (in particular $f(z) = z^k$) satisfies

\[
\iint_{\Omega} f(x + iy) \, dx \, dy = \sum_{i=1}^{n} \sum_{j=0}^{k_j - 1} c_{ij} f^{(j)}(z_i)
\]
D-finite domains

\[ m_{\alpha,\beta} = \int_a^b x^{\alpha} \Psi_\beta(x) = \sum_{j=0}^K \int_{\Delta_j} x^{\alpha} \Psi_{\beta,j}(x) \, dx \]

\[ \Psi_{\beta,j} = \frac{1}{\beta + 1} \sum_{l=1}^{s_j} \left\{ \phi_{j,l}^{\beta+1}(x) - \phi_{j,l}^{\beta+1}(x) \right\} \]

- \( \Psi_\beta \) are piecewise D-finite, are reconstructed via the 1D algorithm.
- \( \{ \phi_{j,l} \} \) are reconstructed pointwise via solving Prony-type system.
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