P-recursive moment sequences of piecewise D-finite functions and Prony-type algebraic systems

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1 Prony-type systems

Linear recurrences with constant coefficients

Definition

The sequence $\{m_k\}_{k=0}^{\infty} \in \mathbb{C}^{\omega}$ is $\mathbb{C}$-**recurrent** if $\exists A_0,\dots,A_d \in \mathbb{C}$ such that $\forall k \in \mathbb{N}$:

$$A_0 m_k + A_1 m_{k+1} + \cdots + A_d m_{k+d} = 0.$$

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General form of solution

Exponential polynomials (Binet's formula)

$$m_k = \sum_{i=1}^{\mathcal{K}} P_i(k)\,\xi_i^k$$

where $\{\xi_i\}$ are the roots of the **characteristic polynomial** $A_0 + A_1 x + \cdots + A_d x^d$.

Prony system

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Reconstruction problem

Given few initial terms $m_0, \dots, m_N$, reconstruct $\{\xi_i, P_i\}$.

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- **Padé approximation:** $\{m_k\}$ are Taylor coefficients of a rational function with poles at $\{\xi_i^{-1}\}$
- High resolution methods in Signal Processing

Example: finite rate of innovation

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- Method: choose a sampling kernel $h(t)$ with certain algebraic properties s.t.

$$y_n=\langle h(t-n),x(t)\rangle=\sum_{j=0}^{\mathscr{K}} a_j\,\mathrm{e}^{-\imath\xi_j n}$$

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- Generalized to piecewise polynomials

Prony solution method

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$$\underbrace{\begin{bmatrix} m_0 & m_1 & \cdots & m_{C-1} \\ m_1 & m_2 & \cdots & m_C \\ \vdots & \vdots & \vdots & \vdots \\ m_{C-1} & m_{d+1} & \cdots & m_{2C-1} \end{bmatrix}}_{\stackrel{\text{def}}{=} M} \times \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{C-1} \end{bmatrix} = - \begin{bmatrix} m_C \\ m_{C+1} \\ \vdots \\ m_{2C} \end{bmatrix}$$

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3. Coefficients of $\{P_i\}$ are found by solving a Vandermonde-type linear system.

Subspace methods

Observations

- $M = V^T B V$, with V-confluent Vandermonde.
- The range of M and V are the same.
- V has the *rotational invariance property*:

$$V^{\uparrow} = V_{\downarrow} J$$

where J is the block Jordan matrix with eigenvalues $\{\xi_j\}$.

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ESPRIT method

1. Compute the SVD $M = W \Sigma V^T$.
2. Calculate $\Phi = W_{\downarrow}^{\#} W^{\uparrow}$.
3. Set $\{\xi_i\}$ to be the eigenvalues of Φ with appropriate multiplicities.

Prony systems - solvability

$$m_k = \sum_{j=1}^{\mathcal{K}} \sum_{i=0}^{l_j-1} a_{i,j} \underbrace{k(k-1)\cdot \dots \cdot (k-i+1)}_{\stackrel{\text{def}}{=} (k)_i} \xi_j^{k-i}; \quad \sum_{j=1}^{\mathcal{K}} l_j = C;\ k = 0,1,\dots,2C-1$$

Theorem

The Prony system ***has a solution*** *if and only if the sequence* $(m_0, \dots, m_{2C-1})$ *is* $\mathbb{C}$*-recurrent of length at most* C.

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Theorem

The parameters $\{a_{i,j}, \xi_j\}$ *can be* ***uniquely*** *recovered from the first* $2C$ *measurements if and only if 1)*$\xi_i \neq \xi_j$ *for* $i \neq j$*, and 2)*$a_{l_j-1,j} \neq 0$ *for all* $j = 1, \dots, \mathcal{K}$.

Prony systems - local stability

Theorem (DB,YY 2010)

Assume that $\max_{k<C} |\Delta m_k| \leq \varepsilon$ *for sufficiently small* ε.
Then there exists a positive constant C_1 *depending only on the nodes* $\xi_1, \dots, \xi_{\mathcal{K}}$ *and the multiplicities* $l_1, \dots, l_{\mathcal{K}}$ *such that for all* $i = 1, 2, \dots, \mathcal{K}$:

$$|\Delta a_{ij}| \leq \begin{cases} C_1 \varepsilon & j = 0 \\ C_1 \varepsilon \left(1 + \frac{|a_{i,j-1}|}{|a_{i,l_i-1}|}\right) & 1 \leq j \leq l_i - 1 \end{cases}$$

$$|\Delta \xi_i| \leq C_1 \varepsilon \frac{1}{|a_{i,l_i-1}|}$$

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- Prony method fails to separate the parameters, worst performance
- ESPRIT is better, but still not optimal

Algebraic Fourier inversion

Problem

Reconstruct a **piecewise** C^d function f from n Fourier samples

$$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{e}^{-\imath kt}\,\mathrm{d}t.$$

- Approximation accuracy $\sim n^{-1}$ - bad!

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Algebraic approach[Eckhoff(1995)]

- Approximate f by a piecewise polynomial Φ
  - jumps at $\{\xi_i\}$ with magnitudes $\{a_{i,j}\}$.
- Recover Φ from the perturbed Prony-type system

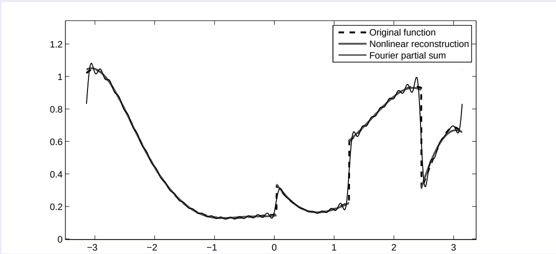
$$c_k(f) = \frac{1}{2\pi}\sum_{j=1}^{\mathscr{K}} \mathrm{e}^{-\imath k\xi_j}\sum_{l=0}^{d}\frac{a_{l,j}}{(\imath k)^{l+1}} + O\left(k^{-d-2}\right)$$

Algebraic Fourier inversion

Theorem (DB,YY 2011)

If f is piecewise-C^{d_1} where $d_1 \geq 2d+1$, then

$$\left|\Delta\xi_j\right| \sim n^{-d-2}$$
$$\left|\Delta a_{l,j}\right| \sim n^{-d-1-l}$$
$$\left|\Delta f\right| \sim n^{-d-1}.$$



2 Piecewise D-finite reconstruction

Piecewise D-finite model

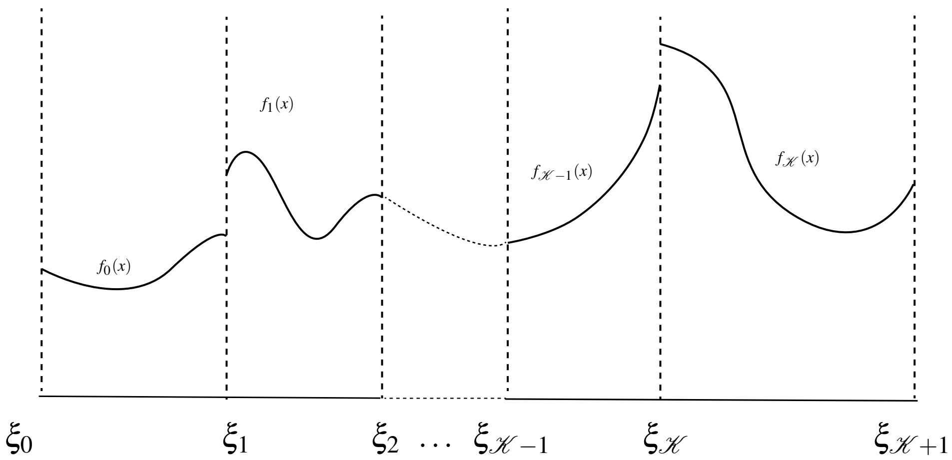


Figure: Piecewise D-finite model

Piecewise D-finite reconstruction

- Every piece satisfies $\mathfrak{D} f_i(x) \equiv 0$, $\mathfrak{D}$ - linear differential operator with polynomial coefficients

$$\mathfrak{D} = \sum_{j=0}^{n}\left(\sum_{i=0}^{d} a_{i,j} x^i\right)\frac{\mathrm{d}^j}{\mathrm{d}x^j} \quad (a_{ij} \in \mathbb{R})$$

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  - Initial values of f at $\{\xi_i\}$.
- Measurements: algebraic moments $m_k(f) = \int_a^b x^k f(x)\,\mathrm{d}x$.

Recurrence relation

$$\underbrace{\sum_{j=0}^{n} \sum_{i=0}^{d} a_{i,j} (-1)^j (i+k)_j m_{i-j+k}}_{\mathfrak{S}\{m_k\}} = \sum_{i=1}^{\mathscr{K}} \sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}$$

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- $c_{i,j}$ - homogeneous bilinear form depending on the values of $\{p_l(x)\}_{l=0}^n$ and the "jump function" $f(x^+) - f(x^-)$ with their derivatives up to order $n-1$ at the point $x = \xi_i$.

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$$\mathscr{E} = \prod_{i=1}^{\mathscr{K}} (\mathrm{E} - \xi_i \mathfrak{I})^n$$

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- Homogeneous recurrence relation for the moments:

$$\mathscr{E}\,\mathfrak{S}\{m_k\} = 0.$$

Reconstruction procedure

$$\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k} = \sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}$$

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1. Operator $\mathfrak{D}$ is known
   - solve the confluent Prony system directly (LHS is known) for $\{\xi_j, c_{i,j}\}$ and fully recover the function.

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   - solve the confluent Prony system directly (LHS is known) for $\{\xi_j, c_{i,j}\}$ and fully recover the function.
2. Operator $\mathfrak{D}$ unknown
   - Solve for coefficients of the difference operator $\mathscr{E}\mathfrak{S}$.
   - Factor out the common roots $\{\xi_j\}$ and the remaining factors $\{a_{i,j}\}$.
   - Finally solve the linear system for $\{c_{i,j}\}$ and fully recover the function.

Moment uniqueness and vanishing

How many moments are necessary for unique reconstruction?

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Definition

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Lemma

$\tau(\mathfrak{D},\mathscr{K}) \le \sigma(\mathfrak{D},2\mathscr{K})$.

Unbouded example

Legendre differential equation

$$\mathfrak{D}_m = \left(1-x^2\right)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + m(m+1)\mathfrak{I}.$$

- For $m \in \mathbb{N}$ solutions are the Legendre orthogonal polynomials $\{L_m\}$
- First $m-1$ moments of L_m are zero
- Conclusion: $\sigma(\mathfrak{D}_m) = m$
- $\Longrightarrow$ No uniform bound in terms of d,n for generic $\mathfrak{D}$!

Regular operators

Theorem (DB,GB 2012)

Assume that the leading coefficient of the operator $\mathfrak{D}$ does not vanish on any two consecutive jump points ξ_j, ξ_{j+1}. Then

$$\sigma(\mathfrak{D}) \le (\mathcal{K}+2)n + d - 1.$$

Proof outline

$$\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{d} a_{i,j}(-1)^j (i+k)_j m_{i-j+k}}_{\mathfrak{S}\{m_k\}} = \underbrace{\sum_{i=1}^{\mathcal{K}}\sum_{j=0}^{n-1} c_{i,j}(k)_j \xi_i^{k-j}}_{\varepsilon_k}$$

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3. $p_n(\xi_j) \neq 0 \Longrightarrow f(\xi_j) = f'(\xi_j) = \cdots = f^{(n-1)}(\xi_j) = 0.$

Resonant Fuchsian operators

Theorem (DB,GB 2012)

Let $\mathfrak{D}$ be of Fuchsian type, and consider moments in $[0,1]$. If $\mathfrak{D}$ has ***at most one negative integer characteristic exponent*** *at the point $z=0$, then*

$$\sigma(\mathfrak{D},0)=2n+d-1.$$

Proof outline

1. Write functional equation for the Mellin transform $M[f](s)=\int_0^1 t^s f(s)\,\mathrm{d}s$.
2. Check analytic continuation to $\Re s<0$.

Moment generating function

$$\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{d}a_{i,j}(-1)^j(i+k)_j m_{i-j+k}}_{\mu_k}=\underbrace{\sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1}c_{i,j}(k)_j\xi_i^{k-j}}_{\varepsilon_k}$$

$$I_g(z)=\sum_{k=0}^{\infty}\frac{m_k}{z^{k+1}}=\int_a^b\frac{f(t)\,\mathrm{d}t}{t-z}$$

Theorem

The Cauchy integral I_g satisfies at the neighborhood of ∞ the inhomogeneous ODE

$$\mathfrak{D}I_g(z)=R(z)$$

where $R(z)$ is the rational function whose Taylor coefficients are given by ε_k.

General Fuchsian operators

$$\underbrace{\sum_{j=0}^{n}\sum_{i=0}^{d}a_{i,j}(-1)^j(i+k)_j m_{i-j+k}}_{\mu_k=\mathfrak{S}\{m_k\}}=\underbrace{\sum_{i=1}^{\mathscr{K}}\sum_{j=0}^{n-1}c_{i,j}(k)_j\xi_i^{k-j}}_{\varepsilon_k}$$

Lemma

Let $\mathfrak{D}$ be a Fuchsian operator. Then the characteristic polynomial of $\mathfrak{D}$ at the point ∞ coincides with the leading coefficient of the difference operator $\mathfrak{S}$.

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Lemma

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Theorem

Let $\mathfrak{D}$ be a Fuchsian operator, and let $\lambda(\mathfrak{D})$ denote the largest positive integer root of its characteristic polynomial at the point ∞. Then

$$\sigma(\mathfrak{D},\mathscr{K})\leq\max\{\lambda(\mathfrak{D}),(\mathscr{K}+2)n+d-1\}.$$

3. D-finite planar domains

2D shapes from complex moments ([Gustafsson et al.(2000)Gustafsson, He, Milanfar, a

- Let $P \subset \mathbb{C}$ be a polygon with vertices $z_1, \dots, z_n$

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$$k(k-1)\mu_{k-2}(\chi_P) = \sum_{i=1}^{n} c_i z_i^k$$

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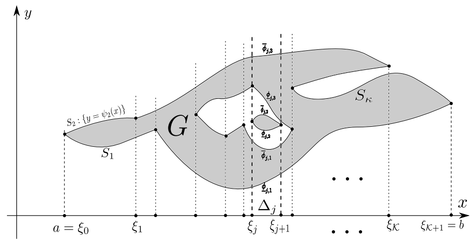
- Special case of *quadrature domains:* any analytic f (in particular $f(z) = z^k$) satisfies

$$\iint_\Omega f(x + \imath y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{i=1}^{n} \sum_{j=0}^{k_j - 1} c_{ij} f^{(j)}(z_i)$$

D-finite domains



$$m_{\alpha,\beta} = \int_a^b x^\alpha \Psi_\beta(x) = \sum_{j=0}^{\mathcal{K}} \int_{\Delta_j} x^\alpha \Psi_{\beta,j}(x)\,\mathrm{d}x$$

$$\Psi_{\beta,j} = \frac{1}{\beta+1} \sum_{l=1}^{s_j} \left\{ \overline{\phi}_{j,l}^{\beta+1}(x) - \underline{\phi}_{j,l}^{\beta+1}(x) \right\}$$

- Ψ_β are piecewise D-finite, are reconstructed via the 1D algorithm.
- $\{\phi_{j,l}\}$ are reconstructed pointwise via solving Prony-type system.

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