Geometric methods for global stability in the Ricker competition model

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ICDEA 2012
Applications of Difference Equations to Biology
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Global Stability Analysis

Program:

- Global analysis of discrete dynamical systems

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- Geometry of Critical sets

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- Global analysis of discrete dynamical systems
- Geometry of Critical sets
- Application to Discrete Planar Systems, in particular **Ricker Competition Model**

Classical RCM

- Ricker competition model (RCM)

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where $F(x, y) = (xe^{K-x-ay}, ye^{L-y-bx})$, where the parameters are positive numbers.

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  - A possible coexistence fixed point (x^*, y^*) (positive).

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We denote LC_{-1} to be the set of **singular points**, that is, the set of points where $J(p)$ vanishes.

Topological Singularity

Classification of Points in Domain: Let $p \in U$.

- If $p \in LC_{-1}$, then p is **singular**.

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Proof: Application of the Implicit Function Theorem.

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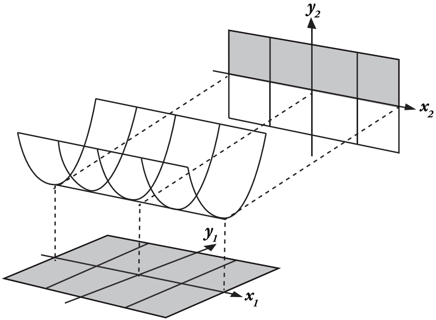
Definition

A point p is an **excellent** point of a good map F if it is a regular, a fold, or a cusp point. We say F is an **excellent** map, if it is excellent at every point.

Geometric Structure near a Fold

Theorem (Whitney, 1955)

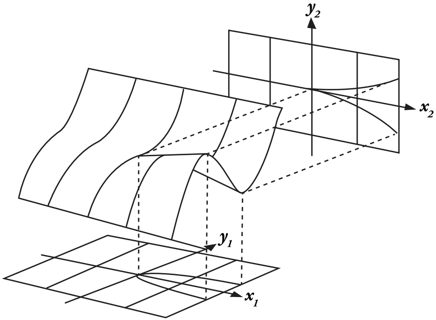
*Let $F : U \to \mathbb{R}^2$ be a smooth map. If $p \in U$ is a **fold** point, then there are smooth coordinates (x_1, y_1) and (x_2, y_2) around p and $F(p)$ such that F takes the form $x_2 = x_1$ and $y_2 = y_1^2$.*



Geometric Structure near a Cusp

Theorem (Whitney, 1955)

*Let $F : U \to \mathbb{R}^2$ be a smooth map. If $p \in U$ is a **cusp** point, then there are smooth coordinates (x_1, y_1) and (x_2, y_2) around p and $F(p)$ such that F takes the form $x_2 = x_1$ and $y_2 = y_1^3 - x_1 y_1$.*



Definition

Let $U \subseteq \mathbb{R}^2$ be a compact region, $p \in U$, and $v \in S^1$ (the unit circle). We say that p is **exposed in the direction of** v if there exists $\varepsilon > 0$ such that $p + tv \in U$ for $t \in (0, \varepsilon)$.

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Geometric and Topological Analysis

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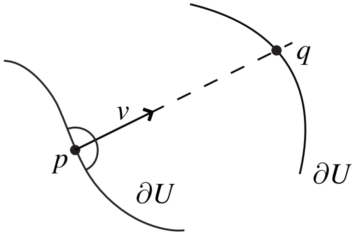
- If $p \in \partial U$ and exposed in direction v, $\exists t > 0$ s.t., $p + tv \in \partial U$.



- Critical curve of the RCM:

$$LC_{-1} = \left\{ (x, y) \in \mathbb{R}^2_+ : y = \frac{1 - x}{1 - (1 - ab)x}, x \neq \frac{1}{1 - ab} \right\}. \tag{1}$$

Geometry and Topology of the Ricker Map

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- LC_{-1} has two connected components: LC^1_{-1} and LC^2_{-1}.

Geometry and Topology of the Ricker Map

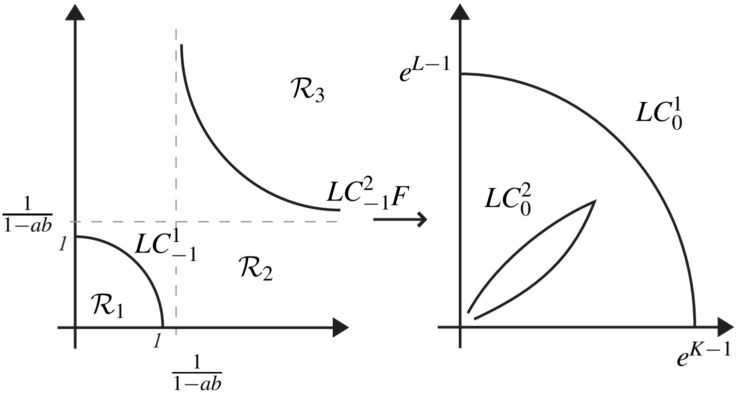

Figure : The subdivision of the Domain of the Ricker competition map by the critical curves LC^1_{-1} and LC^2_{-1} and their respective images LC_0^1 and LC_0^2 showing the typical geometry.

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Let F be the Ricker map. The following are true.

(i) *The x-axis and y-axis are invariant sets.*

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In particular, F has a continuous extension to the one-point compactification and $F(\mathbb{R}^2_+)$ is ***compact****.*

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- This follows because the image of regular points cannot be on the boundary.
- Any new boundary points, must be images of critical points.

Geometry and Topology of the Ricker Map

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- Let $(F \circ \varphi_1)(t) = (\alpha_1(t), \alpha_2(t)) = \alpha(t)$, we must show that

  $\alpha_1'(t)$ and $\alpha_2'(t)$ do not vanish for $t \in [0,1]$.

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where $\rho_1(t) \neq 0$, $\rho_2(t) \neq 0$ for $t \in [0,1]$ and

$$h(t) = (ab-1)^2\, t^3 + \left(-3 - a^2b^2 + 4\,ab\right) t^2 + \left(-2\,ab + 3 - a^2b\right) t - 1$$

$$h(0), h(1) < 0$$

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Goal: Show that $h(t) \neq 0$

Ricker Map is Excellent

Critical points in LC^1_{-1}: Some Geometric Considerations

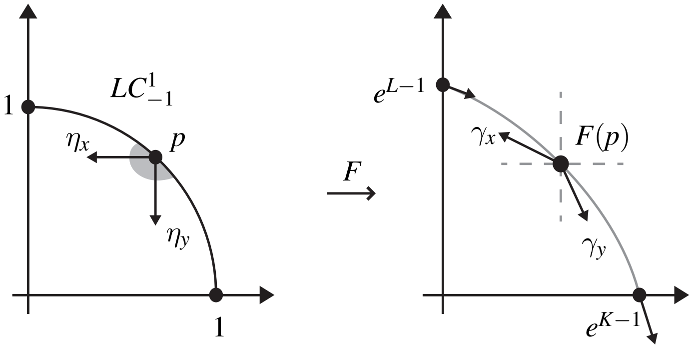

Figure : General directios of rays parallel to axes.

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Suppose it has a root $t_0 \in (0, 1)$

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Look at the behavior of $\alpha(t_0) = q_0$.

Ricker Map is Excellent

Critical points in LC^1_{-1}

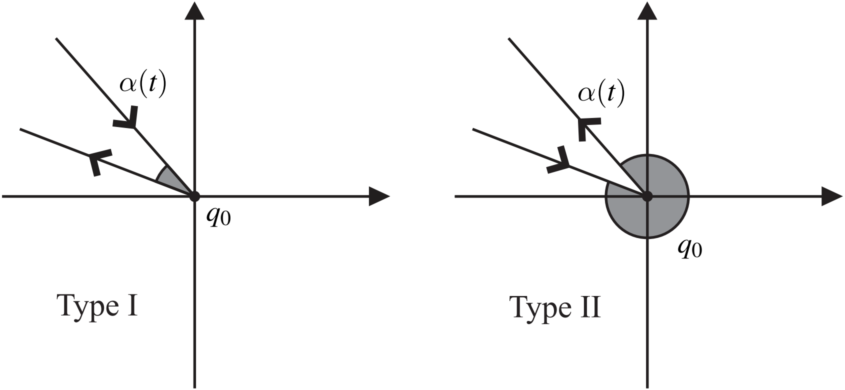

Figure : Possible local behaviors of the curve α at q_0.

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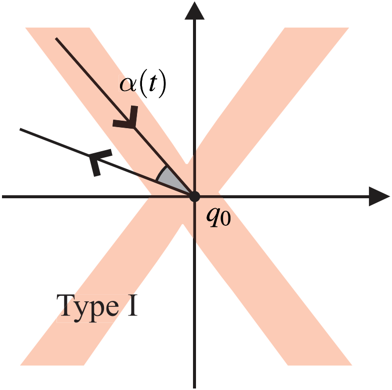
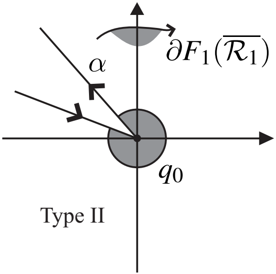


Figure : Only Type II allows for possible locations of γ_x and γ_y.

Ricker Map is Excellent

Critical points in LC^1_{-1}: Behavior of α is Type II

- q_0 is exposed $\rightarrow h(t)$ has another root t_1.



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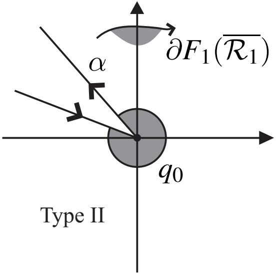

- Suppose t_1 has multiplicity one.

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Ricker Map is Excellent

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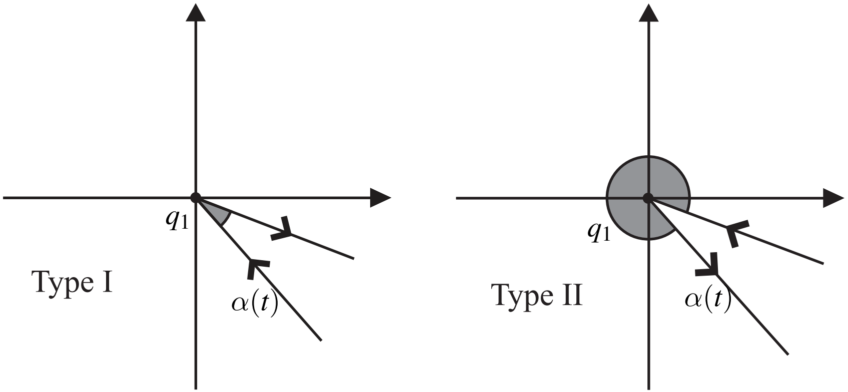

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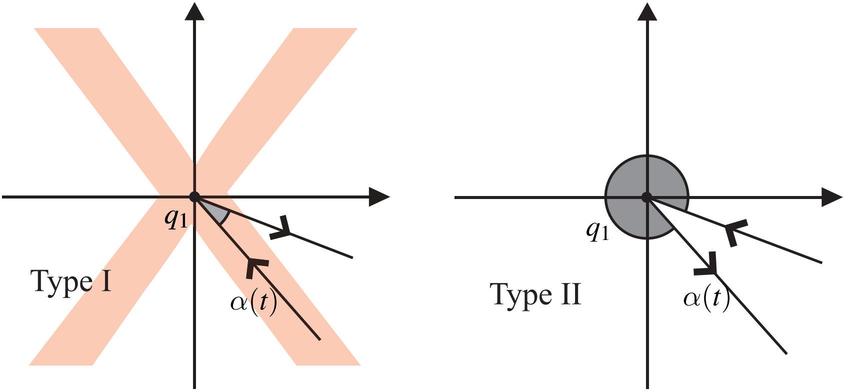
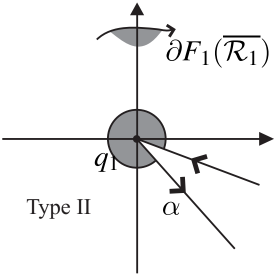


Figure : Only Type II allows for possible locations of γ_x and γ_y.

Ricker Map is Excellent

Critical points in LC^1_{-1}: Behavior of α is Type II

- q_1 is exposed $\rightarrow h(t)$ must change sign at least twice. Contradiction.



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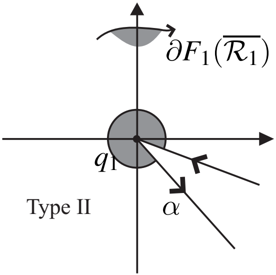

- If t_1 has multiplicity two, $h(t)$ would have to change sign at least two more times, contradiction.

(2) t_0 has multiplicity 2.

Algebraic proof: Root of $h'(t)$ cannot be a root of $h(t)$.

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(3) t_0 has multiplicity 3.

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Ricker Map is Excellent

Critical points in LC^1_{-1}

- All points of LC^1_{-1} are **folds.**
- $\alpha_1'(t)$ and $\alpha_2'(t)$ do not change sign.

- Parametrization of LC_{-1}^2 given by a curve φ_2 as the map $\varphi_2 : (0,1) \to \mathbb{R}^2$ with

$$\varphi_2(t) = \left(\frac{1}{(1-ab)\,t}, \frac{(1-ab)t-1}{(1-ab)(1-t)} \right)$$

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- From Lemma:

$$\lim_{t\to 0} \beta(t) = \lim_{t\to 1} \beta(t) = (0,0).$$

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$$h(t) = (1 - ab)t^3 + (2ab + a^2 b - 3)t^2 + (3 - ab)t - 1 \ .$$

$$h(0) = -1 < 0 \text{ and } h(1) = a^2 b > 0$$

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Ricker Map is Excellent

Critical points in LC^2_{-1}

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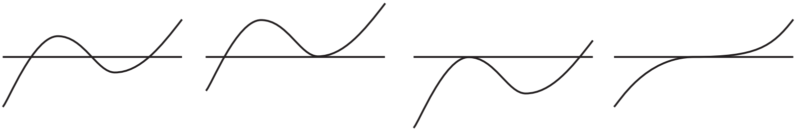
Suppose this is not the case.

- t_0 has mult. one and two roots of mult. one.
- t_0 has mult. one and one root of mult. two.
- t_0 has mult. two and one root of mult. one.
- t_0 has mult. three.



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Conclusion

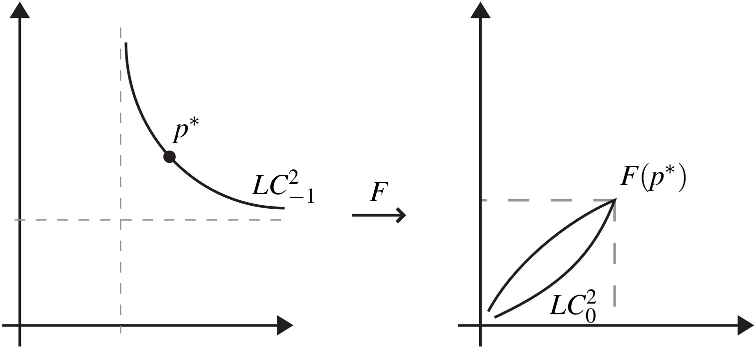
The Ricker Competion Model is Excellent.

Geometry and Topology of the Ricker Map

Corollary

There is one cusp point $p^* \in LC^2_{-1}$ *and*

$$LC^2_0 \subseteq [0, f_1(p^*)] \times [0, f_2(p^*)]$$



Theorem

$F|_{\mathcal{R}_1} : \mathcal{R}_1 \to F(\mathcal{R}_1)$ *is a homeomorphism.*

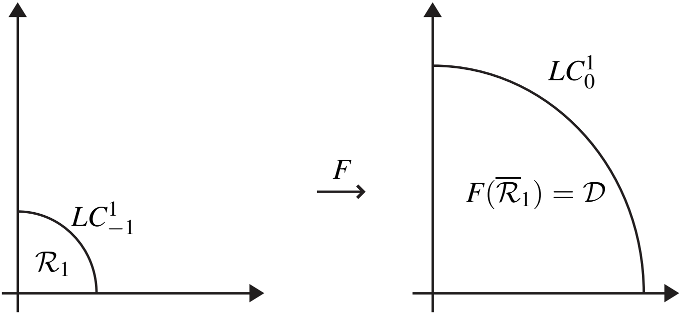


Figure : The image of $\mathcal{R}_1$ is the region $\mathcal{D}$.

A general Topological Result

Theorem (Kestelman, 1971)

Let $F : K \to \mathbb{R}^n$ be an open and locally injective map. If $K \subseteq \mathbb{R}^n$ is a compact set, ∂K is connected, and $F|_{\partial K}$ is injective, then F is injective.

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Use the fold structure to show F is injective on $\partial \mathcal{R}_1$.

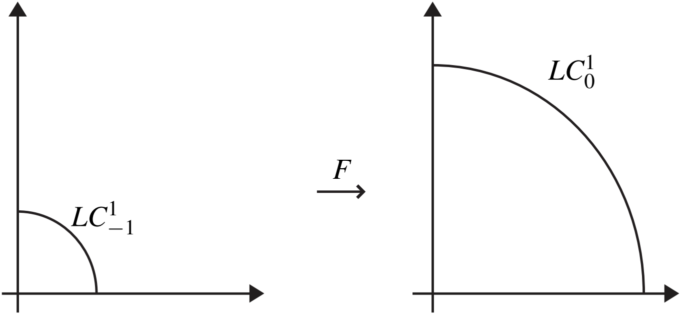


Figure : F is a local diffeomorphism on $int\,(\mathcal{R}_1)$.

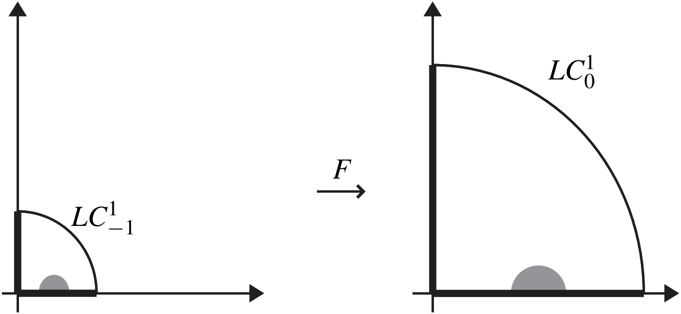


Figure : Axes are invariant and locally injective.

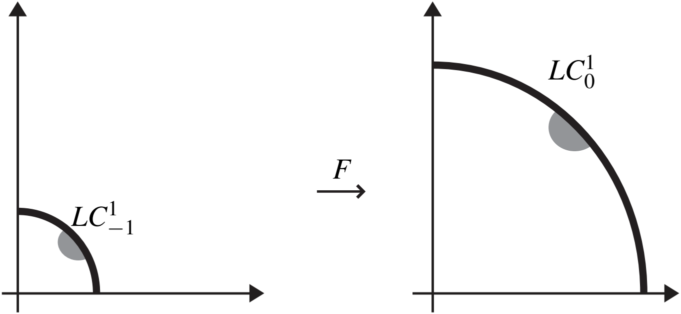
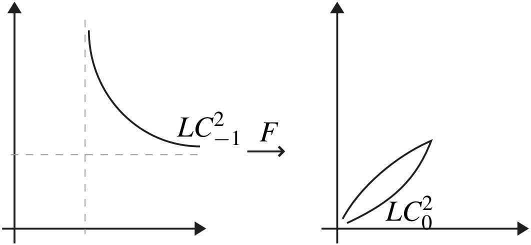


Figure : From the fold structure, F is injective on the boundary.

Geometry and Topology of the Ricker Map

Theorem

$F|_{\mathcal{R}_3} : \mathcal{R}_3 \to F(\mathcal{R}_3)$ *is a homeomorphism.*



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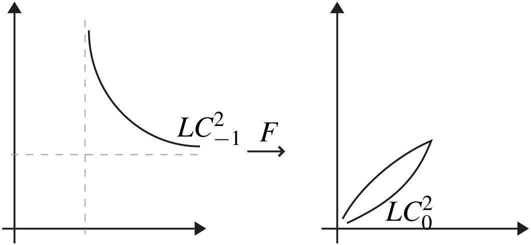
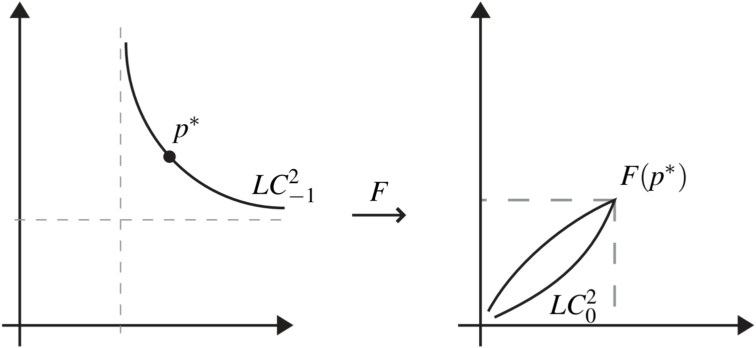


Proof: One compactification and local injectivity at the boundary.



p^*
LC^2_{-1}
F
$F(p^*)$
LC^2_0

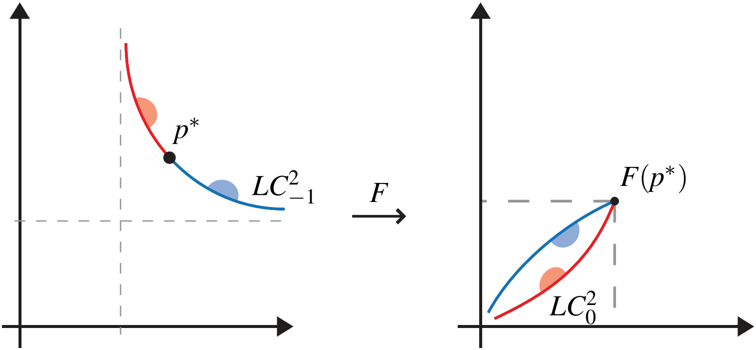


Figure : Except for the cusp p^*, all points are folds.

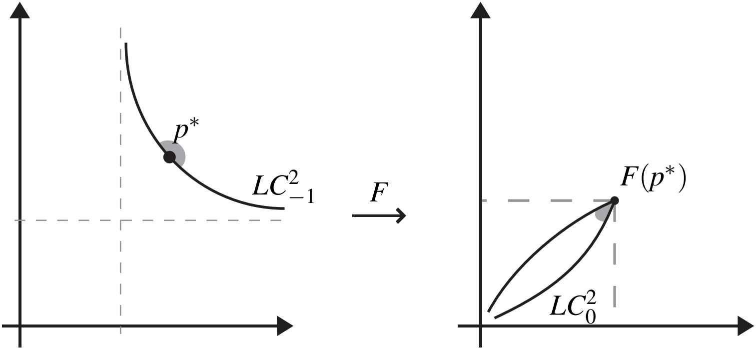


Figure : At the cusp, the local structure theorem yeilds local injectivity.

Theorem

$F(\mathbb{R}^2_+) = \mathcal{D}$, *that is,* $\mathcal{D}$ *is an invariant set.*

Geometry and Topology of the Ricker Map

Theorem

$F(\mathbb{R}^2_+) = \mathcal{D}$, that is, $\mathcal{D}$ is an invariant set.

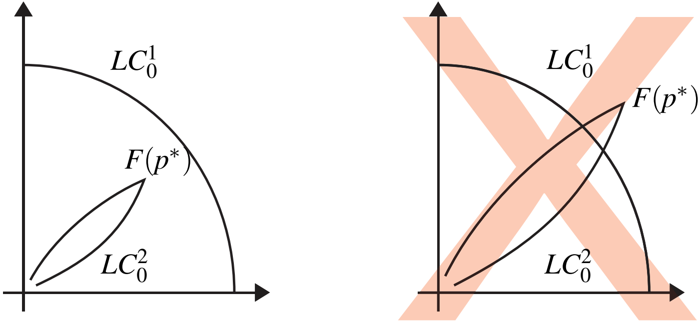


Figure : The only possible location for the image of the cusp is inside the region $\mathcal{D}$.

The image of the cusp is exposed.

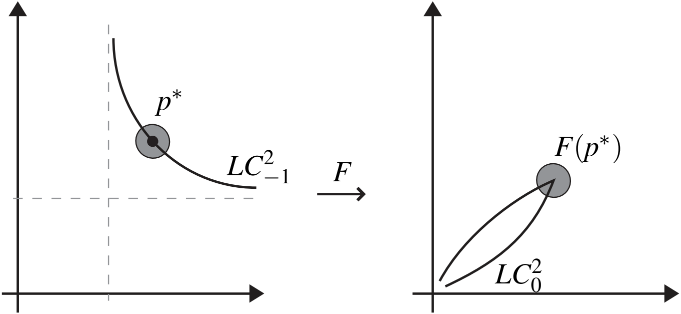
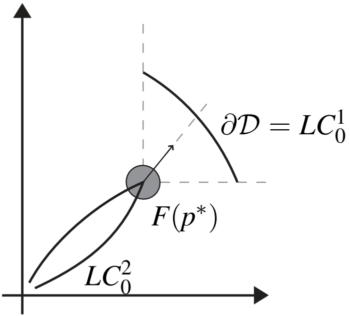
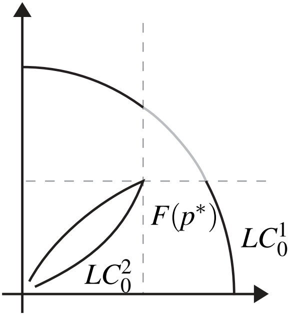


Figure : The cusp point in an interior point of the image, hence exposed.

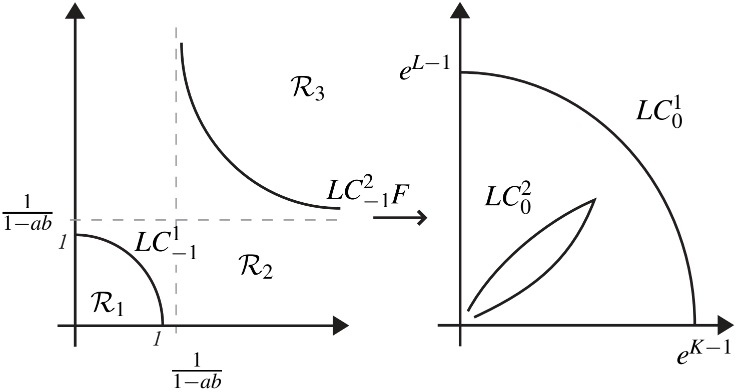
In any direction in the first quadrant, a ray must intersect $\partial\mathcal{D}$.



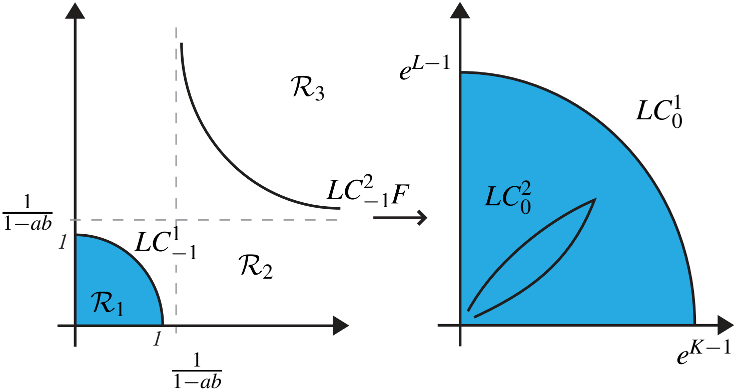
LC_0^1 is above and to the right of LC_0^2.



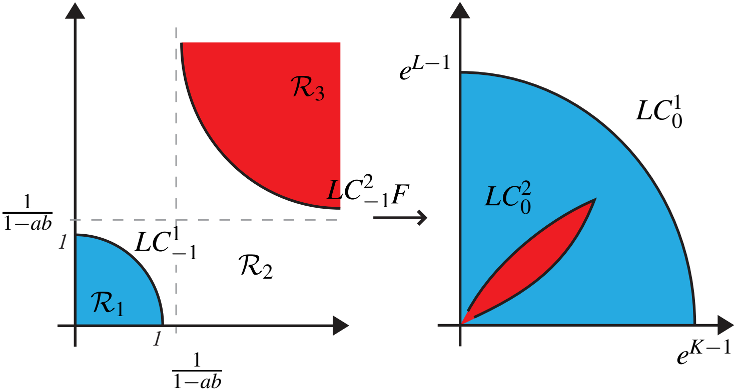
Final Geometric Conclusions



Final Geometric Conclusions



Final Geometric Conclusions



THANK YOU.