Geometric methods for global stability in the Ricker competition model

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Applications of Difference Equations to Biology
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Global Stability Analysis

Program:

- Global analysis of discrete dynamical systems

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- Global analysis of discrete dynamical systems
- Geometry of Critical sets

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Program:

- Global analysis of discrete dynamical systems
- Geometry of Critical sets
- Application to Discrete Planar Systems, in particular **Ricker Competition Model**

Classical RCM

- Ricker competition model (RCM)

$$(x_{n+1}, y_{n+1}) = F(x_n, y_n), n \in \mathbb{Z}^+$$

where $F(x, y) = (xe^{K-x-ay}, ye^{L-y-bx})$, where the parameters are positive numbers.

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  - one extinction fixed point $(0, 0)$

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  - A possible coexistence fixed point (x^*, y^*) (positive).

Singularity theory
Classical work of Whitney

Let U be an open region in $\mathbb{R}^2$ and $F : U \to \mathbb{R}^2$ a smooth map.

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Definition

We say F is **good** at $p \in U$ if either $J(p) \neq 0$ or $\nabla J(p) \neq 0$ and F is **good**, if it is good at every point.

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Definition

We denote LC_{-1} to be the set of **singular points**, that is, the set of points where $J(p)$ vanishes.

Topological Singularity

Classification of Points in Domain: Let $p \in U$.

- If $p \in LC_{-1}$, then p is **singular**.

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The singular points of a good map form smooth curves, called the ***critical curve****, denoted by* LC_{-1}*.*

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Proof: Application of the Implicit Function Theorem.

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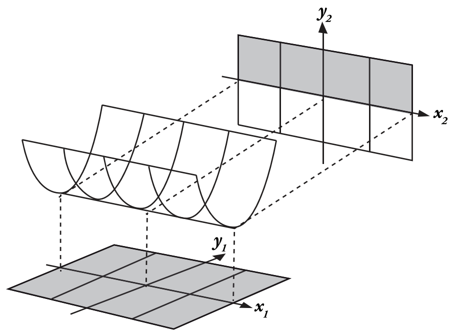
Definition

A point p is an **excellent** point of a good map F if it is a regular, a fold, or a cusp point. We say F is an **excellent** map, if it is excellent at every point.

Geometric Structure near a Fold

Theorem (Whitney, 1955)

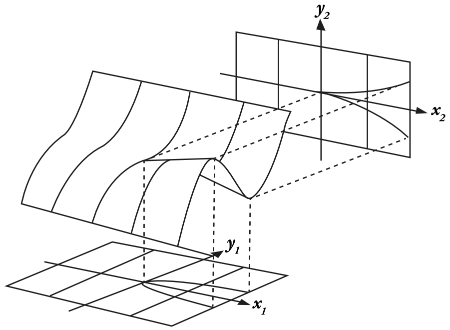
Let $F : U \to \mathbb{R}^2$ be a smooth map. If $p \in U$ is a ***fold*** *point, then there are smooth coordinates (x_1, y_1) and (x_2, y_2) around p and $F(p)$ such that F takes the form $x_2 = x_1$ and $y_2 = y_1^2$.*



Geometric Structure near a Cusp

Theorem (Whitney, 1955)

*Let $F : U \to \mathbb{R}^2$ be a smooth map. If $p \in U$ is a **cusp** point, then there are smooth coordinates (x_1, y_1) and (x_2, y_2) around p and $F(p)$ such that F takes the form $x_2 = x_1$ and $y_2 = y_1^3 - x_1 y_1$.*



Geometric and Topological Analysis

Definition

Let $U \subseteq \mathbb{R}^2$ be a compact region, $p \in U$, and $v \in S^1$ (the unit circle). We say that p is **exposed in the direction of** v if there exists $\varepsilon > 0$ such that $p + tv \in U$ for $t \in (0, \varepsilon)$.

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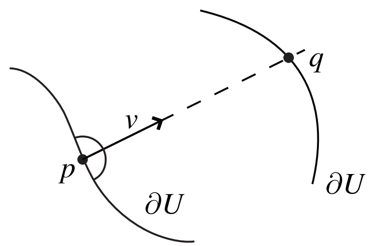
- For all $p \in int(U)$, p is exposed in every direction.
- If $p \in \partial U$ and exposed in direction v, $\exists t > 0$ s.t., $p + tv \in \partial U$.



Geometry and Topology of the Ricker Map

- Critical curve of the RCM:

$$LC_{-1} = \left\{ (x,y) \in \mathbb{R}^2_+ : y = \frac{1-x}{1-(1-ab)x}, x \neq \frac{1}{1-ab} \right\}. \tag{1}$$

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- LC_{-1} has two connected components: LC^1_{-1} and LC^2_{-1}.

Geometry and Topology of the Ricker Map

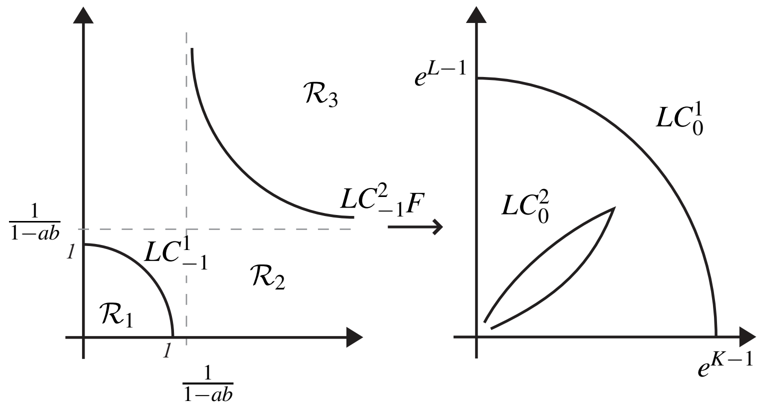

Figure : The subdivision of the Domain of the Ricker competition map by the critical curves LC^1_{-1} and LC^2_{-1} and their respective images LC^1_0 and LC^2_0 showing the typical geometry.

Geometry and Topology of the Ricker Map

Proposition

Let F be the Ricker map. The following are true.

(i) *The x-axis and y-axis are invariant sets.*

(ii) $\lim_{\|p\| \to \infty} F(p) = (0,0).$

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*In particular, F has a continuous extension to the one-point compactification and $F(\mathbb{R}^2_+)$ is **compact**.*

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- This follows because the image of regular points cannot be on the boundary.
- Any new boundary points, must be images of critical points.

Geometry and Topology of the Ricker Map

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The Ricker map is excellent.

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  $\alpha_1'(t)$ and $\alpha_2'(t)$ do not vanish for $t \in [0,1]$.

Ricker Map is Excellent
Critical points in LC^1_{-1}

Direct Computation:

$$\alpha'(t) = \left(\alpha_1'(t), \alpha_2'(t)\right) = \left(\rho_1(t)h(t), \rho_2(t)h(t)\right),$$

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where $\rho_1(t) \neq 0,\ \rho_2(t) \neq 0$ for $t \in [0,1]$ and

$$h(t) = (ab-1)^2\, t^3 + \left(-3 - a^2b^2 + 4\,ab\right) t^2 + \left(-2\,ab + 3 - a^2b\right) t - 1$$

$$h(0), h(1) < 0$$

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Goal: Show that $h(t) \neq 0$

Ricker Map is Excellent
Critical points in LC^1_{-1}: Some Geometric Considerations

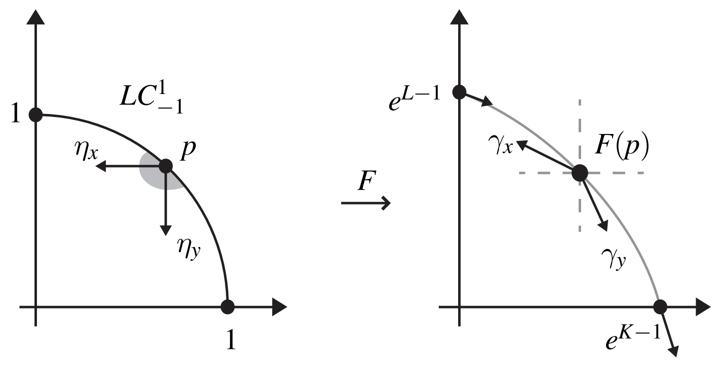

Figure : General directios of rays parallel to axes.

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The cubic polynomial $h(t)$ does not vanish on the interval $[0, 1]$

Suppose it has a root $t_0 \in (0, 1)$

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(1) t_0 has multiplicity 1.

Look at the behavior of $\alpha(t_0) = q_0$.

Ricker Map is Excellent

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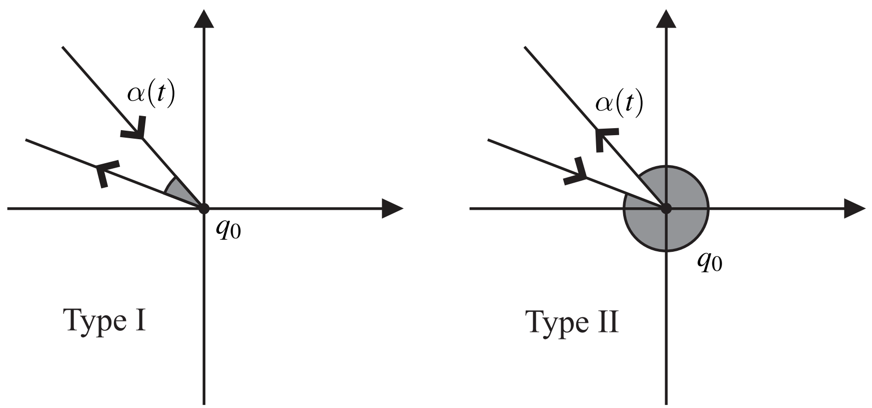

Figure : Possible local behaviors of the curve α at q_0.

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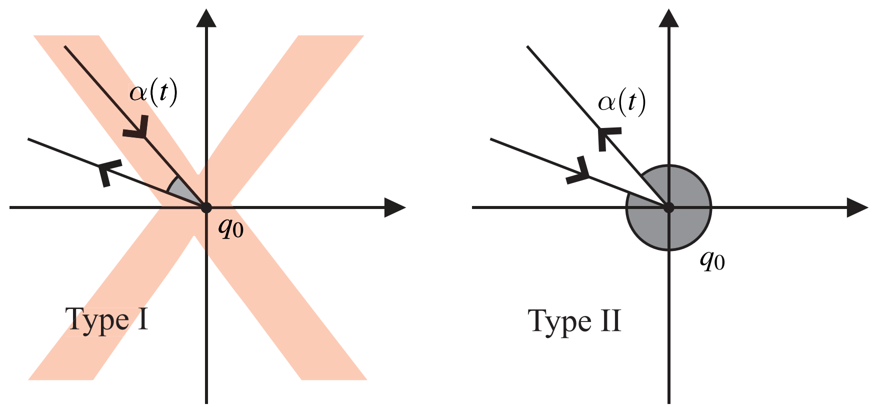
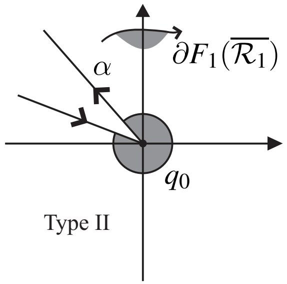


Figure : Only Type II allows for possible locations of γ_x and γ_y.

Ricker Map is Excellent

Critical points in LC^1_{-1}: Behavior of α is Type II

- q_0 is exposed $\rightarrow h(t)$ has another root t_1.



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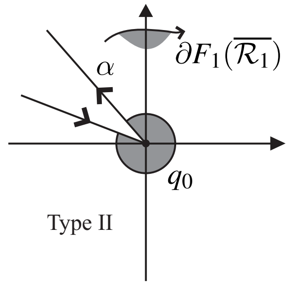
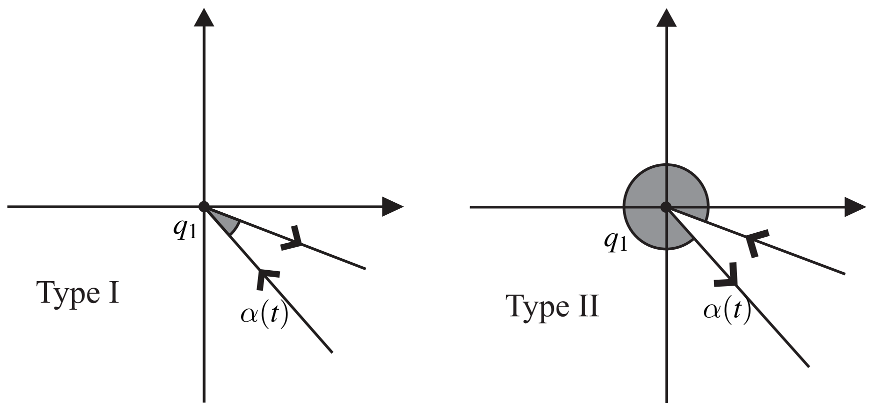


- Suppose t_1 has multiplicity one.

  Look at the behavior of $\alpha(t_1) = q_1$.

Ricker Map is Excellent

Critical points in LC^1_{-1}



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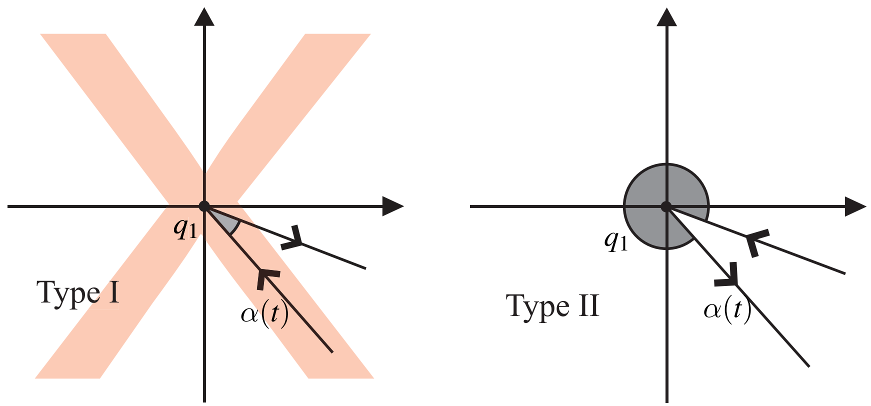

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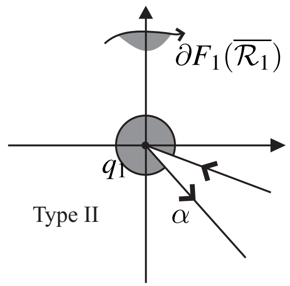
Ricker Map is Excellent

Critical points in LC^1_{-1}: Behavior of α is Type II

- q_1 is exposed $\rightarrow h(t)$ must change sign at least twice. Contradiction.



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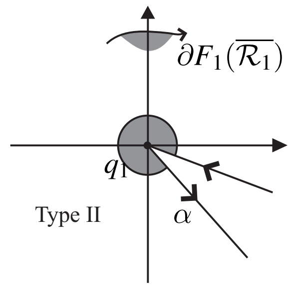

- If t_1 has multiplicity two, $h(t)$ would have to change sign at least two more times, contradiction.

Ricker Map is Excellent

Critical points in LC^1_{-1}

(2) t_0 has multiplicity 2.

Algebraic proof: Root of $h'(t)$ cannot be a root of $h(t)$.

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(3) t_0 has multiplicity 3.

$h(t)$ would have to change sign at least one more time.

Ricker Map is Excellent
Critical points in LC^1_{-1}

- All points of LC^1_{-1} are **folds**.
- $\alpha'_1(t)$ and $\alpha'_2(t)$ do not change sign.

Ricker Map is Excellent
Critical points in LC^2_{-1}

- Parametrization of LC^2_{-1} given by a curve φ_2 as the map $\varphi_2 : (0,1) \to \mathbb{R}^2$ with

$$\varphi_2(t) = \left(\frac{1}{(1-ab)\,t}, \frac{(1-ab)t-1}{(1-ab)(1-t)} \right)$$

Let $F \circ \varphi_2(t) = (\beta_1(t), \beta_2(t)) = \beta(t)$.

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Let $F \circ \varphi_2(t) = (\beta_1(t), \beta_2(t)) = \beta(t)$.
- From Lemma:

$$\lim_{t\to 0} \beta(t) = \lim_{t\to 1} \beta(t) = (0,0).$$

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$$h(t) = (1-ab)t^3 + (2ab + a^2b - 3)t^2 + (3-ab)t - 1 \; .$$

$$h(0) = -1 < 0 \text{ and } h(1) = a^2b > 0$$

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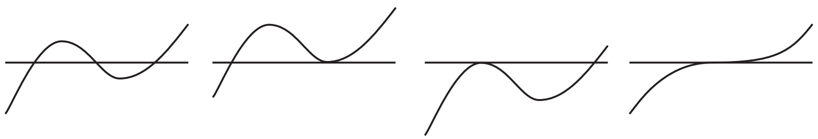
Suppose this is not the case.

- t_0 has mult. one and two roots of mult. one.
- t_0 has mult. one and one root of mult. two.
- t_0 has mult. two and one root of mult. one.
- t_0 has mult. three.



Ricker Map is Excellent
Critical points in LC^2_{-1}

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Algebraic manipulation leads to a contradiction.

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Conclusion

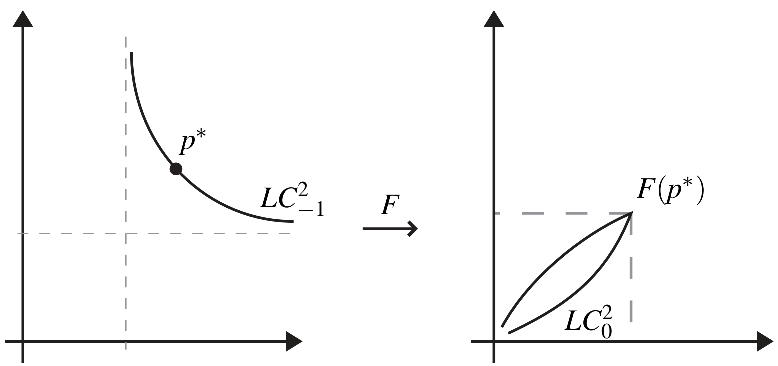
The Ricker Competion Model is Excellent.

Geometry and Topology of the Ricker Map

Corollary

There is one cusp point $p^* \in LC^2_{-1}$ and

$$LC^2_0 \subseteq [0, f_1(p^*)] \times [0, f_2(p^*)]$$



Geometry and Topology of the Ricker Map

Theorem

$F|_{\mathcal{R}_1} : \mathcal{R}_1 \to F(\mathcal{R}_1)$ is a homeomorphism.

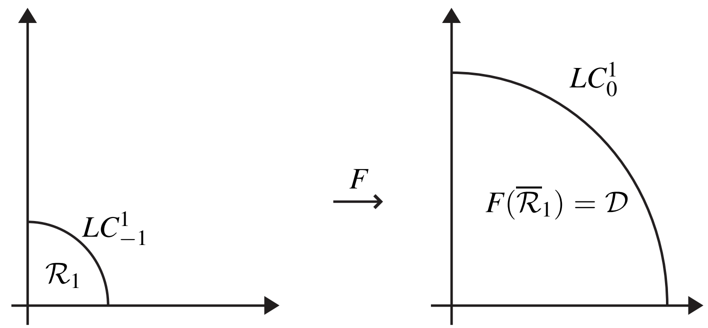


Figure : The image of $\mathcal{R}_1$ is the region $\mathcal{D}$.

Geometry and Topology of the Ricker Map

A general Topological Result

Theorem (Kestelman, 1971)

Let $F : K \to \mathbb{R}^n$ be an open and locally injective map. If $K \subseteq \mathbb{R}^n$ is a compact set, ∂K is connected, and $F|_{\partial K}$ is injective, then F is injective.

Geometry and Topology of the Ricker Map

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Use the fold structure to show F is injective on $\partial \mathcal{R}_1$.

Geometry and Topology of the Ricker Map

Local Injectivity at the boundary

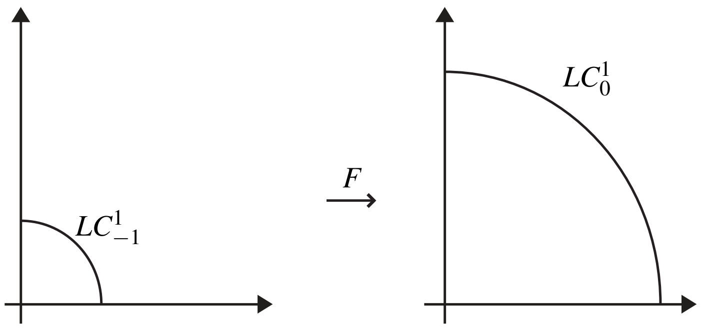

Figure : F is a local diffeomorphism on $int\,(\mathcal{R}_1)$.

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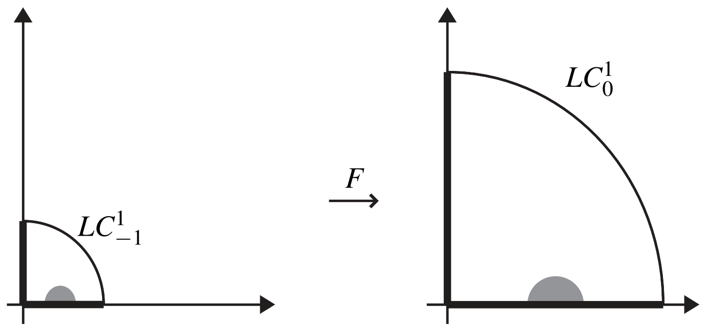


Figure : Axes are invariant and locally injective.

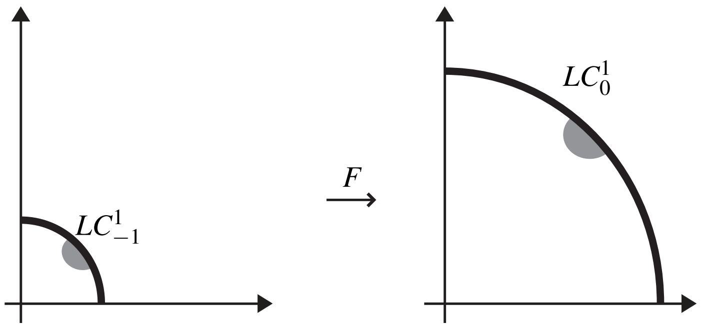
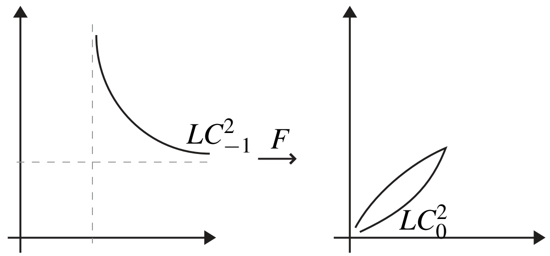


Figure : From the fold structure, F is injective on the boundary.

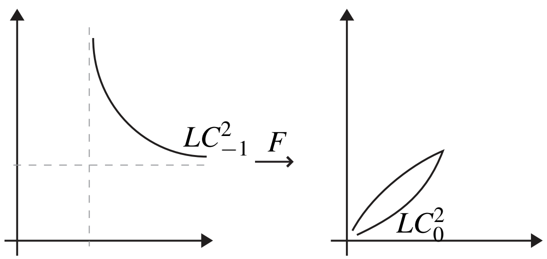
Theorem

$F|_{\mathcal{R}_3} : \mathcal{R}_3 \to F(\mathcal{R}_3)$ *is a homeomorphism.*

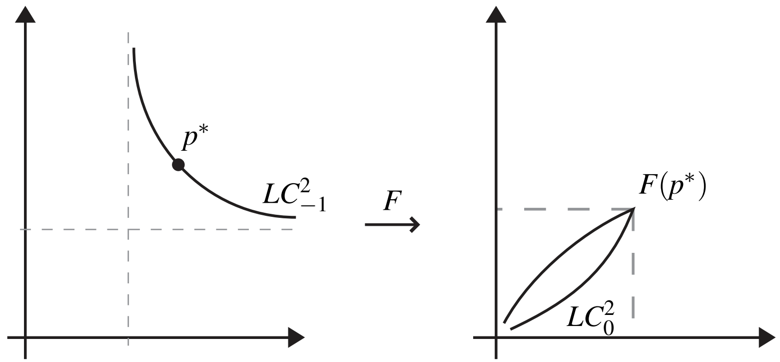


Theorem

$F|_{\mathcal{R}_3} : \mathcal{R}_3 \to F(\mathcal{R}_3)$ *is a homeomorphism.*



Proof: One compactification and local injectivity at the boundary.



Geometry and Topology of the Ricker Map
Local Injectivity at the boundary

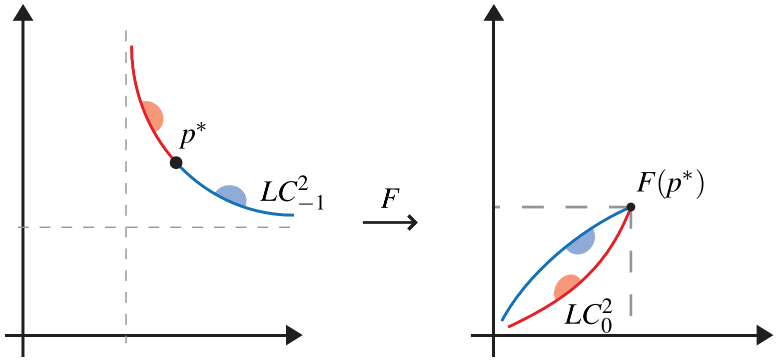


Figure : Except for the cusp p^*, all points are folds.

Geometry and Topology of the Ricker Map
Local Injectivity at the boundary

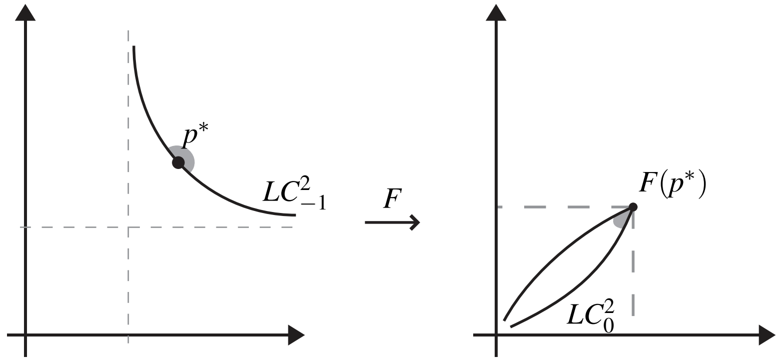


Figure : At the cusp, the local structure theorem yeilds local injectivity.

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Geometry and Topology of the Ricker Map

Theorem

$F(\mathbb{R}^2_+) = \mathcal{D}$, that is, $\mathcal{D}$ is an invariant set.

Geometry and Topology of the Ricker Map

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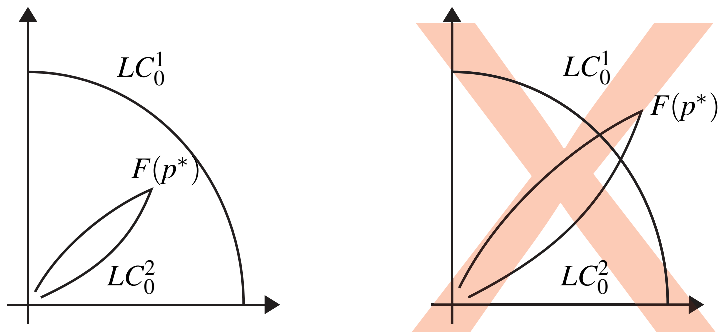

Figure : The only possible location for the image of the cusp is inside the region $\mathcal{D}$.

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Geometry and Topology of the Ricker Map
Proof of Main Geometric Result

The image of the cusp is exposed.

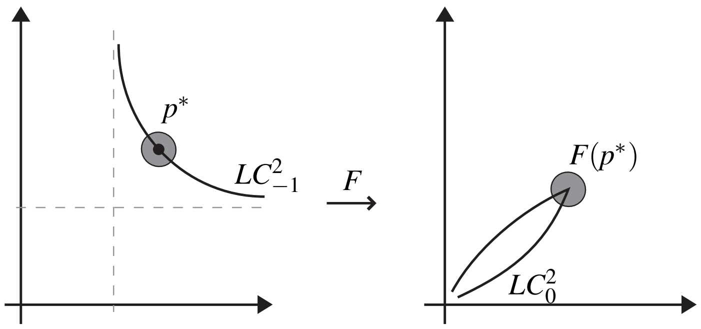
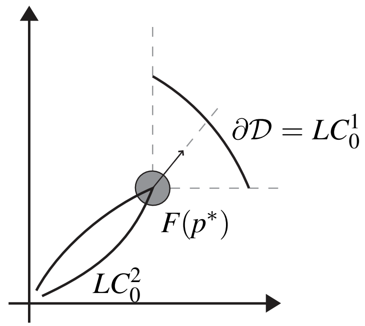


Figure : The cusp point in an interior point of the image, hence exposed.

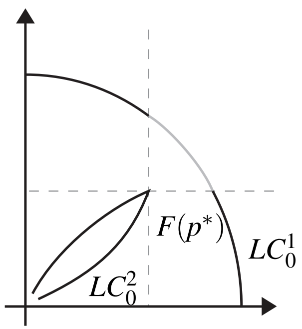
Geometry and Topology of the Ricker Map
Proof of Main Geometric Result

In any direction in the first quadrant, a ray must intersect $\partial\mathcal{D}$.

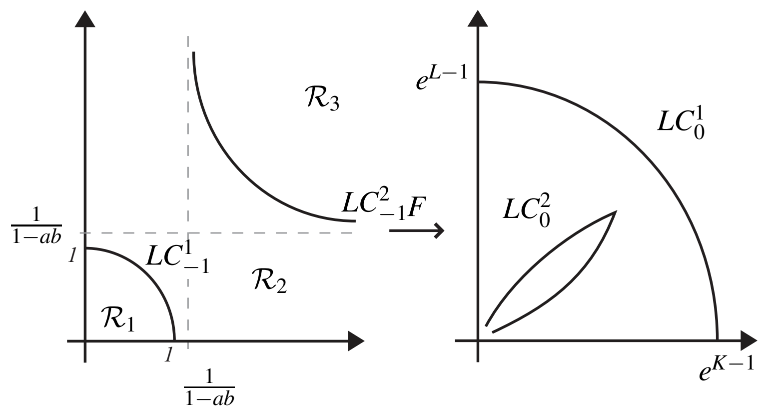


Geometry and Topology of the Ricker Map
Proof of Main Geometric Result

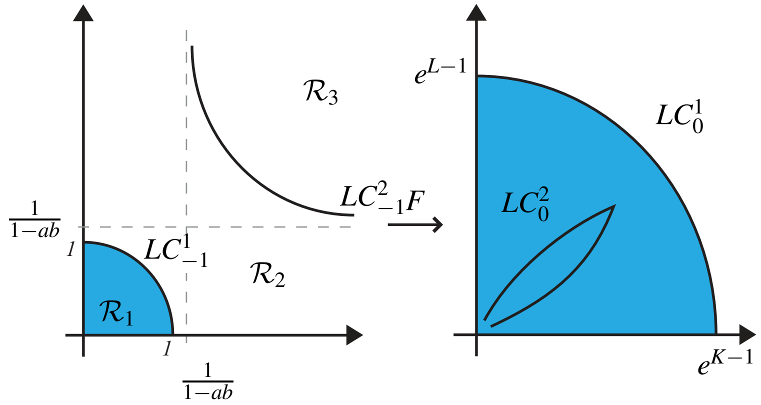
LC^1_0 is above and to the right of LC^2_0.



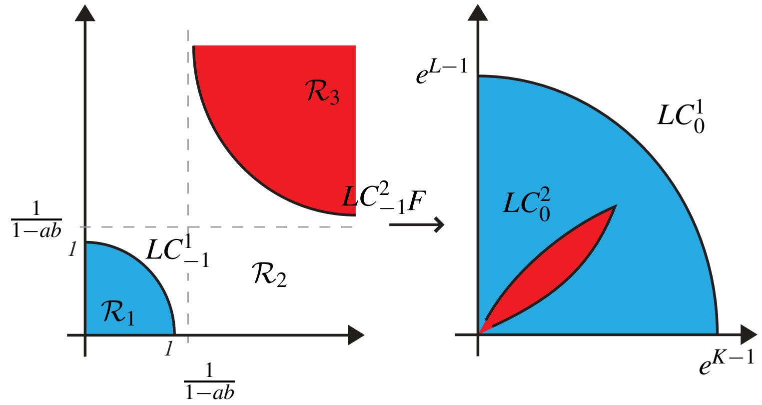
Final Geometric Conclusions



Final Geometric Conclusions



Final Geometric Conclusions



THANK YOU.