Different approaches to the global periodicity problem

ARMENGOL GASULL

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193–Bellaterra, Barcelona SPAIN

E-mail address: `gasull@mat.uab.cat`
URL: `http://www.gsd.uab.cat/personal/agasull`

Let F be a real or complex n-dimensional map. It is said that F is *globally periodic* if there exists some $p \in \mathbb{N}^+$ such that $F^p(x) = x$ for all x for which F^p is well defined, where $F^k = F \circ F^{k-1}, k \geq 2$. The minimal p satisfying this property is called the *period* of F. One problem of current interest is the following: Given a m-dimensional parametric family of maps, say F_λ, determine all the values of λ such that F_λ is globally periodic, together with their corresponding periods, see for instance [1, 7, 10]. Perhaps the paradigmatic example is the 1-parameter family of maps corresponding to the Lyness recurrences

$$F_\lambda(x, y) = \Big(y, \frac{\lambda + y}{x}\Big),$$

for which the solution is well known. The globally periodic cases are F_1 and F_0 and the corresponding periods 5 and 6.

Some people of our research group approaches to the above problem from different points of view. The aim of this talk is to show the techniques that we are using and some of the results that we have obtained. More specifically we approach to it using the following tools:

- Proving its equivalence with the complete integrability of the dynamical system induced by the map F, see [3].
- Using the local linearization given by the Montgomery-Bochner Theorem for characterizing the periodic cases in same families of maps, see [5, 6, 11].
- Using the theory of normal forms, see [4].

- Using properties of the so called *vanishing sums* that are polynomial identities with integer coefficients involving only roots of the unity, see [2, 9].

We will pay special attention to the case where F_λ is a family of rational maps, depending also rationally of the multiparameter λ. Other cases on which we will concentrate are the maps of the form

$$F(x_1,\ldots,x_k)=(x_2,x_3,\ldots,x_k,f(x_1,x_2,\ldots,x_k))\,,$$

that correspond to autonomous k-th order difference equations, and the maps coming from non-autonomous periodic difference equations.

As an example of this last situation consider the 2-periodic Lyness recurrence

$$x_{n+2}=\frac{a_n+x_{n+1}}{x_n},\qquad \text{where}\qquad a_n=\begin{cases} a & \text{for} \quad n=2\ell+1,\\ b & \text{for} \quad n=2\ell,\end{cases}$$

with $\lambda=(a,b)\in\mathbb{C}^2$. Studying the family of maps

$$F_{a,b}(x,y)=\left(\frac{a+y}{x},\frac{a+bx+y}{xy}\right),$$

which describes its behavior, we prove that the only globally periodic recurrences with $a\neq b$ are given by $a=a^*:=(-1\pm i\sqrt{3})/2$ and $b=b^*:=\overline{a^*}=1/a^*$ and are 10-periodic. In fact $F^5_{a^*,b^*}=\mathrm{Id}$. This result can also be obtained following the approach developed in [12] that gives all the globally periodic QRT-maps, see also [8].

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