Dynamical Classification of a family of Birational Maps via Dynamical Degree

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Given complex numbers α_i, γ_i and $\delta_i, i = 0, ..., 2$, consider the family of birational maps $f : \mathbf{C}^2 \to \mathbf{C}^2$ of the following form

$$f(x, y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y, \frac{\gamma_0 + \gamma_1 x + \gamma_2 y}{\delta_0 + \delta_1 x + \delta_2 y}\right). \qquad (1)$$

We consider the imbedding $(x, y) \mapsto (1, x_1, x_2) \in \mathbf{P}^2$ into projective space and consider the induced map $F : \mathbf{P}^2 \to \mathbf{P}^2$ given by

$$F[x_0, x_1, x_2] = [x_0(\delta \cdot x), (\alpha \cdot x)(\delta \cdot x), x_0(\gamma \cdot x)],$$

where $\alpha \cdot x = \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2$. To determine the behavior of iterates, $F^n = F \circ \cdots \circ F$, we will study their degree growth rate particularly we are interested in the quantity

$$D(\alpha, \gamma, \delta) = \lim_{n\to\infty} (\deg(F^n))^{\frac{1}{n}},$$

which is known as the *dynamical degree* in [1] and the logarithm of this quantity has been called the *algebraic entropy* in [6] and [2].

In order to classify our family (1) we first make an identification of two existing cases in (1). For all the values of parameters for which the determinants $(\gamma\delta)_{12}$ and $(\alpha\delta)_{12}$ are zero we call it a *degenerate case* and the values of parameters for which these determinants are non zero we say that the family (1) lies in the *non degenerate case*. In general the family (1) has dynamical degree $D = 2$. The main interest is to identify the possible subcases of (1) for all the parameter values. By the help of the associated characteristic polynomial of each subcase/subfamily we are able to know their growth rate. Therefore we find the dynamical degree D for all the subcases in order to locate the subfamilies with entropy *zero* and the ones where $1 < D < 2$. The subfamilies with zero entropy have rather simpler dynamics than the other subfamilies which have non zero entropy. This talk will focus on providing information of all the existing subcases/subfamilies of (1) in both above mentioned cases. Some families with zero entropy will also be shown.

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