On the generating function of the solution of a multidimensional difference equation

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De Moivre considered power series $F(z) = f(0) + f(1)z + \ldots + f(k)z^k + \ldots$ with recursive constant coefficients $\{f(n)\}_{n=0,1,2,\ldots}$ satisfying a difference equation of the form

$$f(x+m) = c_1 f(x+m-1) + \ldots + c_i f(x+m-i) + \ldots + c_m f(x), 0 \leq i \leq m$$

with some constant coefficients $c_i \in \mathbb{C}$. In 1722 he proved that the power series $F(z)$ are rational functions (see [1]).

Let C be a finite subset of the positive octant $\mathbb{Z}^n_+$ of the integer lattice $\mathbb{Z}^n$ such that, for some $m = (m_1, m_2, \ldots, m_n) \in C$, $\alpha_1 \leqslant m_1, \ldots, \alpha_n \leqslant m_n$ holds for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in C$. The Cauchy problem consists in finding the solution $f(x)$ of the difference equation (we use a multidimensional notation)

$$\sum_{\alpha \in C} c_\alpha f(x+\alpha) = 0, \tag{1}$$

which coincides with the some given function $\varphi : X_m \to \mathbb{C}$ on the set $X_m = \mathbb{Z}^n_+ \setminus \left(m + \mathbb{Z}^n_+\right)$.

The aim of this talk is to find the generating function of the solution of the Cauchy problem for a multidimensional difference equation with constant coefficients of the above form. Namely, under certain restrictions on the difference equation we establish the dependence between the generating function of the initial data $\Phi(z)$ and the generation function $F(z)$ of the solutions to the Cauchy problem of the difference equation under study, where

$$\Phi(z) = \sum_{x \in X_m} \frac{\varphi(x)}{z^{x+1}} \quad \text{and} \quad F(z) = \sum_{x \in \mathbb{Z}^n_+} \frac{f(x)}{z^{x+1}}.$$

As a consequence, we prove that the GF of the solution to the difference equation is rational if and only if the GF of the initial data is rational (see [2, 3]).

These results are used to solve certain problems in enumerative combinatorial analysis.

References

[1] Richard Stanley, *Enumerative Combinatorics, Vol. 1*, Wadsworth & Brooks/Cole, California, 1986.

[2] E. Leinartas, A. Lyapin, *On the rationality of multidimentional recusive series*, *Journal of Siberian Federal University. Mathematics & Physics.*, (2009) **2(4)**, 449-445.

[3] M. Bousquet-Mélou, M. Petkovšek, *Linear recurrences with constant coefficients: the multivariate case*, *DM*, 225, 51-75.