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Abstracts Book

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Special Session: Complex Dynamics

Plenary Lectures

Mandelpinski Necklaces for Singularly Perturbed Rational Maps

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In this lecture we consider rational maps of the form $z^n + C/z^n$ where $n > 2$. When C is small, the Julia sets for these maps are Cantor sets of circles and the corresponding region in the C-plane (the parameter plane) is the McMullen domain. We shall show that the McMullen domain is surrounded by infinitely many simple closed curves called Mandelpinski necklaces. The k^{th} necklace contains exactly $(n-2)n^k + 1$ parameters that are the centers of baby Mandelbrot sets and the same number of parameters that are centers of Sierpinski holes, i.e., disks in the parameter plane where the corresponding Julia sets are Sierpinski curves (sets that are homeomorphic to the Sierpinski carpet fractal). We shall also briefly describe other interesting structures in the parameter plane.

Application of singularity theory to the global dynamics of population models

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The notion of critical curves of two dimensional maps has been around since its introduction by Mira in 1964. The main objective there was to find an absorbing region of the map. Using singularity theory of Hassler Whitney as well as some topological and geometrical results, we will establish the mathematical foundation of Mira's results. The new established theory will be applied to competition models to show that local stability implies global stability. Then we will put forward a new conjecture which says that for proper excellent noninvertible maps (Whitney) with invariant boundary, local stability implies global stability.

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Different approaches to the global periodicity problem

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Let F be a real or complex n-dimensional map. It is said that F is *globally periodic* if there exists some $p \in \mathbb{N}^+$ such that $F^p(x) = x$ for all x for which F^p is well defined, where $F^k = F \circ F^{k-1}, k \geq 2$. The minimal p satisfying this property is called the *period* of F. One problem of current interest is the following: Given a m-dimensional parametric family of maps, say F_λ, determine all the values of λ such that F_λ is globally periodic, together with their corresponding periods, see for instance [1, 7, 10]. Perhaps the paradigmatic example is the 1-parameter family of maps corresponding to the Lyness recurrences

$$F_\lambda(x, y) = \left(y, \frac{\lambda + y}{x}\right),$$

for which the solution is well known. The globally periodic cases are F_1 and F_0 and the corresponding periods 5 and 6.

Some people of our research group approaches to the above problem from different points of view. The aim of this talk is to show the techniques that we are using and some of the results that we have obtained. More specifically we approach to it using the following tools:

- Proving its equivalence with the complete integrability of the dynamical system induced by the map F, see [3].
- Using the local linearization given by the Montgomery-Bochner Theorem for characterizing the periodic cases in same families of maps, see [5, 6, 11].
- Using the theory of normal forms, see [4].

- Using properties of the so called *vanishing sums* that are polynomial identities with integer coefficients involving only roots of the unity, see [2, 9].

We will pay special attention to the case where F_λ is a family of rational maps, depending also rationally of the multiparameter λ. Other cases on which we will concentrate are the maps of the form

$$F(x_1, \ldots, x_k) = (x_2, x_3, \ldots, x_k, f(x_1, x_2, \ldots, x_k)),$$

that correspond to autonomous k-th order difference equations, and the maps coming from non-autonomous periodic difference equations.

As an example of this last situation consider the 2-periodic Lyness recurrence

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n}, \quad \text{where} \quad a_n = \begin{cases} a & \text{for} \quad n = 2\ell + 1, \\ b & \text{for} \quad n = 2\ell, \end{cases}$$

with $\lambda = (a, b) \in \mathbb{C}^2$. Studying the family of maps

$$F_{a,b}(x, y) = \left(\frac{a + y}{x}, \frac{a + bx + y}{xy} \right),$$

which describes its behavior, we prove that the only globally periodic recurrences with $a \neq b$ are given by $a = a^* := (-1 \pm i\sqrt{3})/2$ and $b = b^* := \overline{a^*} = 1/a^*$ and are 10-periodic. In fact $F^5_{a^*,b^*} = \mathrm{Id}$. This result can also be obtained following the approach developed in [12] that gives all the globally periodic QRT-maps, see also [8].

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Homoclinic trajectories of non-autonomous maps

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For time-dependent dynamical systems of the form

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{Z}, \tag{1}$$

homoclinic trajectories are the non-autonomous analog of homoclinic orbits from the autonomous world, cf. [1]. More precisely, two trajectories $(x_n)_{n\in\mathbb{Z}}$, $(y_n)_{n\in\mathbb{Z}}$ of (1) are called homoclinic to each other, if

$$\lim_{n\to\pm\infty} \|x_n - y_n\| = 0.$$

Two boundary value problems are introduced, the solution of which yield finite approximations of these trajectories. Under certain hyperbolicity assumptions, we prove existence, uniqueness and error estimates.

Extending these ideas, we also propose adequate notions for heteroclinic orbits in non-autonomous systems, see [2].

The resulting algorithms and error estimates are illustrated by an example.

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L.A.S. and negative Schwarzian derivative do not imply G.A.S. in Clark's equation

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It has been conjectured for Clark's equation $x_{n+1} = \alpha x_n + (1-\alpha)h(x_{n-k})$ that a locally attracting fixed point is also globally attracting whenever h is a unimodal or decreasing map with negative Schwarzian derivative [1, 2, 3, 4]. In this talk we present some counterexamples to the conjecture when $k \geq 3$. One such counterexample is remarkably provided by Sheperd's function

$$h(x) = \frac{px}{1+x^q}$$

when the positive parameters p, q are appropriately chosen.

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Periodic point free continuous self–maps on graphs and surfaces

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Let $\mathbb{M}$ be either a connected compact graph, or a connected compact surface with or without boundary, orientable or not.

Using the action on the homology of a continuous map, we characterize the continuous maps $f : \mathbb{M} \to \mathbb{M}$ without periodic points, i.e. the so called *periodic point free* continuous self–maps of $\mathbb{M}$.

This talk will be based on the article [1].

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Hyperbolicity in dissipative polygonal billiards

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A billiard is a mechanical system consisting of a point-particle moving freely inside a planar region and being reflected off the perimeter of the region according to some reflection law. The specular reflection law is the familiar rule that prescribes the equality of the angles of incidence and reflection. Billiards with this reflection law are conservative systems, and as such are models for physical systems with elastic collisions. For this reason and their intrinsic mathematical interest, conservative billiards have been extensively studied. Much less studied are dissipative billiards, which originate from reflection laws requiring that the angle of reflection is a contraction of the angle of incidence. These billiards do not preserve the Liouville measure, and therefore can model physical systems with non-elastic collisions. We will present the case of polygonal billiard tables, whose dynamics differs strikingly from the one of its conservative counterparts. Joint work with G. del Magno, P. Duarte, J. P. Gaivão and D. Pinheiro.

Random homeomorphisms of an interval

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We investigate homeomorphisms of a compact interval, applied randomly. We consider this system as a skew product with the two-sided Bernoulli shift in the base. If on the open interval there is a metric in which almost all maps are contractions, then (with mild additional assumptions) there exists a global pullback attractor, which is a graph of a function from the base to the fiber. It is also a forward attractor. However, the value of this function depends only on the past, so when we take the one-sided shift in the base, it disappears. We illustrate those phenomena on an example, where there are two piecewise linear homeomorphisms, one moving points to the right and the other one to the left.

Translation arcs and Lyapunov stability in two dimensions

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Given a closed orbit γ of a system of differential equations in the plane

$$\dot{x} = X(x), \quad x \in \mathbb{R}^2,$$

the index of the vector field X around γ is one. This classical result has a counterpart in the theory of discrete systems in the plane. Consider the equation

$$x_{n+1} = h(x_n), \quad x_n \in \mathbb{R}^2,$$

where h is an orientation-preserving homeomorphism and assume that there is a recurrent orbit that is not a fixed point. Then there exists a Jordan curve γ such that the fixed point index of h around this curve is one. The proof is based on the theory of translation arcs, initiated by Brouwer. This talk is dedicated to discuss some consequences of the above result, specially in stability theory. We will compute the indexes associated to a stable invariant object and show that Lyapunov stability implies persistence (in two dimensions). The invariant sets under consideration will be fixed points, periodic orbits and Cantor sets. The more recent results on Cantor sets are joint work with Alfonso Ruiz-Herrera.

Discrete Dynamics and Spectral Theory

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When dealing with nonautonomous difference equations, it is well-known that eigenvalues yield no information on the stability or hyperbolicity of linear systems. We therefore review several more appropriate spectral notions first.

Our particular focus is on the *dichotomy spectrum* (also called *dynamical* or *Sacker-Sell spectrum*). It is a crucial notion in the theory of dynamical systems, since it contains information on stability, as well as appropriate robustness properties. However, recent applications in nonautonomous bifurcation theory showed that a detailed insight into the fine structure of this spectral notion is necessary. On this basis, we explore a helpful connection between the dichotomy spectrum and operator theory. It relates the asymptotic behavior of linear nonautonomous difference equations to the point, surjectivity and Fredholm spectra of weighted shifts. This link yields several dynamically meaningful subsets of the dichotomy spectrum, which not only allows to classify and detect bifurcations, but also simplifies proofs for results on the long term behavior of difference equations with explicitly time-dependent right-hand side.

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The Evolutionary Robustness of Forgiveness and Cooperation

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A fundamental problem in evolutionary game theory is to explain how cooperation can emerge in a population of self-interested individuals as typically occurs in the prisoner dilemma. Axelrod attributes the reason of emergence of cooperation to the shadow of the future: the likelihood and importance of future interaction.

Since Axelrod, one approach to test the efficiency or robustness of a strategy and further to derive optimal strategies is the evolutionary dynamics (a processes where individuals with low scores die and those with high scores flourish).

A natural question is which strategies or type of strategies are selected by the dynamics equations; in other words, which are the natural attractors and which type of strategies are uniformly selected independently of the strategy set.

We will show that in the context of the evolutionary dynamics associated to the prisoner dilemma, it is possible to identify strategies with uniform local basin of attraction independent of any initial population (provided that "the shadow of the future' is relevant and "mistakes are allowed'). It also proved, as was conjectured by Axelrod, that those strategies are " nice, retaliating, and forgiving".

Double Standard Maps

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We will investigate a two-parameter family of maps of the circle into itself, which we call Double Standard Maps. They are non-invertible analogues of the famous Arnold Standard Maps. I will present some features of this family and analyze the behaviour of their iterates.

Functional envelopes of dynamical systems – old and new results

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The notion of the functional envelope of a dynamical system was introduced by J. Auslander, S. Kolyada and the speaker in 2007. They were inspired mainly by the previous works of A. N. Sharkovsky and his collaborators, and by the notion of the density index of a topological semigroup.

If (X, f) is a dynamical system given by a compact metric space X and a continuous map $f : X \to X$, then the functional envelope of (X, f) is the dynamical system $(S(X), F_f)$ whose phase space $S(X)$ is the space of all continuous self-maps of X and the map $F_f : S(X) \to S(X)$ is defined by $F_f(\varphi) = f \circ \varphi$ for any $\varphi \in S(X)$.

In the first part of the talk we will recall the most interesting facts known on the dynamics of functional envelopes. Then we will speak on recent results due to T. Das, E. Shah and the speaker.

Monotone and slowly oscillating wavefronts of the KPP-Fisher differential-difference equation

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We present recent results on traveling fronts (i.e. positive heteroclinic solutions) for the Kolmogorov-Petrovskii-Piskunov-Fisher differential-difference equation. We discuss such aspects as the existence, uniqueness, approximation, monotonicity and oscillatory properties of the traveling fronts. In the 'non-monotone' part of the work, our approach is based on the construction and subsequent analysis of some auxiliary one-dimensional maps possessing the negative Schwarz derivative. This connects the problem of the existence of traveling fronts to the famous Wright's $3/2$-stability theorem and Wright's global stability conjecture.

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Communications

Global Dynamics for Symmetric Planar Maps

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We consider sufficient conditions to determine the global dynamics for equivariant maps of the plane with a unique fixed point which is also hyperbolic. When the map is equivariant under the action of a compact Lie group, it is possible to describe the local dynamics and – from this – also the global dynamics. In particular, if the group contains a reflection, there is a line invariant by the map. This allows us to use results based on the theory of free homeomorphisms to describe the global dynamical behaviour. In the absence of reflections, we use equivariant examples to show that global dynamics may not follow from local dynamics near the unique fixed point. This talk is based on the papers [1] and [2].

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On the Second Order Rational Difference Equation $x_{n+1} = \beta + \frac{1}{x_n x_{n-1}}$

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The author investigates the local and global stability character, the periodic nature, and the boundedness of solutions of one of the second-order rational difference equation in form

$$x_{n+1} = \beta + \frac{1}{x_n x_{n-1}}, \quad n = 0, 1, ..., \tag{1}$$

with parameter β and with arbitrary initial conditions such that the denominator is always positive. In the paper [1] are given several open problems and conjectures about these equations:
Conjecture 8.1. Every positive solution of (1) has a finite limit.
Open Problem 8.2. Assume that β is a real number. Determine the set G of real initial values x_{-1}, x_0 for which the equation (1) is well defined for all $n \geq 0$, and investigate the character of solutions of (1) with $x_{-1}, x_0 \in G$.

In this talk the author would like to pose some ideas how to solve these problems.

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Initial Condition Problems for Second Order Rational Difference Equations

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Often initial conditions of difference equations have a great influence on existence and behavior of solution. We investigate the initial value effect on the behavior of solutions of second order rational difference equations, for example, in [1, Open Problem 2.9.4]. is as follows: It is known that all solutions of difference equations

$$x_{n+1} = \frac{1 + x_{n-1}}{1 + x_n} \quad \text{and} \quad x_{n+1} = 1 + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, 2, \dots \tag{1}$$

converges to a solution with period two: $\dots, \phi, \psi, \phi, \psi, \dots$. Determine ϕ and ψ in terms of the initial conditions x_{-1} and x_0. Although the given equations seem very simple, however it is very complicated even to calculate numerically some terms of the solution sequence (for example, if we take first equation and initial values $x_{-1} = x_0 = 2$ we get a solution sequence: $2; 2; 1; \frac{3}{2}; \frac{4}{5}; \frac{25}{18}; \frac{162}{215}; \frac{9245}{6786}; \dots$).

Other problems connected with initial values are problems where the set of all initial points $(x_{-1}, x_0) \in R \times R$ through which the given equation is well defined for all $n \geq 0$ has to be gained.

On the basis of obtained results and computational experiments some ideas about initial value problems will be discussed.

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Nonlinear Volterra difference equations with time delays and their applications

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This article studies the boundedness property of the solutions of nonlinear Volterra difference equations with time delays.

The most important result of this paper is a simple new criterion, which unifies and extends several earlier results (see [1, 2, 3]). We show some results on the critical case for the solutions of Volterra difference equations with time delay to be bounded. Examples are also given to illustrate our main theorems.

It is a joint work with Prof. István Győri and Prof. Ferenc Hartung, Department of Mathematics, Faculty of Information Technology, University of Pannonia, Veszprém, Hungary.

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On the structure of Lozi maps kneading curves

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It is well known that the family of Lozi maps plays a key role in our understanding of plane dynamics. Being a two-parameter piecewise linear plane family of maps, therefore a very simple framework to study and understand dynamics on the plane, they are also close to the one-parameter family of tent maps of the interval. Thus, in principle, it seems that we have the chance to study why certain results known for one-dimension dynamics are not true when the dimension of the phase space is larger than one. With this work, we study the relationship between kneading sequences of tent maps, the topological symbolic invariants introduced by Milnor and Thurston, [2], for modal maps of the interval, with Lozi maps kneading sequences, introduced by Yutaka Ishii [3]. Building on the notion of kneading curves on the parameter space, introduced in [1], we characterize the structure of these curves for finite, periodic and aperiodic unimodal kneading sequences. Our results show that there is some strong connection between Lozi dynamics and the dynamics of tent maps.

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P-recursive moment sequences of piecewise D-finite functions and Prony-type algebraic systems

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Algebraic Signal Sampling, an active area of research in Signal Processing, aims at reconstructing an unknown finite-parametric model from a finite number of measurements such as power moments or Fourier coefficients ([2, 3, 4, 6, 7] and references therein). In many cases under consideration, the measurement sequence is P-recursive ([11]), so that the reconstruction process essentially amounts to recovery of the unknown coefficients of the corresponding recurrence relation. Important practical questions such as conditions for unique reconstruction and accuracy (stability) of solutions are directly expressible in the language of these recurrences, their perturbations and the corresponding algebraic systems [1] (so-called "Prony-type" - [8, 9]).

Continuing our previous work [2] on moment inversion for piecewise D-finite ([10]) functions, we have recently obtained a non-trivial upper bound on the number of measurements needed for unique reconstruction in terms of the ODE itself in some "singular" cases. The particular structure of the corresponding finite difference operator played a major role in the proof.

In addition to the above result and related questions, we will also discuss reconstruction of certain 2D domains and the corresponding recurrences [5].

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Discrete Dynamics on Grids with Choice

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In this talk we consider deterministic movement on graphs, integrating local information, memory and choice at nodes. The research is motivated by recent work on deterministic random walks and applications in multi-agent systems. Several results regarding passing messages through toroidal grids are discussed, as well as some open questions.

On the Second Order Quadratic Rational Difference Equation $x_{n+1} = \frac{\alpha}{(1+x_n)x_{n-1}}$

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We investigate the periodic nature of solutions of a rational difference equation

$$x_{n+1} = \frac{\alpha}{(1 + x_n)x_{n-1}}. \tag{1}$$

Classically rational difference equations are explored with nonnegative parameters and nonnegative initial conditions. We show that the rational difference equation (1) with negative initial conditions or/and with negative parameter α have different behaviour from equations with nonnegative parameters and with nonnegative initial conditions.

We explore [1, Open Problem 3.3] that requires to determine all periodic solutions of equation (1). We can assert that, for example, for difference equation (1) with parameter $\alpha > 0$ does not exist initial conditions $x_{-1} > 0$ and $x_0 > 0$ such that solution of equation (1) is periodic with prime period 5 but if $\alpha < 0$, then exist initial conditions $x_{-1} = x_0 > 0$ such that solution of equation (1) is periodic with prime period 5 (for example, $\alpha = -\frac{4}{3}$ and $x_{-1} = x_0 = 1$). Period 7 is first period for which exists nonnegative parameter α and nonnegative initial conditions (for example, $\alpha \approx 1,053218$ and $x_{-1} = 5$, $x_0 = 2$).

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Global bifurcation analysis and applications of a Liénard polynomial system

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We consider Liénard polynomial systems. There are many examples in the natural sciences and technology in which Liénard systems are applied. They are often used to model either mechanical or electrical, or biomedical systems, and in the literature, many systems are transformed into Liénard type to aid in the investigations. They can be used, e. g., in certain mechanical systems, when modeling wind rock phenomena and surge in jet engines. Such systems can be also used to model resistor-inductor-capacitor circuits with non-linear circuit elements. Recently a Liénard system has been shown to describe the operation of an optoelectronics circuit that uses a resonant tunnelling diode to drive a laser diode to make an optoelectronic voltage controlled oscillator. There are also some examples of using Liénard type systems in ecology and epidemiology.

In this talk, we discuss the general Liénard polynomial system with an arbitrary (but finite) number of singular points in the form

$$\dot{x} = y, \quad \dot{y} = -x\,(1 + \beta_1\,x + \ldots + \beta_{2l}\,x^{2l}) + y\,(\alpha_0 + \alpha_1\,x + \ldots + \alpha_{2k}\,x^{2k}). \tag{1}$$

Applying a canonical system with field rotation parameters,

$$\begin{gathered} \dot{x} = y, \\ \dot{y} = -x\,(1 + \beta_1\,x \pm x^2 + \ldots + \beta_{2l-1}\,x^{2l-1} \pm x^{2l}) \\ +\,y\,(\alpha_0 + x + \alpha_2\,x^2 + \ldots + x^{2k-1} + \alpha_{2k}\,x^{2k}), \end{gathered} \tag{2}$$

where $\beta_1,\ \beta_3, \ldots,\ \beta_{2l-1}$ are fixed and $\alpha_0,\ \alpha_2, \ldots,\ \alpha_{2k}$ are field rotation parameters, and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we carry out the global bifurcation analysis of (2) and prove the following theorem.

Theorem. *The general Liénard polynomial system (1) can have at most $k+l$ limit cycles, k surrounding the origin and l surrounding one by one the other its singularities.*

On the existence of a weighted asymptotically constant solutions of Volterra difference equations of nonconvolution type

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We study a Volterra difference equation of the form $x(n+1) = a(n) + b(n)x(n) + c(n)x(n-1) + \sum_{i=0}^{n} K(n,i)x(i)$ where $n \in \mathbb{Z}$, $a, b, c, x\colon \mathbb{Z} \to \mathbb{R}$ and $K\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$. For every admissible constant $c^* \in \mathbb{R}$, sufficient conditions for the existence of a solution $x = x(n)$ of the above equation such that $x(n) \sim c^* n\beta(n)$, where $\beta(n) = \frac{1}{2^n}\prod_{j=0}^{n-1} b(j)$ are presented. Next, sufficient conditions for the existence of an eventually positive, oscillatory, and quickly oscillatory solution of this equation are obtained, as a corollary of the main result. Finally, a conditions under which considered equation possesses an asymptotically periodic solution are given.

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Continuous Limit in Dynamics with Choice

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We are interested in time evolution of systems that can and do switch their modes (regimes) of operation at discrete moments of time. The intervals between switching may, in general, vary. The number of modes (regimes) may be finite or infinite. Such systems are very common in life. Every living organism is like that. In papers [1], [2], [3] we have developed a theory that models such systems and call it dynamics with choice (DWC). We have studied the long term behavior and, in particular, the existence and properties of global compact attractors in DWC. In this paper, we define and study a continuous time dynamics whose trajectories are limits of trajectories of discrete DWC as time step goes to zero. Under certain conditions we are able to prove the semi-group property for the continuous limit, and we study such semi-dynamical systems. In a special case of a switched system, i.e., when the DWC is generated by switching between solutions of a finite number of systems of ODEs, we show that the reachable set in the continuous limit DWC coincides with the reachable set of a certain differential inclusion.

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W-maps and harmonic averages

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The W-map [1] is a transformation $\tau : [0,1] \to [0,1]$ with a graph in the shape of letter W. We assume that it is continuous, piecewise linear with four branches: τ_1, τ_2, τ_3 and τ_4. The τ_1 and τ_3 are decreasing, the τ_2 and τ_4 increasing. The first and the last branches are onto, while the middle branches meet at point $(1/2, 1/2)$, i.e., $1/2$ is a turning fixed point of τ. The modulus of the slope of τ_i is $s_i > 1$, $i = 1, 2, 3, 4$, so τ is piecewise expanding and as such admits an absolutely continuous invariant measure (acim) μ. Let us consider a family of small perturbations τ_n of map τ, such that $\tau_n \to \tau$ as $n \to \infty$. Let τ_n have acim μ_n. If $\mu_n \to \mu$, we call τ "acim-stable". It was shown in [1] that W-map with $s_2 = s_3 = 2$ is not acim-stable. It turns out that acim-stability of τ depends on $1/s_2 + 1/s_3$ which is related to harmonic average of s_2 and s_3 equal $\frac{2}{1/s_2+1/s_3}$. If $1/s_2 + 1/s_3 < 1$, then τ is acim-stable. We prove this slightly improving the classical Lasota-Yorke inequality [2]. If $1/s_2 + 1/s_3 > 1$, then we can produce a family of τ_n's such that $\mu_n \to \delta_{\{1/2\}}$ weakly. If $1/s_2+1/s_3 = 1$, then we can produce a family of τ_n's exact on $[0, 1]$ such that μ_n's converge weakly to a combination of μ and $\delta_{\{1/2\}}$.

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On some stochastic competition models

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In his recent review, Schreiber [1] describes the state of the art for stochastic competition models of the general form

$$X_{t+1}^i = f_i(X_t, \xi_t)X_t^i, \;\; i = 1, \dots k, \;\; t = 0, 1, 2, \dots$$

where the state space **S** is a subset of $\Re_+^k$ and the union of the coordinate axes in $\Re_+^k$ forms the *extinction set* $\mathbf{S}_0$; the ξ's represent a randomly evolving environment.

I will look at some variants and special cases of this model. Specifically, we will limit ourselves to two populations, $k = 2$, and the functions f_i are chosen to be of the *Ricker* type:

$$\exp(r_t^i - K_t^i(X_t^i + \alpha^j X_t^j)), \;\; i, j = 1, 2, \;\; i \neq j.$$

Here the r_t^i's model the average intrinsic per capita rate of growth of population i at time t. The growth is attenuated by the negative term where K_t^i describes the intra-specific competition at time t and α^j (assumed constant over time) the relative importance of the inter-specific competition.

Another variant of the model is concerned with *demographic* stochasticity. Here our aim is to study the evolution of two finite populations as size-dependent branching processes, which on average follow the above Ricker-type model, with non-random r_i, K_i. We want to describe the long-term behavior and compare it with that of the corresponing deterministic model. The size-dependent branching processes will necessarily have finite life-times. Can anything be said about these?

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Global Dynamics of Anti-Competitive Systems in the Plane

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We give some general results about global dynamics of an anti-competitive system of the form

$$\begin{cases} x_{n+1} &= T_1(x_n, y_n) \\ y_{n+1} &= T_2(x_n, y_n) \end{cases}, \quad n = 0, 1, 2, \ldots$$

where

$$T_1 : \mathcal{I} \times \mathcal{J} \to \mathcal{I},\ T_2 : \mathcal{I} \times \mathcal{J} \to \mathcal{J} \ \text{ and } \ (x_0, y_0) \in \mathcal{I} \times \mathcal{J},$$

and functions T_1 and T_2 are continuous and $T_1(x, y)$ is non-increasing in x and non-decreasing in y while $T_2(x, y)$ is non-decreasing in x and non-increasing in y. We illustrate our results by means of an example which shows wide variety of typical dynamical behavior for an anti-competitive system.

Study of Velocity Control Algorithm of Vehicle Platoon

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Grouping vehicles into platoons is a method of increasing the capacity of roads. Platoons decrease the distances between vehicles using electronic, and possibly mechanical, coupling. The automated highway system is a proposal for one such system, where cars organize themselves into platoons.

The key-point of the vehicle platoon is to adequately control the vehicle velocity. The velocity control model is defined as the ordinary differential equation of the information from the nearest leader vehicle (the distance or the velocity). In this study, the platoon of five vehicles is considered as the example. Each follower vehicle has one, two, three or four leader vehicles. The velocity control model of the follower vehicle is defined so that the velocity depends on the velocity of the all leader vehicles[1, 2]. It is considered as the steady state that all vehicles move at identical velocity. Model stability analysis around the steady state gives the stable condition of the sensitivities from a vehicle to its leader vehicles. In this study, maximization of the sensitivities revels that the vehicle velocity depends only on the nearest leader and the lead vehicles of the platoon. Finally, traffic simulations of the vehicles platoon are shown in order to discuss the validity of the model.

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Asymptotic behaviour of random tridiagonal Markov chains in biological applications

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Discrete-time discrete-state random Markov chains with a tridiagonal generator are shown to have a random attractor consisting of singleton subsets, essentially a random path, in the simplex of probability vectors. The proof uses the Hilbert projection metric and the fact that the linear cocycle generated by the Markov chain is a uniformly contractive mapping of the positive cone into itself. The proof does not involve probabilistic properties of the sample path and is thus equally valid in the nonautonomous deterministic context of Markov chains with, say, periodically varying transitions probabilities, in which case the attractor is a periodic path.

Emden-Fowler type difference equations of the fourth-order

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This is a joint work with Prof. Zuzana Došlá. We consider the nonlinear difference equation

$$\Delta\left(a_n\left(\Delta b_n\left(\Delta c_n\left(\Delta x_n\right)^{\gamma}\right)^{\beta}\right)^{\alpha}\right)+d_n x_{n+\tau}^{\lambda}=0,$$

where α, β, γ, λ are the ratios of odd positive integers, $\tau\in\mathbb{Z}$ and $\{a_n\},\{b_n\}$, $\{c_n\},\{d_n\}$ are positive real sequences defined for all $n\in\mathbb{N}$.

We state new oscillation theorems and we complete the existing results in the literature. Our approach is based on considering our equation as a system of the four-dimensional difference system and on the cyclic permutation of the coefficients in the difference equations.

On Rational Difference Equations with Periodic Coefficients

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We investigate the global stability, periodic character, and the boundedness nature of the solutions of several special cases which are contained in the difference equation

$$x_{n+1} = \frac{\alpha_n + \beta_n x_n x_{n-1} + \gamma_n x_{n-1}}{A_n + B_n x_n x_{n-1} + C_n x_{n-1}}, n = 0, 1, \ldots$$

where the parameters $\alpha_n, \beta_n, \gamma_n, A_n, B_n, C_n$ are nonnegative periodic sequences, and the initial conditions x_{-1}, x_0 are nonnegative real numbers, such that the denominators are always positive.

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Efficient synchronization of one-dimensional chaotic quadratic maps coupled without symmetry

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We consider synchronization phenomena of chaotic discrete dynamical systems with unidirectional and bidirectional coupling mechanisms. It is studied a nonlinear coupling scheme that appears in natural a family of analytic complex quadratic maps (Isaeva *et al.* [1]). It is an asymmetric coupling between two real quadratic maps in which we use different values of the control parameters chosen in the region of chaos. The map obtained by coupling two chaotic quadratic maps exhibits a richer dynamics that the single one, but it is still possible to study its behaviour. We are not aware about any studies of this type of coupling. When practical synchronization (in the Kapitaniak sense) is not achieved, but the difference between the dynamical variables of the systems is bounded, we still can apply to the coupled system a chaos control technique based on the well-know OGY-method [2] (Ott-Grebogy-Yorke), the pole-placement control technique, developed by Romeiras *et al.* [3], in order to decrease the difference between the dynamical variables. Moreover, we obtain stable identical and generalized synchronization with some versions of the original coupling, highlighting the absence of symmetry. Two of them are generalizations promoting the use of different parameters coupling. By analyzing the difference between the dynamical variables of the systems, we obtain some results leading to stable synchronization. In case of coupling with two different coupling parameters, the linear stability of the synchronous state is ensured when some relations are guaranteed between the coupling parameters and the initial conditions.

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On the generating function of the solution of a multidimensional difference equation

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De Moivre considered power series $F(z) = f(0) + f(1)z + \ldots + f(k)z^k + \ldots$ with recursive constant coefficients $\{f(n)\}_{n=0,1,2,\ldots}$ satisfying a difference equation of the form

$$f(x+m) = c_1 f(x+m-1) + \ldots + c_i f(x+m-i) + \ldots + c_m f(x), 0 \leq i \leq m$$

with some constant coefficients $c_i \in \mathbb{C}$. In 1722 he proved that the power series $F(z)$ are rational functions (see [1]).

Let C be a finite subset of the positive octant $\mathbb{Z}^n_+$ of the integer lattice $\mathbb{Z}^n$ such that, for some $m = (m_1, m_2, \ldots, m_n) \in C$, $\alpha_1 \leqslant m_1, \ldots, \alpha_n \leqslant m_n$ holds for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in C$. The Cauchy problem consists in finding the solution $f(x)$ of the difference equation (we use a multidimensional notation)

$$\sum_{\alpha \in C} c_\alpha f(x+\alpha) = 0, \tag{1}$$

which coincides with the some given function $\varphi : X_m \to \mathbb{C}$ on the set $X_m = \mathbb{Z}^n_+ \setminus \left(m + \mathbb{Z}^n_+\right)$.

The aim of this talk is to find the generating function of the solution of the Cauchy problem for a multidimensional difference equation with constant coefficients of the above form. Namely, under certain restrictions on the difference equation we establish the dependence between the generating function of the initial data $\Phi(z)$ and the generation function $F(z)$ of the solutions to the Cauchy problem of the difference equation under study, where

$$\Phi(z) = \sum_{x \in X_m} \frac{\varphi(x)}{z^{x+1}} \quad \text{and} \quad F(z) = \sum_{x \in \mathbb{Z}^n_+} \frac{f(x)}{z^{x+1}}.$$

As a consequence, we prove that the GF of the solution to the difference equation is rational if and only if the GF of the initial data is rational (see [2, 3]).

These results are used to solve certain problems in enumerative combinatorial analysis.

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Attractivity in Nonautonomous Periodic and Random Difference Equations on Compact Spaces

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Discrete-time dynamical systems driven by periodic and random inputs arise as models in many areas such as population biology, epidemiology, neural networks etc.. We consider periodic difference equations and random difference equations which arise respectively when the driving sequence acting as an input is periodic and when as a stationary stochastic process. Among the fundamental entities in understanding the asymptotic behavior of such systems are nonautonomous attractors like pullback, forward and uniform attractors [1].

A difficulty some of the nonautonomous attractors pose is that their existence is unknown. We present some results on the existence of a notion of uniform attractivity for random difference equations on a compact space. In particular, with a typical path-wise consideration, we define certain autonomous attracting sets and show that each such set contains a local positively-invariant uniform attractor. In the case of periodic difference equations we relate the existence of a *globally asymptotically stable periodic solution* to nonautonomous attractors and also to what is known as the *echo state property* of a driven system [2].

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On periodic solutions of 2–periodic Lyness difference equations

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We study the existence of periodic solutions of the *non–autonomous periodic Lyness' recurrence*

$$u_{n+2} = \frac{a_n + u_{n+1}}{u_n}, \tag{1}$$

where $\{a_n\}_n$ is a cycle with positive values a,b and with positive initial conditions.

It is known that for $a = b = 1$ all the sequences generated by this recurrence are 5–periodic. Among other results concerning periodic solutions, we prove:

Proposition *Consider the* 2*–periodic Lyness' recurrence (1) for* $a > 0$, $b > 0$ *and positive initial conditions* u_1 *and* u_2.

(i) If $(a, b) \neq (1, 1)$*, then there exists a computable value* $p_0(a, b) \in \mathbb{N}$ *such that for any* $p > p_0(a, b)$ *there exist continua of initial conditions giving rise to* $2p$*–periodic sequences.*

(ii) The set of prime periods arising when $(a, b) \in (0, \infty)^2$ *and positive initial conditions are considered, contains all the even numbers except* 4, 6, 8, 12 *and* 20*. If* $a \neq b$*, then it does not appear any odd period, except* 1.

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Properties of non-commuting cycles in matrix algebras iteration

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Combining the properties of the algebraic theory with the theory of the quadratic map in the real and complex cases, we explore the properties of the non-commuting cycles arising from the iteration in matrix algebras, under the action of quadratic maps with matrix parameter. The stability domains in the subalgebra where the dynamics occurs is also studied.

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Existence, uniqueness and attractivity of prime period two solution for a difference equation of exponential form

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In [1] the authors studied the existence of the equilibrium and the boundedness of solutions of the difference equation

$$x_{n+1} = a + bx_{n-1}e^{-x_n} \tag{1}$$

where a, b are positive constants and the initial values x_{-1}, x_0 are positive numbers. Moreover the authors gave a conjecture concerning the existence, the uniqueness and the attractivity of prime period two solution.

In this paper we give an answer concerning the existence and the uniqueness of a prime period two solution for the equation (1). Moreover we find solutions of (1) which converge to the unique periodic solution of period two.

Equation (1) may have applications in Biology if we consider a as the immigration rate and b as the population growth rate.

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Asymptotics for second-order linear q-difference equations

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We will work on the q-uniform lattice $q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$ or, possibly, on $q^{\mathbb{Z}}$. We will introduce the class of functions satisfying the relation

$$y(qt)/y(t) \sim \omega(t) \quad \text{as } t \to \infty,$$

where ω is a nonzero function. We will study its properties and show how this class is related e.g. to the class of q-regularly functions or to the class of q-hypergeometric functions. Then we will consider the second-order linear q-difference equation

$$y(q^2t) + a(t)y(qt) + b(t)y(t) = 0,$$

where $b(t) \neq 0$ and $a(t)$ are real functions. Sufficient and necessary conditions will be presented for this equation to have solutions in the above mentioned class. Related results concerning estimates for solutions and (non)oscillation of all solutions will also be discussed.

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Decoupling and simplifying of difference equations in the neighbourhood of invariant manifold

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We consider systems of nonautonomous difference equations in Banach space. For these systems, sufficient conditions under which there is an local Lipschitzian invariant manifold are obtained. Using this result and nonexponential Green type maps we find sufficient conditions of partial decoupling and simplifying for systems of invertible and noninvertible difference equations.

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Chaotic models stabilized by stochastic perturbations with nonzero expectation

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A map which experiences a period doubling route to chaos, under a stochastic perturbation with a positive mean, can have a stable blurred 2-cycle for large enough values of the parameter. The limit dynamics of this cycle is described. It was shown that well-known population dynamics models, like Ricker, truncated logistic, Hassel and May, and Bellows maps, have this stable blurred 2-cycle and belong to one of the three described types. In addition, there may be a blurred stable area near the equilibrium.

Chaos in discrete structured population models

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We prove analytically the existence of chaotic dynamics in some classical discrete-time age-structured population models. Our approach allows us to estimate the sensitive dependence on the initial conditions, regions of initial data with chaotic behavior, and explicit ranges of parameters where the considered models display chaos. These properties have important implications to evaluate the influence of a chaotic regime in the predictions based on mathematical models. We illustrate through particular examples how to apply our results.

On the dynamics of two exponential type systems of difference equations

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In this paper we investigate the boundedness and the persistence of the positive solutions, the existence of a unique positive equilibrium and the global asymptotic stability of the equilibrium of the following systems of difference equations

$$x_{n+1} = a + by_{n-1}e^{-x_n}, \quad y_{n+1} = c + dx_{n-1}e^{-y_n}, \tag{1}$$

$$x_{n+1} = a + by_{n-1}e^{-y_n}, \quad y_{n+1} = c + dx_{n-1}e^{-x_n}, \tag{2}$$

where the constants a, b, c, d are positive real numbers and the initial values x_{-1}, x_0, y_{-1}, y_0 are also positive real numbers. We note that if $x_{-1} = y_{-1}$, $x_0 = y_0$ then $x_n = y_n$, for all $n = -1, 0, \ldots$ and so both systems reduce to the difference equation $x_{n+1} = \alpha + \beta x_{n-1}e^{-x_n}$ which has been studied in [1]. System (2) represents the rule by which two discrete, competing populations reproduce from one generation to the next. Variables x_n, and y_y, denote population sizes during the n-th generation and the sequence or orbit (x_n, y_n), $n = 0, 1, 2, \ldots$ describes how the populations evolve over time.

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Existence of a bounded solution of Volterra difference equations via Darbo's fixed point theorem

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We study linear Volterra difference equation of nonconvolution type on the form

$$x(n+1) = a(n) + b(n)x(n) + \sum_{i=0}^{n} K(n,i)x(i),$$

where $x\colon \mathbb{N} \to \mathbb{R}$, $a\colon \mathbb{N} \to \mathbb{R}$, $K\colon \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ and $b\colon \mathbb{N} \to \mathbb{R}$. Sufficient conditions for an existence of bounded solution of this equation are presented. Using this result, an asymptotic equivalence of a solution and of the given sequence, dependent on terms of sequence b, is obtained.

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Oscillation theory of discrete symplectic systems with nonlinear dependence on the spectral parameter

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Symplectic systems represent a discrete time analogue of the linear Hamiltonian systems. They contain as special cases many important difference equations and systems, namely the Sturm–Liouville difference equations, symmetric three-term recurrence equations, Jacobi difference equations, and linear Hamiltonian difference systems. Following our recent work in [3] and [2], we introduce a new theory of discrete symplectic systems, in which the dependence on the spectral parameter is nonlinear. This requires to develop new definitions of (finite) eigenvalues and (finite) eigenfunctions and their multiplicities for such systems. Our main results include the corresponding oscillation theorems, which relate the number of (finite) eigenvalues with the number of focal points of the principal solution in the given discrete interval. The present theory generalizes several known results for discrete symplectic systems which depend linearly on the spectral parameter, such as in [1]. We also show that our results are new even for the above mentioned special discrete symplectic systems.

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An answer to some problems on self-similar sets and the open set condition

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The subject of this talk are self-similar sets (a *self-similar* set E is a fixed point of map $\varphi(E) = \cup\varphi_i(E)$, where φ_i are contractive similitudes on $\mathbb{R}^n$, see [3]), especially those satisfying the so-called open set condition (OSC). These sets have many "nice" properties, but they are still raising many questions too.

The OSC requires existence of an open set G, such that

$$\varphi(G) \subset G \text{ and } \varphi_i(G) \cap \varphi_j(G) = \emptyset.$$

Such a set G is called a feasible set.

In the talk I will consider some open problems formulated by L. Feng and Z. Zhou in their papers [1] and [2]. Namely I will present a counterexample to the existence of connected (or even convex) feasible sets and i will prove that, if the self-similar set satisfies OSC, the fixed points of maps φ_i must be distinct for different indeces.

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Transport Equation on Semidiscrete Domains

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In this talk, we analyze the transport equation on semidiscrete domains. We consider discrete, discrete-continuous and discrete-time scale domains. We discuss its relationship with nonlinear hyperbolic problems and corresponding subjects from numerical analysis (semidiscrete methods, conservation laws, ...). Analysing integral and sign conservation, we disclose an interesting relationship of the transport equation with counting stochastic processes (Poisson and Bernoulli processes) and the corresponding probability distributions. Consequently, we mention possible application of the transport equation as a generator of mixed probability distributions.

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Dynamical Classification of a family of Birational Maps via Dynamical Degree

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Given complex numbers α_i, γ_i and $\delta_i, i = 0, ..., 2$, consider the family of birational maps $f : \mathbf{C}^2 \to \mathbf{C}^2$ of the following form

$$f(x, y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y, \frac{\gamma_0 + \gamma_1 x + \gamma_2 y}{\delta_0 + \delta_1 x + \delta_2 y}\right). \tag{1}$$

We consider the imbedding $(x, y) \mapsto (1, x_1, x_2) \in \mathbf{P}^2$ into projective space and consider the induced map $F : \mathbf{P}^2 \to \mathbf{P}^2$ given by

$$F[x_0, x_1, x_2] = [x_0(\delta \cdot x), (\alpha \cdot x)(\delta \cdot x), x_0(\gamma \cdot x)],$$

where $\alpha \cdot x = \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2$. To determine the behavior of iterates, $F^n = F \circ \cdots \circ F$, we will study their degree growth rate particularly we are interested in the quantity

$$D(\alpha, \gamma, \delta) = \lim_{n\to\infty} (\deg(F^n))^{\frac{1}{n}},$$

which is known as the *dynamical degree* in [1] and the logarithm of this quantity has been called the *algebraic entropy* in [6] and [2].

In order to classify our family (1) we first make an identification of two existing cases in (1). For all the values of parameters for which the determinants $(\gamma\delta)_{12}$ and $(\alpha\delta)_{12}$ are zero we call it a *degenerate case* and the values of parameters for which these determinants are non zero we say that the family (1) lies in the *non degenerate case*. In general the family (1) has dynamical degree $D = 2$. The main interest is to identify the possible subcases of (1) for all the parameter values. By the help of the associated characteristic polynomial of each subcase/subfamily we are able to know their growth rate. Therefore we find the dynamical degree D for all the subcases in order to locate the subfamilies with entropy *zero* and the ones where $1 < D < 2$. The subfamilies with zero entropy have rather simpler dynamics than the other subfamilies which have non zero entropy. This talk will focus on providing information of all the existing subcases/subfamilies of (1) in both above mentioned cases. Some families with zero entropy will also be shown.

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Special Session

Asymptotic Behavior and Periodicity of Difference Equations

I. Győri and M. Pituk

New stability conditions for linear delay difference equations

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The Bohl-Perron result on exponential dichotomy for a linear difference equation

$$x(n+1) - x(n) = -\sum_{l=1}^{m} a_l(n)x(h_l(n)), \quad h_l(n) \le n,$$

states (under some natural conditions) that if all solutions of the non-homogeneous equations with a bounded right hand side are bounded, then the relevant homogeneous equation is exponentially stable. According to its corollary, if a given equation is *close* to an exponentially stable comparison equation (the norm of some operator is less than one), then the considered equation is exponentially stable.

For a difference equation with several variable delays and coefficients we obtain new exponential stability tests using the above results, representation of solutions and comparison equations with a positive fundamental function.

Main results of the talk where published in [1].

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On Poincaré–Perron theorems for systems of linear difference equations

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The classical theorems of Poincaré and Perron are concerned with the asymptotic behavior of solutions of scalar kth order linear difference equations

$$y(n+k) + [c_1 + p_1(n)]y(n+k-1) + \cdots + [c_k + p_k(n)]y(n) = 0,$$

with $p_i(n) \to 0$ as $n \to \infty$, $1 \leq i \leq k$.

More recently, generalizations to the asymptotic behavior of solutions of systems of first order difference equations

$$\vec{y}(n+1) = [A + P(n)]\,\vec{y}(n), \qquad P(n) \to 0 \quad \text{as } n \to \infty,$$

have attracted significant interest. Contributions include works by Máté and Nevai, Trench, Pituk, and M. Pinto.

In our talk, we will briefly review some of the established results in this field before presenting further generalizations derived in our work.

Stability of difference equations with an infinite delay

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The relation between the exponential stability of linear difference equations with infinite delay and the ℓ^p-input ℓ^q-state stability (Perron's property) is investigated. The Perron's property means that solutions of the non-homogeneous equation with zero initial data belong to ℓ^q when non-homogeneous terms are in ℓ^p. The two properties are equivalent in a wide range of spaces, under some conditions, which include uniform boundedness of operators and exponential memory fading. We demonstrate [1] that these conditions are to some extent necessary.

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Admissibility of linear stochastic discrete Volterra operators applied to an affine stochastic convolution equation

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The long run behaviour of a finite dimensional finite delay summation equation with an additive Gaussian noise is analysed. This equation may be considered as a generalisation of an autoregressive process of arbitrary, but finite, order. It is known from existing theory that the asymptotic behaviour of the resolvent function of such an equation may be expressed in terms of the roots of its characteristic equation, c.f. e.g. [1]. It is shown that the solution of the stochastic equation is also reliant upon the leading order roots of the characteristic equation.

Admissibility theory of deterministic equations has been studied in connection with the asymptotic theory of such equations, in e.g. [2]. The authors develop a stochastic admissibility theory of linear Volterra operators to obtain their results. While the asymptotic results described in this presentation hold almost surely it is shown that this mode of convergence implies convergence in mean square.

In addition to the finite delay stochastic equation, a Volterra summation equation is also discussed. Some examples of the results are sketched which illustrate the differing types of long run behaviour which may occur depending upon the order of the leading roots and whether the leading roots are purely real or complex.

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Positive and oscillating solutions of discrete linear equations with delay

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A linear $(k+1)$th-order discrete delayed equation $\Delta x(n) = -p(n)x(n-k)$ where $p(n)$ is a positive sequence is considered for $n \to \infty$. This equation is known to have a positive solution if the sequence $p(n)$ satisfies an inequality. Recently it was proved that if

$$p(n) \le \left(\frac{k}{k+1}\right)^k \times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n \ln n)^2} + \dots + \frac{k}{8(n \ln n \dots \ln_q n)^2}\right], \qquad (1)$$

where $q \in \mathbb{N}_0$, then there exists a positive vanishing solution of the considered equation and the upper bound was found. We improve this result by finding even the lower bound for the positive solution, supposing the function $p(n)$ is bounded above and below by certain functions. As well we show that, in the case of an opposite inequality to (1) for $p(n)$, all solutions of the equation considered are oscillating for $n \to \infty$.

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Asymptotic behavior and oscillation of fourth-order difference equations

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We present some our recent results on asymptotic and oscillatory properties of solutions for the nonlinear difference equations of the fourth order

$$\Delta\left(a_n\left(\Delta b_n\left(\Delta c_n\left(\Delta x_n\right)^{\gamma}\right)^{\beta}\right)^{\alpha}\right)+d_n f(x_{n+\tau})=0, \quad (n\in\mathbb{N})$$

where α, β, γ are the ratios of odd positive integers, $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ are positive real sequences defined for $n\in\mathbb{N}$ and $\tau\in\mathbb{Z}$ is a deviating argument.
The role of the deviating argument to oscillation will be given, too. This is a joint work with Jana Krejčová.

Periodic symplectic difference systems

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We consider symplectic difference systems

$$z_{k+1} = S_k(\lambda) z_k, \quad S_k \in \mathbb{R}^{2n \times 2n},\ z \in \mathbb{R}^{2n}, \tag{1}$$

depending on a (generally complex valued) parameter λ. We suppose that the matrices S_k are J-unitary, i.e.

$$S^*(\lambda)\mathcal{J}S(\lambda) = \mathcal{J}, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

and N-periodic, i.e., $S_{k+N}(\lambda) = S_k(\lambda)$, $k \in \mathbb{N}$. We show that some previous results on periodic Hamiltonian difference systems [2, 3] (which are a special case of (1)) can be extended to (1). In particular, we demonstrate that the classical Krein's traffic rules for multipliers of the monodromy matrix of periodic Hamiltonian *differential* systems, cf. [1], remain to hold also for periodic symplecitc difference systems.

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Moving average network examples for asymptotically stable periodic orbits of strongly monotone maps

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Given a graph Γ, the discrete–time dynamical system

$$(x_{1,t}, \ldots, x_{I,t}) = \mathbf{x}_t \;\rightarrow\; \mathbf{x}_{t+1} = \mathbf{F}(\mathbf{x}_t)\;, \quad t = 0, 1, \ldots$$

on the I–dimensional unit cube $[0,1]^I$ is considered where

$$(\mathbf{F}(\mathbf{x}_t))_i = F(\bar{x}_{i,t})\;, \quad i = 1, \ldots, I\;,$$

$F : [0,1] \rightarrow [0,1]$ is a strictly increasing continuous function,

$$\bar{x}_{i,t} = \frac{1}{n_i} \sum_{j \in N_i} x_{i,t}\;,$$

i is a vertex of Γ, and N_i is the set of its neighbouring vertices with $n_i = \mathrm{card}(N_i) \neq 0$.

Conditions for the existence of a globally asymptotically stable fixed point as well as a variety of examples for asymptotically stable nontrivial periodic orbits is presented. The motivation comes from modelling local interactions in tax evasion [1].

The talk is based on joint work with *Judit Várdai*.

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Utilization of the circulant matrix theory in periodic higher order autonomous difference equations

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In this talk we develop easily verifiable tests that we can apply to determine whether or not a higher order autonomous difference equation has a p-periodic solution. One of the main tools in our investigations is a transformation, recently introduced by the authors, which formulates a given higher order difference equation as a first order recursion. The second important tool is the theory of circulant matrices. The periodicity conditions are formulated in terms of the coefficients of the higher order equation, along with examples showing that they have nontrivial applications.

Asymptotic behavior of nonlinear difference equations

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In this talk we investigate the growth/decay rate of solutions of a class of nonlinear Volterra difference equations. Our results can be applied for the case when the characteristic equation of an associated linear difference equation has complex dominant eigenvalue with higher than one multiplicity. Illustrative examples are given for describing the asymptotic behavior of solutions in a class of linear difference equations and in several discrete nonlinear population models.

Sharp algebraic periodicity conditions for linear higher order difference equations

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It will be derived new necessary and sufficient, and sufficient algebraic conditions on the periodicity of the solutions of the d-dimensional system of the sth order difference equations

$$x(n) = \sum_{i=1}^{s} A_i(n)x(n-i), \quad n \geq 0,$$

where

(C$_1$) $s \geq 1$ is a given integer, and $A_i(n) \in \mathbb{R}^{d\times d}$ for every $1 \leq i \leq s$ and $n \geq 0$.

The main tool in our investigation is a transformation, recently introduced by Győri and Horváth in [1], which formulates a given higher order recursion as a first order difference equation in the phase space. The periodicity conditions are formulated in terms of the so called companion matrices and the coefficients of the given higher order equation, as well (see [2]).

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Asymptotic formula for solutions of Volterra difference equations with infinite delay

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For linear Volterra difference equations with infinite delay, we obtain an explicit asymptotic representation formula of solutions by utilizing a part of some bases belonging to the phase space, together with a part of the dual bases for the formal adjoint equation associated with a certain bilinear form. Our technique employed here is similar to the standard one in the theory of linear functional differential equations.

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A continuous separation of tipe II. Applications to nonautonomous delay differential equations.

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Poláčik and Tereščák proved the existence of a continuous separation for a general abstract mapping Φ satisfying the usual strong monotonicity, smoothness and compactness condition. In many important applications the origin of this map is a homeomorphism F of a compact metric subset K in a Banach space X and Φ is defined by $\Phi : K \times X \to K \times X, (\omega, x) \to (F(\omega), dF(\omega)(x))$.

For nonautonomus cooperative ordinary and parabolic equations we prove that if the constant matrix defined by the superior of the partial derivatives of the vector field on a minimal set K is irreducible then the flow map of the linearized equation at certain time t_1 admits a continuous separation on $K \times X$. However this kind of results are no longer valid for nonautonomous cooperative delay differential equations. In this case the linear flow map is not eventually strongly positive but satisfies a dichotomy behavior which provides a dynamical scenario that we define as a continuous separation of type II. This scenario preserves many of the previous dynamical properties which are relevant in the applications of the theory.

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A variant of the Krein-Rutman theorem for Poincaré difference equations

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Let $\mathbf{x}_n$, $n \in \mathbb{N}$, be a nonvanishing solution of the Poincaré difference equation

$$\mathbf{x}_{n+1} = A_n \mathbf{x}_n, \qquad n \in \mathbb{N},$$

where A_n, $n \in \mathbb{N}$, are $k \times k$ real matrices such that the limit $A = \lim_{n\to\infty} A_n$ exists (entrywise). According to a Perron type theorem, the limit $\rho = \lim_{n\to\infty} \sqrt[n]{\|\mathbf{x}_n\|}$ exists and is equal to the modulus of one of the eigenvalues of A. In this talk, we show that if the solution belongs to a given order cone K in $\mathbb{R}^k$, then ρ is an eigenvalue of A with an eigenvector in K. In the case of constant coefficients, this result implies the finite-dimensional version of the Krein-Rutman theorem.

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Boundedness Character of solutions, Monotonic Character of solutions and Existence of Periodic Solutions of a Non-Autonomous Rational Difference Equation

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Our aim is to investigate the *boundedness character*, the *periodic character* and the *monotonic character* of the non-negative solutions of the following non-autonomous rational difference equation:

$$x_{n+1} = \frac{A_n x_{n-l}}{1 + \sum_{i=0}^{k} B_i x_i}, \; n = 0, 1, \dots \; ,$$

where $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive real numbers, $\sum_{i=0}^{k} B_i > 0$, $l = 0, 1, 2, \dots$, and $k = 1, 2, 3, \dots$. We will examine how the different periods of the sequence and the relationship of the terms of the sequence affect the longer term behavior of the solutions.

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A QRT-system of two order one homographic difference equations: conjugation to rotations, periods of periodic solutions, sensitiveness to initial conditions

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We study the QRT-system of order one homographic difference equations in ${\mathbb{R}_*^+}^2$

(1) $$u_{n+1}u_n = 1 + \frac{d}{v_n} \quad v_{n+1}v_n = 1 + \frac{d}{u_{n+1}}, \text{ for } d > 0,\ u_0 > 0\ v_0 > 0.$$

Using some tools presented in the references we prove the following results.

(1) The invariant $G(x, y) = x + y + \frac{1}{x} + \frac{1}{y} + \frac{d}{xy}$ has a strict minimum K_m at the equilibrium $L = (\ell, \ell)$ (where $\ell^3 - \ell - d = 0$), and so the solutions of (1) are permanent and L is locally stable. The orbit of a point $M_0 = (u_0, v_0) \in {\mathbb{R}_*^+}^2$ is included in the positive component $\mathcal{C}_K^+$ of the cubic curve $\mathcal{C}_K$ with equation $(xy + 1)(x + y) + d - Kxy = 0$ passing through M_0 (for a unique $K > K_m$).

(2) With the use of the group law on the cubic $\mathcal{C}_K$ and with the use of Weierstrass' function $\wp$ one can see that the restriction to $\mathcal{C}_K^+$ of the map F defined on ${\mathbb{R}_*^+}^2$ by

$$F(x, y) = (X, Y), \text{ where } Xx = 1 + \frac{d}{y},\ Yy = 1 + \frac{d}{X} \quad \text{with } F(u_n, v_n) = (u_{n+1}, v_{n+1})$$

is conjugated to a rotation on the circle $\mathbb{T}$ of an angle $2\pi\theta_d(K)$, where $\theta_d(K) \in]0, 1/2[$ is given explicitely by the ratio of two integrals. There is a non-empty open interval $I \subset]0, +\infty[$ such that for each $d \in I$ the map $K \mapsto \theta_d(K)$ is not one-to-one.

(3) The set of starting points with periodic orbits is dense in ${\mathbb{R}_*^+}^2$, and the only integers which are not periods of some solution of (1) for some $d > 0$ are 2, 3, 4, 6 and 10.

(4) For every compact $\mathcal{K} \subset {\mathbb{R}_*^+}^2$ not containing the equilibrium, it exists a number $\delta_{\mathcal{K}} > 0$ such that $F_{|\mathcal{K}}$ has $\delta_{\mathcal{K}}$-sensitiveness to initial conditions: in every neighborhood of $M \in \mathcal{K}$ it exists a point M' such that $||F^n(M) - F^n(M')|| \geq \delta_{\mathcal{K}}$ for infinitely many integers n .

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Averaging theorems for dynamic equations on time scales

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Classical averaging theorems for ordinary differential equations are concerned with the initial-value problem

$$x'(t) = \varepsilon f(t, x(t)) + \varepsilon^2 g(t, x(t), \varepsilon), \quad x(t_0) = x_0,$$

where $\varepsilon > 0$ is a small parameter. According to these averaging theorems, a good approximation of the solution can be obtained by considering the autonomous differential equation

$$y'(t) = \varepsilon f^0(y(t)), \quad y(t_0) = x_0,$$

where $f^0(y) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(t, y)\,\mathrm{d}t$.
The aim of the talk is to present time scale analogues of both periodic and non-periodic averaging theorems, as well as a related theorem on the existence of periodic solutions of dynamic equations (see [1, 2]). We make use of the correspondence between dynamic equations and generalized ordinary differential equations (see [3]).

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Oscillation theorems for second-order nonlinear difference equations of Euler type

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This talk deals with the oscillatory behavior of the difference equation which corresponds to the nonlinear differential equation of Euler type $x'' + f(x)/t^2 = 0$, where $f(x)$ is continuous on $\mathbb{R}$ and satisfies the signum condition $xf(x) > 0$ if $x \neq 0$. To give the oscillation theorem for the nonlinear difference equation, we consider the linear difference equation corresponding to the Riemann-Weber version of the Euler differential equation.

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Turan-type inequalities and Taylor domination for solutions of linear ODE's

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Let a family of analytic functions $f_\lambda(z) = \sum_{k=0}^{\infty} a_k(\lambda) z^k$ be given, and let $R(\lambda)$ be the radius of convergence of f_λ. The family f_λ possesses a property of an (N, C)-uniform Taylor domination if

$$|a_k(\lambda)| R^k(\lambda) \leq C \max_{i=0,\ldots,N} |a_i(\lambda)| R^i(\lambda), k = N, N+1, \ldots, \tag{1}$$

with N and C not depending on λ. Taylor domination provides, in particular, a uniform bound on the number of zeroes of f_λ in each disk strictly contained in the disk of convergence.

An important example is the family R_λ^d of all rational functions of degree d. Here uniform Taylor domination follows from the classical Turan lemma ([3]). Equivalently, Taylor domination holds for solutions of a linear recurrence relation with constant coefficients.

In this talk we discuss uniform Taylor domination for solutions of linear differential equations with polynomial coefficients, or for linear recurrence relations with the coefficients polynomially depending on the index. This is the situation of the classical Poincaré-Perron theorem ([2]), however, we are not aware of any generalization of the Turan lemma to this case. We prove a weaker version, where C in (1) is replaced by C^k, and discuss a "dynamical" approach (as in some proofs of the Poincaré-Perron theorem - see [2, 1]) to the original case. We also consider Taylor domination for the sequences of the moments of a given algebraic function.

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Special Session

Applications of Difference Equations to Biology

J. Cushing, S. Elaydi and J. Li

The dynamics of some contest-competition population models with the effect of harvesting and stocking

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In this talk, we consider contest-competition population models of the form $x_{n+1} = x_n f(x_{n-k})$, where $k = 0, 1$ and the map $f(x)$ is decreasing on $[0, \infty)$. We investigate the dynamics under different harvesting or stocking strategies. In particular, we give some results concerning periodicity, stability, invariants and persistent sets.

Geometric methods for global stability in the Ricker competition model

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It is an important problem to determine when local conditions can be globally verified. In [2] and [3], the authors investigated the local stability of the equilibrium points of the logistic competition model and the Ricker competition model, respectively. It was shown that the coexistence equilibrium point of the Ricker competition model is locally asymptotically stable if the parameters lie in a certain stability region in the parameter space.

Later in [1] it was conjectured that the coexistence (positive) equilibrium is indeed globally asymptotically stable in the hypotheses above. In this talk, we will discuss the geometric and topological tools that allow us to completely describe the geometry of the image of the Ricker map. We will use singularity theory to describe the relative position of the images of the critical curves and, using methods from covering space theory, we will describe the regions where the cardinality of the pre-images of points are constant. Finally we will describe how these methods are used to show global stability.

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Difference equations arising in evolutionary population dynamics

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Difference (matrix) equation models in population dynamics that arise in the modeling of certain life history strategies, namely semelparity, give rise to an interesting dynamic dichotomy. This dichotomy consists of two invariant sets, each of which is a potential attractor (but never both) [1]. One is an equilibrium interior to the positive cone and the other lies on the boundary of the positive cone (and yields synchronized periodic orbits). Which is the attractor depends on the nature of the nonlinearities (specifically the strengths of the nonlinear interactions between and within age classes) [2, 3]. I will describe the difference equations that arise when such a population is subject Darwinian evolution [4] and give theorems that describe the nature of the dynamic dichotomy in an evolutionary context.

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Allee effect in two interacting species

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This work extends the notion of Allee effect to two-dimensional setting. In this setting, Allee (threshold) point will be replaced by Allee (threshold) curve. The case we examine here is when one or both species possess the Allee effect. The notion of critical curves and singularity theory will be used to understanding the global dynamics of certain competition models with Allee effect.

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Global asymptotically stable of a generalized discrete Lotka-Volterra competition system

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Sufficient conditions for the global asymptotic stability of one equilibrium point of a generalized Lotka-Volterra competition system, which appears as a model for dynamics with one extinct species , is obtained by applying the technique of average functions and limiting equations.

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Local stability implies global stability in the Ricker competition model

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In [2] and [3] the authors investigated the local stability of the equilibrium points of the logistic competition model and the Ricker competition model, respectively. It was shown that in each model, the coexistence equilibrium point is locally asymptotically stable if the parameters lie in a certain stability region in the parameter space.

We conjectured in [1] that in this stability region, the coexistence (positive) equilibrium is indeed globally asymptotically stable with respect to the interior of the first quadrant. The proof of this conjecture follow a complex set of tools. It includes singularity theory of planar maps, the notion of critical curves, one-point compactification of the positive quadrant, the dynamics of the local slow manifold of the coexistence fixed point and the global unstable manifold of the exclusion fixed point.

In this talk we will focus our attention in the dynamics of the manifolds. We will present the principal tolls that we use to show global stability of the coexistence fixed point.

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Phase as determined by Correlation is irrelevant for Resonance versus Attenuation in the Beverton-Holt model

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An exact expression is derived relating the state average of the periodic solution x_j to the average of the environmental carrying capacities K_j for the periodic Beverton-Holt equation for arbitrary period. By studying numerically the period 3 case we show that the correlation coefficient of the intrinsic growth rates u_j and K_j is not relevant in determining attenuation or resonance. Instead a new criterion is presented.

Special Session

Complex Dynamics

B. Devaney, N. Fagella and X. Jarque

Rigidity for non-recurrent exponential maps

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An exponential map $f(z) = e^z + c$ is called non-recurrent if the asymptotic value c is not in the accumulation set of its own forward orbit. We will present the result that whenever two non-recurrent exponential maps satisfy some combinatorial equivalence, then they are conjugate by a quasiconformal map. If moreover c has a bounded orbit, the conjugation can be made conformal.

On the Tongues of a Degree 4 Blashcke Product

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The family of Blashcke products $B_a(z) = z^3(z-a)/(1-\bar{a}z)$ is the rational analogous of the double standard family given by $h(z) = e^{i\alpha}z^2e^{\beta/2(z-1/z)}$. Both families restrict, for certain parameters, to degree 2 coverings of the unit circle. This fact leads to some interesting phenomena like the existence of tongues in the parameter plane. This tongues were studied for the first time by M. Misiurewicz and A. Rodrigues [1] and are a degree 2 analogous of the Arnold Tongues.

During the talk we will introduce the concept of tongue for the Blashcke family and we will study what occurs around the tongues. We will also study some phenomena which take place because of not having an holomorphic dependence with respect to parameters, like the existence of small copies of the Mandelbar set (see [2]).

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Conformal dimension and combinatorial modulus: applications to rational maps

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A fundamental quasisymmetry numerical invariant of a compact metric space X is its conformal dimension $\dim_{AR} X$. It was introduced by P. Pansu in order to classify, up to quasi-isometry, homogeneous spaces of negative curvature [3, 2]. Motivated by Sullivan's dictionary [4, 1], which establishes a fundamental correspondence between the properties of hyperbolic groups and of a particular class of finite branched coverings, I will define this invariant in the context of rational maps. I will show how to compute $\dim_{AR} X$ using the critical exponent Q_M associated to the combinatorial modulus, which is a discrete version of the conformal modulus from complex analysis. Finally, I will apply this result to compute the conformal dimension of the Julia sets of some hyperbolic rational maps.

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Perturbed Polynomial Maps with Small Perturbation

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We consider the family of rational maps $F_\lambda(z) = z^n + \frac{\lambda}{z^n}$ with $\lambda \in \mathbb{C}$. For $n > 2$, Julia sets for maps corresponding to parameters near the origin are Cantor sets of simple closed curves. For $n = 2$, however, as λ approaches zero, it is known that the Julia sets for these maps converge to the closed unit disc in the Hausdorff metric. In this talk, we give a description of a ”ring structure” in the parameter plane for $n = 2$ near $\lambda = 0$, identifying a pattern of rings of alternating Sierpiński holes and Mandelbrot sets surrounding the origin.

Convergence of Rational Rays in Hyperbolic Components

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We are interested in entire transcendental maps with two singular values, one of which is a fixed critical value, and the other is a free asymptotic value, with only one finite preimage. The family of such maps can be parametrized as:

$$f_a(z) = a(e^z(z-1)+1)$$

where the critical value is fixed at $z = 0$, and the asymptotic value is at $z = a$, with finite preimage at $z = 1$.
In this talk, we illustrate in this family, proof of a landing theorem of rational rays in hyperbolic components by means of Carathéodory Convergence Theory, and discuss further results in parameter spaces.

Dynamics of different classes of meromorphic functions with a finite set of singular values

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We will define some classes of meromorphic functions which have a finite set of singular values. We will state some results in complex dynamics which can be applied to all of these classes of functions. We will also give some new results.

Quadratic Mating Discontinuity

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According to Milnor, the mating operation is interesting because it has none of the usual good properties. It is not injective, surjective, everywher defined, or continuous. We give a survey of discontinuity mechanisms, with special attention to the quadratic case.

On the existence of absorbing domains in Baker domains

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Let U be a hyperbolic domain in $\mathbb{C}$ and let $f : U \to U$ be a holomorphic map. An invariant domain $W \subset U$ is called *absorbing in U for f* if for every compact set $K \subset U$ there exists $n = n(K) > 0$, such that $f^n(K) \subset W$. The problem of existence of absorbing domains for a given f and U has a long history, since such sets are useful in many problems in dynamics.

Based in the Denjoy-Wolf Theorem (on dynamics of holomorphic maps on the unit disc), Cowen proved the existence of a simply connected absorbing domain $V \subset \mathbb{H}$ for holomorphic maps $G : \mathbb{H} \to \mathbb{H}$ (where $\mathbb{H}$ denotes the right half plane) such that $G^n \to \infty$ as $n \to \infty$. He also showed the existence of a semiconjugacy (actually a conjugacy on V) of the map G to a Möbius transformation T acting on Ω where $\Omega \in \{\mathbb{H}, \mathbb{C}\}$.

Later, König used Cowen's Theorem to prove the existence of simply connected absorbing domains in Baker domains of meromorphic maps with finitely many poles. Moreover König also showed, by means of an example, that simply connected absorbing domains do not always exist.

The main result we present here is the existence of (possibly multiply connected) absorbing domains in the general case (putting especial attention on the case of Baker domains for transcendental meromorphic functions). In another talk, Xavier Jarque will show how to use this result to prove the connectedness of the Julia set for transcendental meromorphic maps having no weakly repelling fixed points.

Homeomorphisms Between Julia Sets for Rational Maps

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We consider the family of rational maps $F_\lambda(z) = z^2 + \frac{\lambda}{z^2}$ with $\lambda \in \mathbb{C}$. In particular, we choose λ from the main cardioid of an accessible period-k baby Mandelbrot set with $k \geq 2$. When $k = 2$, there exists a dynamical invariant (identical to that used in the discussion of checkerboard Julia sets) to determine when the dynamics of two such maps are conjugate. When $k > 2$, we discuss the existence of a topological invariant for homeomorphisms between Julia sets of the same prime period.

Sierpiński curve Julia sets for quadratic rational maps

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The Sierpiński carpet fractal is one best known planar, compact, connected sets. It is a universal plane continuum and there is a topological characterization of this set. Any planar set homeomorphic to a Sierpiński carpet is called a Sierpiński curve. In recent years several authors have shown that Sierpiński curve can arise as the Julia set of certain holomorphic functions. We make an attempt towards a more systematic approach to the problem of existence of Sierpiński curves as Julia sets of rational maps. Our goal is to find dynamical conditions under which we can assure that the Julia set of a certain quadratic rational map is a Sierpiński curve.

A family of rational maps with buried Julia components

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A Julia component is said buried if it has no intersection with the boundary of any Fatou domain. It is well known that may not arise for polynomial maps. The first example of such Julia components is due to Curtis McMullen [1] who provided a family of rational maps for which the disconnected Julia set is a Cantor of Jordan curves. However all known examples of buried Julia components, up to now, are wandering Jordan curves and comes from rational maps of degree at least 5.

I will introduce a family of degree 3 rational maps whose disconnected Julia set contains buried Julia components of all types which may occur a priori according to a result of Kevin Pilgrim and Tan Lei [2]: wandering points, wandering Jordan curves but also preperiodic infinitely connected Julia components. That totally answers a question McMullen raised since 3 is the minimal degree expected for rational map with buried Julia components [3].

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Dynamic Rays for Transcendental Holomorphic Self-maps of $\mathbb{C}^*$

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I am interested in the iteration of holomorphic self-maps of the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ for which both zero and infinity are essential singularities. These maps are of the form $f(z) = z^n \exp(g(z) + h(1/z))$ with $n \in \mathbb{Z}$ and $g(z), h(z)$ non-constant entire functions. In particular, I would like to understand what is the structure of the escaping set $I(f)$, the set of points whose orbit accumulates to zero and/or infinity.

In the setting of transcendental entire functions, A. Eremenko conjectured that every $z \in I(f)$ could be joined with ∞ by a curve in $I(f)$. In analogy to what G. Rottenfußer, J. Rückert, L. Rempe and D. Schleicher proved in [1] for functions in class $\mathcal{B}$, we show that this property holds for a class of functions whose singular set is bounded away from zero and from infinity and satisfy some technical conditions which are related to the notion of finite order.

References

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A combinatorial invariant for escape time Sierpinski rational maps

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An escape time Sierpinski map is a rational map drawn from the McMullen family $z \mapsto z^n + \lambda/z^n$ with escaping critical orbits and Julia set homeomorphic to the Sierpinski curve continuum. We address the problem of characterizing (postcritically finite) escape time Sierpinski maps in a combinatorial way. To accomplish this, we define a combinatorial model given by a planar tree whose vertices come with a pair of combinatorial data that codifies the dynamics of critical orbits. We show that each escape time Sierpinski map realizes a subgraph of the combinatorial tree and the combinatorial information is a complete conjugacy invariant.

Some results on the size of escaping sets

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Let f be a transcendental entire function in the class B, that is, the set of singular values of f is bounded. We present some new results about the Hausdorff dimension and Hausdorff measure of the escaping set $I(f)$ and various subsets of it.

We show that for any given sequence (p_n) tending to ∞, the set of escaping points with $|f^n(z)| \leq p_n$ always has Hausdorff dimension at least 1, and that there are functions f for which this set can be 'larger' than the fast escaping set of f (in a certain sense). This result contrasts with the situation for exponential maps, since in this case it is known that the fast escaping set has a larger Hausdorff dimension than the set points that escape slowly.

Further, we set $B_\rho := \{f \in B : f \text{ has order } \rho\}$. We show that the set $I(f)$ has infinite Hausdorff measure with respect to a certain gauge function h_ρ for every $\rho \geq 1/2$ and $f \in B_\rho$. On the other hand, for $\tilde{\rho}$ large enough, we prove the existence of a function $f \in B_{\tilde{\rho}}$ such that $I(f)$ has zero measure with respect to h_ρ. This means that the escaping sets of functions in class B of finite order can become smaller as the order increases.

Regularity and fast escaping points of entire functions

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We give a new sufficient condition for a point to be in the fast escaping set $A(f)$ of a transcendental entire function f; see [2] for details and results about $A(f)$. More precisely, we show that the 'quite fast escaping set' $Q(f)$, introduced in [3], is equal to $A(f)$ if and only if the maximum modulus of f satisfies a certain regularity condition called 'weak regularity'. We show that weak regularity holds, as does an even stronger regularity condition called 'log-regularity', whenever the minimum modulus of f is not too large, in particular whenever f belongs to the Eremenko-Lyubich class $\mathcal{B}$; see [1].

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About rigidity

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We will present how rigidity results can be used to study the boundary of hyperbolic components.

Singular perturbations in the quadratic family

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In recent years, families of complex rational maps that result from perturbing well known quadratic maps such as $Q_0(z) = z^2$ and $Q_c(z) = z^n + c$, where c is the center of the corresponding Multibrot set, have been of interest. In this presentation, we consider maps of the form $P_c(z) = z^2 + c$, where c is the center of a hyperbolic component of the Mandelbrot set, that have been perturbed by the addition of a pole or multiple poles which affect the superattracting cycle of the unperturbed map. We will focus on the topological and dynamical characteristics of the resulting Julia sets. In particular, we will give conditions which guarantee that the corresponding Julia set contains homeomorphic copies of the unperturbed Julia set, Cantor sets of quasicircles, and Cantor sets of point components that accumulate on these curves.

Spiders web escaping sets

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We discuss examples of functions for which the escaping set $I(f)$ has the structure of a 'spider's web'. In particular, we consider the case that a subset $A_R(f)$ of the fast escaping set $A(f)$ has this structure. In this case the function has many strong dynamical properties, and both Eremenko's conjecture and Baker's conjecture hold as discussed in [2]. Further, the escaping set has a very intricate structure as described in [2] and [1]. We conclude by giving examples for which the 'quite fast escaping set' $Q(f)$ is a spider's web (by results in [3]) but $A(f)$ is not, as shown in [4].

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Regluing and topological models for quadratic rational functions

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We consider rational functions of one complex variable as topological dynamical systems on the sphere. These maps can be given by explicit formulas. However, explicit algebraic descriptions do not provide a good understanding of topological dynamics. Thus one needs descriptions, which we call *topological models*, that would be explicit in a different sense, i.e. from the viewpoint of topological dynamics.

Topological models for polynomials with locally connected Julia sets can be described in terms of *Thurston's laminations* in the disk. There are several remarkable constructions that build topological models for rational functions out of topological models for polynomials, for example, *matings* and *captures*.

We discuss another surgery tool called *regluing* [1]. It can be used to build new topological models for rational functions out of the already known topological models. E.g. captures can be understood as regluings. Moreover, regluing produces many matings out of every capture. In this way, we can prove that the boundaries of many hyperbolic components in parameter slices of degree two rational functions consist of matings [2].

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Connectivity of Julia sets of meromorphic maps with Baker domains

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In this talk we show that if a meromorphic transcendental map has a multiply connected Baker domain, then it must also have at least one weakly repelling fixed point (i.e. repelling or with derivative equal to one). This was the last remaining case in the proof of the following result (which was proven by Shishikura for rational maps): If f is a meromorphic transcendental map with a disconnected Julia set, then f has a weakly repelling fixed point. The historical motivation of this theorem was its corollary, namely that the Julia set of Newton's method of every entire map is connected or, equivalently, all its Fatou components are simply connected. To prove this theorem we use a result explained in Núria Fagella's talk, which shows the existence of absorbing regions in Baker domains, a question which has been open for some time.

Some applications of approximation theory to complex dynamics

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Let f be a transcendental entire function. For $n \in \mathbb{N}$, let f^n denote the n^{th} iterate of f. The set $F(f) = \{z \in \mathbb{C} : (f^n)_{n \in \mathbb{N}}$ is normal in some neighbourhood of $z\}$ is called the Fatou set of f, and the set $\mathbb{C} \setminus F(f)$ denoted by $J(f)$ is called the Julia set of f.

Let U be a component of $F(f)$, then by complete invariance of the Fatou set, $f(U)$ lies in some component V of $F(f)$. If $U_n \cap U_m = \phi$ for $n \neq m$, where U_n denotes the component of $F(f)$ which contains $f^n(U)$, then U is called a wandering domain, else U is called a pre-periodic domain, and if $U_n = U$ for some $n \in \mathbb{N}$, then U is called periodic domain.

Here we present some of the results that we have obtained on the wandering and periodic domains using approximation theory.

Special Session

Chaotic Linear Dynamics

J. P. Bès, P. Oprocha and A. Peris

Graph theoretic structure of maps of the Cantor space

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We develop unifying graph theoretic techniques to study the dynamics and the structure of the spaces $H(X)$ and $C(X)$, the space of homeomorphisms and the space of continuous self-maps of the Cantor space X, respectively. Using our methods, we give characterizations which determine when two homeomorphisms of the Cantor space are conjugate to each other. We also give a new characterization of the comeager conjugacy class of the space $H(X)$. The existence of this class was established by Kechris and Rosendal in [9] and a specific element of this class was described concretely by Akin, Glasner and Weiss in [1]. Our characterization readily implies many old and new dynamical properties of elements of this class. For example, we show that no element of this class has a Li-Yorke pair, implying the well known Glasner-Weiss result [8] that there is a comeager subset of $H(X)$ each element of which has topological entropy zero. Our analogous investigation in $C(X)$ yields a surprising result: there is a comeager subset of $C(X)$ such that any two elements of this set are conjugate to each other by an element of $H(X)$. Our description of this class also yields many old and new results concerning dynamics of a comeager subset of $C(X)$.

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Li-Yorke and distributional chaos for linear operators

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We show the equivalence between Li-Yorke chaos and the existence of an irregular vector and the equivalence between distributional chaos and the existence of a distributionally irregular vector for a linear continuous operator in a Banach spaces.

Moreover we give sufficient conditions in order to obtain dense distributional chaos in Frechet spaces. As consecuence we obtain:

A)Let T be a linear and continuous operator on X. If there exists a dense set X_0 such that $\lim_{n\to\infty} T^n x = 0$, for all $x \in X_0$ and one of the following conditions is true:

a) X is a Fréchet space and there exists a eigenvalue λ with $|\lambda| > 1$.

b) X is a Banach space and $\sum \frac{1}{\|T^n\|} < \infty$ (in particular if $r(T) > 1$).

c) X is a Hilbert space and $\sum \frac{1}{\|T^n\|^2} < \infty$(in particular if $\sigma_p(T) \cap \mathbb{T}$ has positive Lebesgue measure).

then T is densely distributionally chaotic.

B) All operator that satisfies the Frequent Hypercyclic criterion is dense distributionally chaotic.

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Distributional chaos for the solutions of certain partial differential equations

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In the study of the dynamics of linear operators defined on Banach spaces, interesting phenomena appear when we consider an underlying infinite-dimensional setting. Several notions of chaos, such as the ones of Devaney and Auslander & Yorke (hypercyclicity), have been already considered for linear operators and C_0-semigroups of operators that give the solution of certain abstract Cauchy problems. We refer to the monograph by Grosse-Erdmann and Peris [2] for further information on these topics.

The notion of distributional chaos has been recently added to the study of the chaoticity of linear operators [3]. We will report some results concerning how does it works on C_0-semigroups of operators. In particular several examples of partial differential equations that present this behaviour will be provided. Moreover, we will provide an example of a C_0-semigroup with a full scrambled set.

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Chaotic solution for the Black-Scholes equation

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The Black-Scholes semigroup is studied on spaces of continuous functions on $[0, \infty)$ which may grow at both 0 and at ∞, which is important since the standard initial value is an unbounded function. We prove that in the Banach spaces

$$Y^{s,\tau} := \{u \in C((0,\infty)) \; : \; \lim_{x\to\infty} \frac{u(x)}{1+x^s} = \lim_{x\to 0} \frac{u(x)}{1+x^{-\tau}} = 0\}$$

with norm $\|u\|_{Y^{s,\tau}} = \sup_{x>0} \left| \frac{u(x)}{(1+x^s)(1+x^{-\tau})} \right| < \infty$, the Black-Scholes semigroup is strongly continuous and chaotic for $s > 1, \tau \geq 0$ with $s\nu > 1$, where $\sqrt{2\nu}$ is the volatility. The proof relies on the Godefroy-Shapiro hypercyclicity criterion.

Hypercyclic and topologically mixing properties of certain classes of abstract time-fractional equations

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In recent years, considerable effort has been directed toward the topological dynamics of abstract PDEs whose solutions are governed by various types of operator semigroups, fractional resolvent operator families and evolution systems. In this talk, we shall present the most important results about hypercyclic and topologically mixing properties of some special subclasses of the abstract time-fractional equations of the following form:

$$\begin{aligned}&\mathbf{D}_t^{\alpha_n}u(t)+c_{n-1}\mathbf{D}_t^{\alpha_{n-1}}u(t)+\cdots+c_1\mathbf{D}_t^{\alpha_1}u(t)=A\mathbf{D}_t^{\alpha}u(t),\quad t>0,\\&u^{(k)}(0)=u_k,\quad k=0,\cdots,\lceil\alpha_n\rceil-1,\end{aligned}$$

where $n \in \mathbb{N} \setminus \{1\}$, A is a closed linear operator acting on a separable infinite-dimensional complex Banach space E, $c_1, \cdots, c_{n-1}$ are certain complex constants, $0 \leq \alpha_1 < \cdots < \alpha_n$, $0 \leq \alpha < \alpha_n$, and $\mathbf{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order α. We slightly generalize results from [1] and provide several applications, including those to abstract higher order differential equations.

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Distributional chaos for operators with full scrambled sets

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In recent years many researchers were looking for conditions that yield complex, nontrivial dynamics of linear operators (note that, to admit such behaviour, the space must be infinite dimensional). Probably the most studied is the notion of hypercyclicity, that is, the existence of vectors $x \in X$ such that the orbit of this vector $x, T(x), T^2(x), \ldots$ under the action of a continuous and linear operator $T\colon X \to X$ on a topological vector space (most often Banach or Fréchet space) X forms a dense subset of X. Distributional chaos was introduced by Schweizer and Smital as a natural extension of Li-Yorke chaos and we consider its strongest notion of *uniform distributional chaos*, which requires the existence of an uncountable set $D \subset X$ and $\varepsilon > 0$ such that for every $t > 0$ and every distinct $x, y \in D$ the upper densities of the sets $\{i \in \mathbb{N};\ \|T^i x - T^i y\| \geq \varepsilon\}$ and $\{i \in \mathbb{N};\ \|T^i x - T^i y\| < t\}$ are equal to 1. The set D is called a *distributionally ε-scrambled set*.

We answer in the negative the question of whether hypercyclicity is sufficient for distributional chaos for a continuous linear operator (we even prove that the mixing property does not suffice). Moreover, we show that a extremal situation is possible: There are (hypercyclic and non-hypercyclic) operators such that the whole space consists, except zero, of distributionally irregular vectors.

Extension problem and fractional powers of generators

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We extend results of Caffarelli-Silvestre and Stinga-Torrea regarding a characterization of fractional powers of differential operators via an extension problem. Conversely, a solution to the extension problem is given in terms of the fractional power. Our main result applies to generators A of integrated semigroups, in particular to operators with purely imaginary symbol. We also give a result on the growth of perturbated tempered α-times integrated semigroups, that could be of independent interest.

Chaoticity and invariant measures for two population models

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First, we present a short introduction to an ergodic approach to chaotic systems based on [1]. The main idea is to show that a semiflow possesses an invariant mixing measure positive on open sets. From this it follows that the system is topological mixing and its trajectories are turbulent in the sense of Bass. Then we present two cell population models which lead to chaotic systems. The first one [2] describes the evolution of maturity of blood cells in the bone marrow and is given by a partial differential equation. The equation generates a semiflow acting on densities (i.e. integrable functions with the integral one). Next, we consider a classical structured model of cells reproduction system [3] given by a partial differential equations with a non-local division term. This equation generates semiflows acting on some subspaces of locally integrable functions. We show that our semiflows are isomorphic to the shift semiflow $f(x) = f(x+t)$ on properly chosen spaces of functions Y. The second step is to construct a mixing and invariant measure m supported on the space Y. We can do it, if we find a Gaussian process with trajectories from the space Y. Then the measure m of a Borel subset A of Y is the probability that trajectories of the process are from the set A. It should be noted that most of the recent papers concerning chaos for semigroups of operators are based on studying spectral properties of their infinitesimal generators. This approach seems to be easier than ours. But, in our opinion, the approach based on the isomorphism with shift semigroups and using invariant measures reveals why our semiflows are chaotic. The second advantage of the ergodic theory approach is that we can prove much stronger results concerning chaos.

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Special Session

Combinatorial and Topological Dynamics

S. Kolyada and Ľ. Snoha

Zeta functions and periodic entropy of nonautonomous dynamical systems

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Periodic sequences of continuous self mappings on a compact topological space, $F = (f_i)_{i\in\mathbb{N}}$, are commonly regarded as periodic nonautonomous difference equations or deterministic periodic nonautonomous dynamical systems (periodic dynamical systems for simplicity).

The study of the periodic entropy, $\mathbf{h}_{\text{per}}(F)$, of a p-periodic dynamical system is the main topic of this talk.

Special attention will be paid to the formula

$$\mathbf{h}_{\text{per}}(F) = \frac{\mathbf{h}_{\text{per}}(f_{p-1} \circ \cdots \circ f_1 \circ f_0)}{p}, \tag{1}$$

which, in contrast with topological entropy, fails in general.

Our main goal is to provide sufficient conditions in order to get the equality in (1). Naturally the study of the analytic properties of the zeta function of F will play a central role in this discussion.

Perturbations of autonomous systems and Lyapunov exponents in non-autonomous systems

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We will consider relations between small perturbations of non-chaotic autonomous dynamical systems and the chaotic behavior of the non-autonomous systems in which they are transformed.

This is made using a extended notion of Lyapunov exponents in non-autonomous systems and that of chaoticity as equivalent to have sensible dependence on initial conditions. As a first example we use the easy difference equation of first order $x_{n+1} = ax_n$ when we perturb the real parameter a and other examples obtained perturbing the same parameter in the logistic equations of first and second order $x_{n+1} = ax_n(1 - x_n)$ and $x_{n+1} = ax_n(1 - x_{n-1})$.

Additionally we will study some relations between the stability and instability of solutions of difference equations with the value of Lyapunov exponents (when they exist), particularly with periodic perturbations or doubling periodic using Jacobi functions. As a consequence, we will construct some pathological examples.

Iterated function systems on the circle

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Some dynamical properties such as transitivity, minimality, density of periodic orbits, can be also studied for iterated function systems (IFS). Blending regions are introduced as open sets which are minimal sets for an IFS under small C^1-perturbations. Duminy's Lemma shows examples of blending regions for an IFS generated by two maps on the real line close enough to the identity. An extension of this lemma allows us to study the dynamics of IFS of generic diffeomorphisms on the circle close enough to the identity. As in the Denjoy's Theorem, no invariant minimal Cantor sets appear under conditions of regularity in the IFS. In this setting, it is characterized when S^1 is a minimal set of an IFS and it is obtained an spectral decomposition result about of the dynamic of the limit set of an IFS.

On the dynamics of Cournot–Puu oligopoly

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Cournot–Puu oligopoly is a market consisting on n firms producing the same, or perfect substitutes, goods with demand function $p = 1/Q$, where p is the price, $Q = q_1 + ... + q_n$ is the total output and q_i is the output of firm i. We consider constant marginal costs c_i for each firm i, which implies that cost functions are $C_i = c_i q_i$ for $i = 1, ..., n$. Under naive expectations, each firm will plan its production at time $t + 1$ as

$$q_i(t+1) = f_i(Q_i(t)) = \max\left\{0, \sqrt{Q_i(t)/c_i} - Q_i(t)\right\},$$

where $Q_i = Q - q_i$ is the residual supply and f_i is the reaction function of each firm. In this talk we summarize the dynamics in duopoly case, that is, when $n = 2$ (see e.g. [2, 1]) and analyze some phenomena of interest from the point of view of economic dynamics like conditions which guarantees the disappearing of firms in the market.

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Snap-back repellers in rational difference equations

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We are dealing with the existence of chaos, in the Li-Yorke sense, in rational difference equations. This problem arise, for example, when we apply the Newton's method to polynomials obtaining rational difference equations of order one.

In the paper we review ideas and results from Marotto in [1] based on a subtle study of the dynamics near a special kind of equilibrium (snap-back repeller). We find that some of them remain true in the non-continuous setting, for example in rational difference equations , and how some others can be proved via the introduction of an additional condition, what we call the *compact pre-image property*.

As applications, we estimate numerically the snap-back repellers of the families $x_{n+1} = \frac{1}{x_n^2 - c}$ (*inverse parabolas*) and $x_{n+1} = \frac{1}{rx_n(1-x_n)}$ (*inverse logistic equations*). Such estimations are obtained using the basins of attraction, in $\mathbb{C}$, of the reciprocal difference equations. Additionally we use them to estimate also *their forbidden sets*.

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On cascades of elliptic periodic points in area-preserving maps with homoclinic tangencies

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We use qualitative and topological methods to study the orbit behaviour near nontransversal homoclinic orbits in area-preserving maps which are not necessarily orientable. Let f_0 be such C^r-smooth map $(r \geq 3)$ having a saddle fixed point O whose stable and unstable invariant manifolds have a quadratic tangency at the points of some homoclinic orbit Γ_0. Let f_ε be a family (unfolding) of area-preserving maps containing the map f_0 at $\varepsilon = 0$.

Our aim is to study bifurcations of the so-called single-round periodic points in the family f_ε. Every point of such an orbit can be considered as a fixed point of the corresponding *first return map*. We study bifurcations of the fixed points and prove the existence of cascades of generic elliptic periodic points for one and two parameter unfoldings f_ε. Thus, we generalize the results obtained in [1] where only the symplectic (area-preserving and orientable) case was analyzed.

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Dense orbits of flows and maps — misunderstandings and new results

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Existence of dense orbit (topological transitivity) or a stronger property of every orbit being dense (topological minimality) belong to the core questions of topological dynamics. In this talk I discuss these notions for continuous as well as discrete time systems in a general setting without assuming compactness of the underlying space.

Besides well known results, there are common and frequent misunderstandings related to the mentioned two properties. In the talk I briefly mention a few of them and then proceed to the main part of the talk - new results from the joint paper with Ľubomír Snoha, see [1]. Among others, I present results relating density of orbits of flows and corresponding t-maps, and density of full orbits versus density of forward or backward semi-orbits.

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On some properties of discrete dynamical systems on dendrites

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We consider some properties of discrete dynamical systems on compact metric spaces, such as positive topological entropy, the existence of a horseshoe, Lyapunov stability on the set of periodic points or the existence of a homoclinic trajectory. Here we survey the relations between the properties in case of systems on dendrites.

Modified Lotka-Volterra maps and their interior periodic points

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Consider the plane triangle

$$\Delta = \{ [x, y] : 0 \leq x,\ 0 \leq y,\ x + y \leq 4 \}$$

and the map

$$F : \Delta \to \Delta,\ [x, y] \mapsto [x(4 - x - y),\ xy]\ .$$

(We denote by $[x, y]$ a point in the plane, while (α, β) and $\langle \alpha, \beta \rangle$ are open and closed intervals on the real line.) In [4] A. N. Sharkovskiĭ formulated some problems about properties of a map which is conjugated with the map F. This map was studied in [1, 2, 3, 5] and is called a Lotka-Volterra map (in [1, 2, 3]). It is easy to show that a point $P = [x, 0] \in \Delta$ is a periodic point of the map F if and only if $x = 4 \sin^2 \frac{k\pi}{2^n \pm 1}$, where $n \geq 1$ and k are integers with $0 \leq 2k < 2^n \pm 1$. We are interested in *interior* periodic points of the map F. Our main result of [3] is a relation between lower and interior periodic points. Namely, if a point $[4 \sin^2 \frac{k\pi}{2^n \pm 1}, 0]$ is a saddle point of the map F^n then there is a interior periodic point with the same itinerary with respect to the sets

$$\Delta_L = \{[x, y] : 0 \leq x < 2\,, 0 \leq y \leq 4 - x\}$$

and

$$\Delta_R = \{[x, y] : 2 < x \leq 4\,, 0 \leq y \leq 4 - x\}\,.$$

We extend this result for modifications of the map F which are defined as follows. Assume that for any $x \in (0,\ 4)$ we have an increasing homeomorphism φ_x of the interval $\langle 0,\ 4 - x \rangle$ onto itself. Moreover let the function $\varphi(x, y) = \varphi_x(y)$ be continuous in the domain

$$\widehat{\Delta} = \{ [x, y] : 0 < x < 4\,, 0 \leq y \leq 4 - x \}\ .$$

Let $G : \Delta \to \Delta$ be defined by

$$G[x, y] = \begin{cases} [0, 0] & \text{if } x = 0 \text{ or } x = 4 \ , \\ [x(4 - x - \varphi(x, y)), x\varphi(x, y)] & \text{otherwise.} \end{cases}$$

Then the map G is called a modified Lotka-Volterra map. Note that $F[x, 0] = G[x, 0]$ for all $x \in \langle 0, \ 4 \rangle$. We construct two modifications G for which all lower fixed points of the map G^n are repulsive and saddle points respectively. We give an example of a modification G such that for any $n \geq 1$ all repulsive lower fixed points of the map F^n are saddle points of G^n and vice versa.

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Average shadowing properties: a few sufficient conditions

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Average shadowing properties generalize standard shadowing properties (e.g. shadowing, limit shadowing) by considering averages of errors in consecutive steps of pseudo-orbit rather than ordinary error in each step. This modification enables application of average shadowing in dynamical systems where we cannot control error in each step, but we can ensure that average error is sufficiently small. In particular, there are maps with average shadowing property but without shadowing property.

In this talk we will survey recent results on average shadowing properties. We will present a few sufficient conditions that ensure average shadowing and comment on relations between average shadowing and notions from topological dynamics, like shadowing property, mixing, specification, proximality and the like.

Combination of scaling exponents and geometry of the action of renormalization operators

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Renormalization group ideas are important for describing universal properties of different routes to chaos: period-doubling in unimodal maps, quasiperiodic transitions in circle maps, dynamics on the boundaries of Siegel disks, destruction of invariant circles of area-preserving twist maps, etc. The universal scaling exponents for each route are related to the properties of the corresponding renormalization operators.

In [1, 2], we proposed the *Principle of Approximate Combination of Scaling Exponents* (PACSE) that organizes the scaling exponents for different transitions to chaos. Roughly speaking, if the combinatorics of a transition is a composition of two simpler combinatorics, then the scaling exponent of this transition is approximately equal to the product of the scaling exponents corresponding to the two simpler transitions. In [1, 2], we stated PACSE quantitatively as precise asymptotics of the scaling exponents for the combined combinatorics, and gave convincing numerical evidence for it for the four dynamical systems mentioned above.

We propose an explanation of PACSE in terms of the dynamical properties of the renormalization operators—in particular, as a consequence of certain transversal intersections of the stable and unstable manifolds of the operators corresponding to different transition to chaos. As an essential ingredient in this picture, we prove [3] a general shadowing theorem that works for infinite dimensional discrete dynamical systems that are not necessarily invertible (which is the case of the renormalization operators acting in appropriate function spaces).

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Asymptotic recurrence quantification analysis

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The notion of recurrence plays a fundamental role in the theory of dynamical systems. A powerful way for visualization of recurrence, called recurrence plot, was introduced by Eckman, Kamphorst and Ruelle in 1987. A few years later, Zbilut and Webber established the recurrence quantification analysis (RQA) by introducing measures of complexity, such as recurrence rate or determinism. Since then, RQA proved to be useful in a wide area of disciplines, ranging from life and earth sciences, engineering, material sciences, finance and economics, to chemistry and physics.

In the talk we introduce asymptotic RQA characteristics and we show their basic properties. Then we focus on the determinism and its relationships to various properties of dynamical systems. We also present some examples with nontrivial determinism.

Minimal sets and free intervals

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We will talk about minimal sets of dynamical systems defined on a continuum with a free interval, i.e. with an open subset homeomorphic to the open interval $(0, 1)$. Dirbák et al. showed in [1] that every minimal set M intersecting a free interval J is either a finite set or a finite union of disjoint circles or a nowhere dense cantoroid. We prove that such a minimal set must satisfy a property, which we call J-clipping (i.e. there is an arc in $\overline{J}$ containing $M \cap J$). As an application of this result we obtain a topological characterization of minimal sets on the Warsaw circle.

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On sensitivity and some classes of induced maps

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The talk will consists of two independent parts. Firstly we will speak on some qualitative properties of sensitive topological dynamical systems. Some results from a joint paper with Oleksandr Rybak will be presented. In the second part we will consider some induced dynamical systems, in particular we will speak on ω-limit sets of induced triangular maps (based on a joint work with Damoon Robatian) and on the topological entropy of functional envelope of some dynamical systems (based on a joint work with Iuliia Semikina).

Special Session

Economic Dynamics and Control

A. A. Pinto and A. Yannacopoulos

Anosov diffeomorphisms on surfaces

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We consider a hyperbolic matrix

$$\tilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$$

and the hyperbolic toral automorphism $A : \mathbb{T} \to \mathbb{T}$ induced by $\tilde{A}$, where $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}$. Then $\tilde{A}$ has two eigenvalues λ and μ such $|\lambda| > 1$ and $|\mu| < 1$. Furthemore, $\gamma = |\mu|$ is a quadratic surd and the continued fraction expansion of $\gamma = 1/(a_1 + 1/(a_2 + 1/\ldots))$ is M-periodic, for some $M = M(\gamma) \in \mathbb{N}$.

We prove the existence of a one-to-one correspondence between C^{1+} conjugacy classes of C^{1+} Anosov diffeomorphisms G that are topologically conjugate to A and pairs of C^{1+} *stable* and *unstable self-renormalizable sequences*.

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On the convergence to Walrasian prices in random matching Edgeworthian economies

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We show that for a specific class of random matching Edgeworthian economies, the expectation of the limiting equilibrium price coincides with the equilibrium price of the related Walrasian economies. This result extends to the study of economies in the presence of uncertainty within the multi-period Arrow-Debreu model, allowing to understand the dynamics of how beliefs survive and propagate through the market.

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Stability of the Cournot equilibrium for a Cournot oligopoly model with n competitors

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It was stated by [7] (see also [3] page 237) that the oligopoly model produced under constant marginal costs with a linear demand function is neutrally stable for three competitors and unstable for more than three competitors. As discussed in [6], linear demand functions are very easy to use, but they do not avoid negative supplies and prices, so it is possible to use them only for the study of local behavior. This problem can be solved by using nonlinear demand functions such as piecewise linear functions or other more complex functions, one of which was suggested by [4] for a duopoly and later by [5] for a triopoly using iso-elastic demand functions. These types of demand function were later studied by [1] and [2] for a nonlinear (iso-elastic) demand function and constant marginal costs and it was concluded that this Cournot model for n competitors is neutrally stable if $n = 4$ and is unstable if the number of competitors is greater than five (see also [6]).

The main aim is to consider Cournot points and discuss their stability while the number of players is increasing for the model with an iso-elastic demand function and under the assumption that the firms' costs are identical. The terminology of dynamical systems is used, that is the Cournot point is identified as a fixed one. Finally, it is proved that for identical unit costs the Cournot point is a sink for two or three competitors and a saddle for more than four players.

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An economical model for dumping in a duopoly market

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We consider an international trade economical model where two firms of different countries compete in quantities and can use three different strategies: (i) repeated collusion, (ii) deviation from the foreigner firm followed by punishment by the home country and then followed by repeated Cournot, or (iii) repeated deviation followed by punishment. In some cases (ii) and (iii) can be interpreted as dumping. We compute the profits of both firms for each strategy and we characterize the economical parameters where each strategy is adopted by the firms (see also [1]).

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Cournot duopoly games with heterogeneous players

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The main aim of this paper is to analyze the dynamics of nonlinear discrete-time maps generated by duopoly games in which players are heterogeneous and the reaction functions are non-monotonic and asymmetric. We discusses here two cases: in the first one we introduce games with boundedly rational players and in the second one games with adaptive expectations. The dynamics and the topological entropy are mainly analyzed by numerical simulations. There are always multiple equilibria, and the significance of the Nash equilibria is pointed out.

Rational Bubbles and Economic Policy

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We are living in dangerous times, hardly conceivable just fifteen years ago. Bubbles in the housing market in various countries, bubbles in the financial markets, bank runs that put to the ground some of biggest international banks in no more than a couple of days, panic in many sovereign bond markets (negative bubbles), central banks running out of policy tools to manage monetary policy efficiently, are just some of the features of the tormented world we are currently facing. Is this world governed by irrational behavior? Or is it the case that such fantastic behavior may arise from fully rational agents? In an influential review article, Stephen Le Roy (2004) stated very clearly that "under rational asset pricing, including rational expectations, such biased expectations [bubbles, panic, bank runs] cannot occur: absence of arbitrage implies that the expected (risk-adjusted and discounted) gain on any security or portfolio is zero. Thus in this usage bubbles are synonymous with irrationality." In this paper, we analyze the dynamics of an asset pricing model subject to complete markets and rational expectations, firstly developed by Aiyagari (1988). The model displays endogenous cycles and high volatility even in the case of no exogenous shocks hitting the economy. Contrary to the dominant view in economics and finance — "the preliminary conclusion seems to be that when endogenous fluctuations exist in optimizing models, the associated policy advice is laissez-faire", Bullard and Butler (1993) — we argue that economic policy may lead to an increase in social welfare in such a context by ruling out rational bubbles.

Strategic optimization in R&D Investment

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We use d'Aspremont and Jacquemin's strategic optimal R&D investment in a duopoly Cournot competition model to construct myopic optimal discrete and continuous R&D dynamics. We show that for some high initial production costs, the success or failure of a firm is very sensitive to small variations in its initial R&D investment strategies.

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Resort Pricing and Bankruptcy

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We introduce a resort pricing model, where different types of tourists choose between different resorts. We study the influence of the resort prices on the choices of the different types of tourists. We characterize the coherent strategies of the tourists that are Nash equilibria.We find the prices that lead to the bankruptcy of the resorts and, in particular, their dependence on the characteristics of the tourists.

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Dynamics of Human decisions

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We study a dichotomous decision model, where individuals can make the decision yes or no and can influence the decisions of others. We characterize all decisions that form Nash equilibria. Taking into account the way individuals influence the decisions of others, we construct the decision tilings where the axes reflect the personal preferences of the individuals for making the decision yes or no. These tilings characterize geometrically all the pure and mixed Nash equilibria. We show, in these tilings, that Nash equilibria form degenerated hystereses with respect to the replicator dynamics, with the property that the pure Nash equilibria are asymptotically stable and the strict mixed equilibria are unstable. These hystereses can help to explain the sudden appearance of social, political and economic crises. We observe the existence of limit cycles for the replicator dynamics associated to situations where the individuals keep changing their decisions along time, but exhibiting a periodic repetition in their decisions.

Timing structures in economics and games

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In this talk, we study the role of different timing structures in economic modelling. We focus on the models in which the various underlying timing structures have some real-world interpretation and could be changed so that the desired outcomes (like low inflation, balanced budgets etc.) prevail. A major example of this phenomenon are rigidities in monetary-fiscal interactions.

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Neimark-Sacker bifurcation in a discrete-time Goodwin model

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This paper presents Goodwin's growth cycle model [1] in its discreet version [2] that has been obtained by means of a non-standard Micken's discretization method. Based on explicit Neimark-Sacker bifurcation, normal form method and center manifold theory [3], the system's existence, stability and direction of Neimark-Sacker bifurcation are studied. Numerical simulations are employed to validate the main results of this work. Some comparison of bifurcation between the discrete-time Goodwin model and its continuous-time system is given.

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