A QRT-system of two order one homographic difference equations: conjugation to rotations, periods of periodic solutions, sensitiveness to initial conditions

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We study the QRT-system of order one homographic difference equations in ${\mathbb{R}_*^+}^2$

$$(1) \qquad u_{n+1}u_n = 1 + \frac{d}{v_n} \quad v_{n+1}v_n = 1 + \frac{d}{u_{n+1}}, \text{ for } d > 0,\ u_0 > 0\ v_0 > 0.$$

Using some tools presented in the references we prove the following results.

(1) The invariant $G(x, y) = x + y + \frac{1}{x} + \frac{1}{y} + \frac{d}{xy}$ has a strict minimum K_m at the equilibrium $L = (\ell, \ell)$ (where $\ell^3 - \ell - d = 0$), and so the solutions of (1) are permanent and L is locally stable. The orbit of a point $M_0 = (u_0, v_0) \in {\mathbb{R}_*^+}^2$ is included in the positive component $\mathcal{C}_K^+$ of the cubic curve $\mathcal{C}_K$ with equation $(xy + 1)(x + y) + d - Kxy = 0$ passing through M_0 (for a unique $K > K_m$).

(2) With the use of the group law on the cubic $\mathcal{C}_K$ and with the use of Weierstrass' function $\wp$ one can see that the restriction to $\mathcal{C}_K^+$ of the map F defined on ${\mathbb{R}_*^+}^2$ by

$$F(x, y) = (X, Y), \text{ where } Xx = 1 + \frac{d}{y},\ Yy = 1 + \frac{d}{X} \quad \text{with } F(u_n, v_n) = (u_{n+1}, v_{n+1})$$

is conjugated to a rotation on the circle $\mathbb{T}$ of an angle $2\pi\theta_d(K)$, where $\theta_d(K) \in]0, 1/2[$ is given explicitely by the ratio of two integrals. There is a non-empty open interval $I \subset]0, +\infty[$ such that for each $d \in I$ the map $K \mapsto \theta_d(K)$ is not one-to-one.

(3) The set of starting points with periodic orbits is dense in $\mathbb{R}_*^{+2}$, and the only integers which are not periods of some solution of (1) for some $d > 0$ are 2, 3, 4, 6 and 10.

(4) For every compact $\mathcal{K} \subset \mathbb{R}_*^{+2}$ not containing the equilibrium, it exists a number $\delta_{\mathcal{K}} > 0$ such that $F_{|\mathcal{K}}$ has $\delta_{\mathcal{K}}$-sensitiveness to initial conditions: in every neighborhood of $M \in \mathcal{K}$ it exists a point M' such that $||F^n(M) - F^n(M')|| \geq \delta_{\mathcal{K}}$ for infinitely many integers n .

References

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