Periodic solution orbits of Hamiltonian systems via index iteration theory for symplectic paths —a survey

Yiming Long
Chern Institute of Mathematics
Nankai University
Tianjin, China

at Centre de Recerca Matemática, Bellaterra
June 3rd, 2014

Contents

1. A brief review on the Maslov-type index and its iteration theory for symplectic matrix paths

2. Applications to the linear stabilities of periodic solutions of the 3-body problems

3. Applications to the closed characteristics on compact convex hypersurfaces in $\mathbf{R}^{2n}$

$*$ This is an enlarged version of my lecture on June 3rd, 2014 at CRM.

A brief review on ω-index theory of symplectic matrix paths

Consider the Hamiltonian system:

$$\begin{cases} \dot{x}(t) = JH'(t,x(t)), & \forall t \in \mathbf{R}, \\ x(\tau) = x(0), \end{cases} \qquad \text{(HS)}$$

where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, $H \in C^2(\mathbf{R}/(\tau\mathbf{Z}) \times \mathbf{R}^{2n}, \mathbf{R})$, $x : \mathbf{R}/(\tau\mathbf{Z}) \to \mathbf{R}^{2n}$.

The variational structure

$$f(x) = \int_0^\tau (-\frac{1}{2}J\dot{x}(t) \cdot x(t) - H(t,x(t)))dt,$$

for $x \in \mathrm{dom}(-J\frac{d}{dt}) \subset L^2(\mathbf{R}/(\tau\mathbf{Z}), \mathbf{R}^{2n}) \equiv L^2$.

$f'(x) = 0 \Leftrightarrow x$ is a τ-periodic solution of (HS).

Let $x = x(t)$ be a solution of (HS). Then

$$\langle f''(x)y, z\rangle = \int_0^\tau (-J\dot{y} \cdot z - H''(t,x(t)))y \cdot z dt, \ \forall\, y, z \in \mathrm{dom}(-J\frac{d}{dt}).$$

Morse indices: $m^+(x) = m^-(x) = +\infty$, $0 \le m^0(x) \le 2n$.

Consider the linearized Hamiltonian system at x:

$$\begin{cases} \dot{y}(t) = JH''(t, x(t))y(t) & \forall t \in \mathbf{R}, \\ y(\tau) = y(0). \end{cases} \qquad \text{(LHS)}$$

Its fundamental solution $\gamma(t) = \gamma_x(t)$ is defined by

$$\begin{cases} \dot{\gamma}(t) = JH''(t, x(t))\gamma(t) & \forall t \geq 0, \\ \gamma(0) = I. \end{cases} \qquad \text{(LHS)}$$

Then γ is a path in $\mathrm{Sp}(2n) = \{M \in GL(\mathbf{R}^{2n}) \,|\, M^t JM = J\}$ with $\gamma(0) = I$.
(LHS) has a solution $y \not\equiv 0$ $\Longleftrightarrow$ $1 \in \sigma(\gamma(\tau))$ $\Longleftrightarrow$ $\det(\gamma(\tau) - I) = 0$.
Thus we consider the following degenerate hypersurface in $\mathrm{Sp}(2n)$:

$$\mathrm{Sp}(2n)_1^0 = \{M \in \mathrm{Sp}(2n) \,|\, \det(M - I) = 0\}.$$

An intuitive model

For each $M \in \mathrm{Sp}(2)$, we have:

$$M = \begin{pmatrix} r & z \\ z & \frac{1+z^2}{r} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \leftrightarrow (r, \theta, z) \in \mathbf{R}^3 \setminus \{z-\mathrm{axis}\}.$$

Matrices in $\mathrm{Sp}(2)$ are one-to-one correspondent to points in $\mathbf{R}^3 \setminus \{z-\mathrm{axis}\}$ in cylindrical coordinates.

$$\det(M - I) = 0 \Leftrightarrow (r^2 + z^2 + 1)\cos\theta = 2r.$$

$$\begin{aligned} \mathrm{Sp}(2)_1^0 &= \{M \in \mathrm{Sp}(2) \mid 1 \in \sigma(M)\} \\ &= \{(r, \theta, z) \in \mathbf{R}^3 \setminus \{z-\mathrm{axis}\} \mid (r^2 + z^2 + 1)\cos\theta = 2r\}. \end{aligned}$$

$\mathrm{Sp}(2)_1^0$ forms a singular surface in $\mathrm{Sp}(2)$ as shown below in the cylindrical coordinates of $\mathbf{R}^3$.

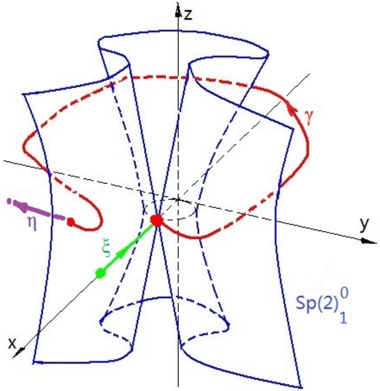


Figure: Graph of γ and $\mathrm{Sp}(2)^0_1$

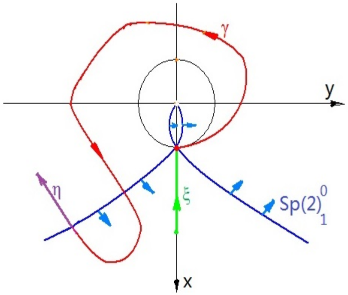


Figure: Illustrations on the graphs of γ and $\mathrm{Sp}(2)^0_1$ when $z = 0$

In $\mathrm{Sp}(2)$ let ξ be the segment path connecting $\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ to I_2.

Let $\eta(t) = \gamma(\tau)e^{-t\epsilon J}$ with $t \in [0, \tau]$ and $\epsilon > 0$ small.

We define the orientation of $\mathrm{Sp}(2)^0_1$ as shown in the Figure.

Definition For $\gamma \in C([0, \tau], \mathrm{Sp}(2))$ with $\gamma(0) = I$, we define

$$
\begin{aligned}
i_1(\gamma) &= [\eta * \gamma * \xi : \mathrm{Sp}(2)^0_1], \\
\nu_1(\gamma) &= \dim \ker(\gamma(\tau) - I).
\end{aligned}
$$

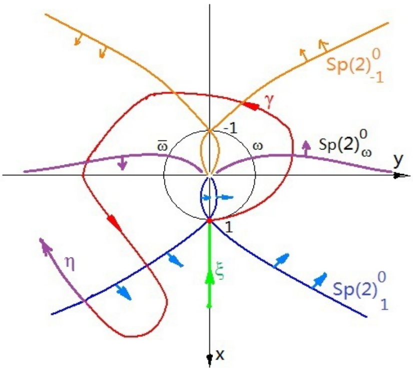


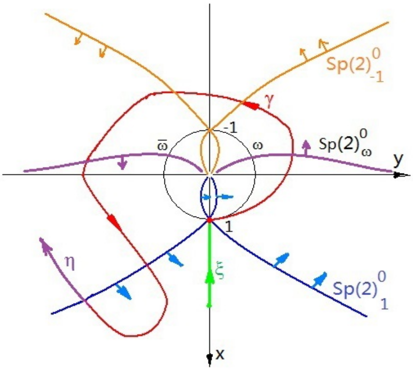
Figure: Graph of $\mathrm{Sp}(2)^0_\omega$ when $z = 0$

For $\omega \in \mathbf{U} = \{z \in \mathbf{C} \mid |z| = 1\}$ and $M \in \mathrm{Sp}(2n)$, we let

$$D_\omega(M) = (-1)^{n-1}\omega^{-n}\det(M - \omega I),$$

and define degenerate hypersurfaces

$$\mathrm{Sp}(2n)^0_\omega = \{M \in \mathrm{Sp}(2n) \mid D_\omega(M) = 0\}.$$



Definition For $\gamma \in C([0,\tau], \mathrm{Sp}(2n))$ with $\gamma(0) = I$, and every $\omega \in \mathbf{U}$ we define

$$
\begin{aligned}
i_\omega(\gamma) &= [\eta * \gamma * \xi : \mathrm{Sp}(2n)^0_\omega], \\
\nu_\omega(\gamma) &= \dim_{\mathbf{C}} \ker_{\mathbf{C}}(\gamma(\tau) - \omega I).
\end{aligned}
$$

Then $\quad (i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbf{Z} \times \{0, 1, \ldots, 2n\}, \qquad \forall\, \omega \in \mathbf{U}.$

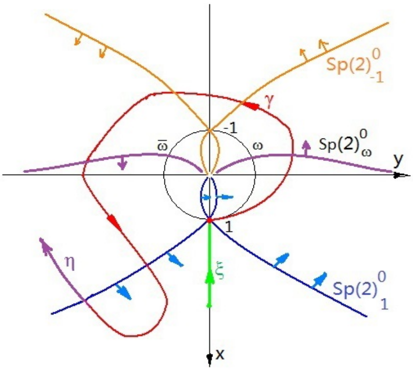


Figure: Graph of $\mathrm{Sp}(2)^0_\omega$ when $z = 0$

$$i_1(\gamma) = 1, \qquad i_\omega(\gamma) = i_{-1}(\gamma) = 2, \quad \text{for } \omega \in \mathbf{U} \setminus \{1\} \text{ in the figure.}$$

For a given solution $x = x(t)$ of (HS):

$$\begin{cases} \dot{x}(t) = JH'(t, x(t)), & \forall t \in \mathbf{R}, \\ x(\tau) = x(0). \end{cases} \qquad \text{(HS)}$$

Viewing $x = x(t)$ as a critical point of the functional

$$f(x) = \int_0^\tau \left(-\frac{1}{2}J\dot{x}(t) \cdot x(t) - H(t, x(t))\right)dt,$$

defined on $L^2(\mathbf{R}/(\tau\mathbf{Z}), \mathbf{R}^{2n}) \equiv L^2$, by using saddle point (Lyapunov-Schmidt) reduction to reduce the problem to a space Z with $\dim Z = 2d$, a functional a and $z \in \mathrm{Crit}(a)$ corresponding to L^2, f and x respectively, we obtain

$$\begin{aligned} m^-(a, z) &= d + i_1(\gamma_x), \\ m^0(a, z) &= \nu_1(\gamma_x), \\ m^+(a, z) &= d - i_1(\gamma_x) - \nu_1(\gamma_x). \end{aligned}$$

ω-index theory for symplectic paths in $\mathrm{Sp}(2n)$:

1984, C. Conley-E. Zehnder: for $\omega = 1$ and any 1-non-degenerate path γ in $\mathrm{Sp}(2n)$ with $n \geq 2$, i.e., $(i_1(\gamma), \nu_1(\gamma))$ with $\nu_1(\gamma) = 0$;

1990, Y. Long-E. Zehnder: for $\omega = 1$ and any 1-non-degenerate path γ in $\mathrm{Sp}(2)$, i.e., $(i_1(\gamma), \nu_1(\gamma))$ with $\nu_1(\gamma) = 0$;

1990, Y. Long, C. Viterbo (independently): for $\omega = 1$ and any path γ in $\mathrm{Sp}(2n)$ and γ may be 1-degenerate, i.e., $(i_1(\gamma), \nu_1(\gamma))$ with $\nu_1(\gamma) \geq 0$;

1999, Y. Long: for any $\omega =\in$ **U** and any path γ in $\mathrm{Sp}(2n)$, i.e., $(i_\omega(\gamma), \nu_\omega(\gamma))$ with $\nu_\omega(\gamma) \geq 0$.

Index iteration theory for symplectic paths

For $\gamma \in \mathcal{P}_\tau(2n) = \{\xi \in C([0,\tau], \mathrm{Sp}(2n)) \mid \xi(0) = I\}$, define

$$\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j, \quad \text{for } j\tau \le t \le (j+1)\tau, 0 \le j \le m-1.$$

Basic problem: find precise values of $(i_1(\gamma^m), \nu_1(\gamma^m))$ for all $m \in \mathbf{N}$, based on information of $\gamma(\tau)$ and $(i_1(\gamma), \nu_1(\gamma))$.

1. Bott-type iteration formula (L. 1999) $i_1(\gamma^m) = \sum_{\omega^m=1} i_\omega(\gamma)$.
2. Precise iteration formula (L. 2000)
   $i_1(\gamma^m) = m\, c_1(M, i_1(\gamma), \nu_1(\gamma)) + \sum_{j=1}^{q} E(\frac{m\theta_j}{2\pi}) + c_2(M, i_1(\gamma), \nu_1(\gamma))$,

   where $E(a) = \min\{k \in \mathbf{Z} \mid k \ge a\}$ for $a \in \mathbf{R}$.
3. Abstract precise iteration formula (L-Zhu, 2002)
4. Various index inequalities and estimates (Liu-L., L.-Zhu, 2000-2002)
5. Common index jump theorem (L.-Zhu, 2002) (On common properties of finitely many symplectic paths γ_js).

Applications of the index iteration theory for symplectic paths

1. Rabinowitz conjecture on prescribed minimal period solution of (HS).
2. Conley's conjecture on multiplicity of periodic sol. orbits of (HS) on T^n.
3. Multiplicity and stability of closed characteristics on compact convex hypersurfaces in $\mathbf{R}^{2n}$.
4. Seifert's conjecture on brick orbits on compact domain diffeo. to the unit ball in $\mathbf{R}^n$.
5. Multiplicity and stability of closed geodesics on Finsler manifolds.
6. Stability of periodic solutions of the N-body problems.

   ....

Y. Long, "Index Theory for Symplectic Paths with Applications". Progress in Math. 207, Birkhäuser. Basel. 2002.

Applications to the linear stability of the elliptic Lagrangian solutions of the 3-body problem

For second order Hamiltonian systems

$$\ddot{q}(t) + V'(q(t)) = 0, \quad q(\tau) = q(0), \quad \dot{q}(\tau) = \dot{q}(0), \tag{LS}$$

the corresponding functional f defined for $q \in W^{1,2}(\mathbf{R}/(\tau\mathbf{Z}), \mathbf{R}^n)$ is given by

$$f(q) = \int_0^\tau \left(\frac{1}{2}|\dot{q}(t)|^2 - V(q(t)) \right) dt.$$

Then we have

$$m^-(f, q) = i_1(\gamma_{(q,\dot{q})}), \quad m^0(f, q) = \nu_1(\gamma_{(q,\dot{q})}).$$

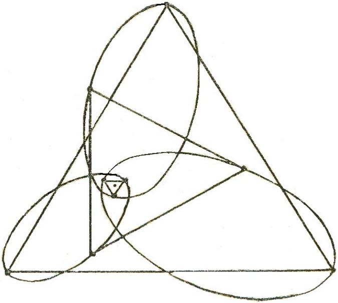
We consider the classical planar three-body problem in celestial mechanics. Denote by $q_1, q_2, q_3 \in \mathbf{R}^2$ the position vectors of three particles with masses $m = (m_1, m_2, m_3) \in (\mathbf{R}^+)^3$ respectively. By Newton's second law and the law of universal gravitation, the system of equations for this problem is

$$m_i \ddot{q}_i = \frac{\partial U(q)}{\partial q_i}, \qquad \text{for} \quad i = 1, 2, 3, \tag{1}$$

where

$$U(q) = U(q_1, q_2, q_3) = \sum_{1 \le i < j \le 3} \frac{m_i m_j}{|q_i - q_j|}$$

is the potential function by using the standard norm $|\cdot|$ of vector in $\mathbf{R}^2$.



In 1772, J. Lagrange discovered his τ-periodic elliptic solutions of the 3-BP (ELS for short): $q(t) = r(t)R(\theta(t))q(0)$, with $q(0) \in (\mathbf{R}^2)^3$, $r(t) > 0$, and $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for $\theta \in \mathbf{R}$.

Here, when $q(0)$ is not collinear, $q(0)$ and consequently $q(t)$ always form an equilateral triangle (central configuration) at every time t, and each point runs along an ellipse with the same eccentricity $e \in [0, 1)$. We denote these τ-periodic ELS by $q_{m,e}(t)$.

We write the 3-BP system (1) into a Hamiltonian system:

$$\dot{z} = JH'(z), \quad z(\tau) = z(0). \tag{2}$$

with $z = (p, q) = (p_1, p_2, p_3, q_1, q_2, q_3) \in (\mathbf{R}^2)^6$, $p(t) = \bar{M}\dot{q}(t)$, and

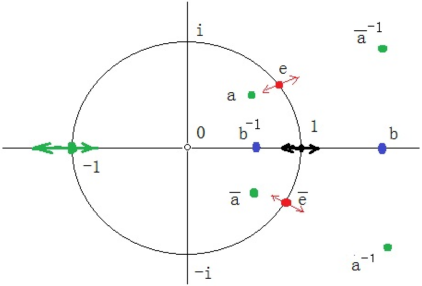
$$H(z) = H(p, q) = \sum_{i=1}^{3} \frac{|p_i|^2}{2m_i} - U(q), \quad J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix},$$

with $\bar{M} = \mathrm{diag}(m_1, m_1, m_2, m_2, m_3, m_3)$. The linearized Hamiltonian system at $z_{m,e}(t) = (\bar{M}\dot{q}_{m,e}(t), q_{m,e}(t)) \in (\mathbf{R}^2)^6$ is given by

$$\dot{y}(t) = JH''(z_{m,e}(t))y(t), \quad y(\tau) = y(0), \tag{3}$$

whose fundamental solution $\psi = \psi_{m,e}(t)$ satisfies $\psi(0) = I_{12}$ and $\psi_{m,e}(t) \in \mathrm{Sp}(12) = \{M \in \mathrm{GL}(\mathbf{R}^{12}) \,|\, M^T JM = J\}$ for all $t \in [0, \tau]$.

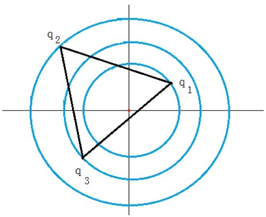
Our main concern is the linear stability of these ELS, which is determined by $\psi_{m,e}(\tau)$ and its eigenvalues. Let $\mathbf{U} = \{z \in \mathbf{C} \mid |z| = 1\}$.



Let $M \in \mathrm{Sp}(2n)$. Then possible eigenvalue distributions of M are:

$$1 \quad \text{is of even multiplicities;} \qquad -1 \quad \text{is of even multiplicities;}$$
$$e, \ \bar{e} \quad \in \quad \mathbf{U} \setminus \mathbf{R}; \qquad b, \ b^{-1} \quad \in \quad \mathbf{R} \setminus \{0, \pm 1\};$$
$$a, \ a^{-1}, \ \bar{a}, \ \bar{a}^{-1} \quad \in \quad \mathbf{C} \setminus (\mathbf{U} \cup \mathbf{R}).$$

Thus $\exists$ 3 possible ways for eigenvalues to escape from $\mathbf{U}$.



Earlier studies on the linear stability:

M.Gascheau (1843), E.Routh (1875)   for circular orbits, i.e., $e = 0$.

J.Danby (1964), G.Roberts (2003), K.Meyer-D.Schmidt (2005): for $e \geq 0$ sufficiently small, by perturbation method.

R.Martínez, A.Samà and C.Simó (2004-2006): (see below for more details)

Let $\gamma_{\beta,e}(t)$ be the fundamental solution of the essential part of the linearized H.S. at $z_{m,e}(t)$ (by the Meyer-Schmidt result quoted below).

Theorem. (X.Hu and S.Sun, 2010) *(I)* $2 \le i_1(z^2_{m,e}) \le 4$ *holds always;* *Suppose* $\gamma_{\beta,e}(2\pi)^2$ *is non-degenerate, i.e.,* $1 \notin \sigma(\gamma_{\beta,e}(2\pi)^2)$. *Then*

(II-1) If $i_1(z^2_{\beta,e}) = 4$, *then* $\gamma_{\beta,e}(2\pi) \approx R(\theta_1)\diamond R(\theta_2)$ *holds for some* θ_1 *and* $\theta_2 \in (\pi, 2\pi)$, *and ELS is linearly stable;*

(II-2) If $i_1(z^2_{\beta,e}) = 3$, *then* $\gamma_{\beta,e}(2\pi) \approx D(\lambda)\diamond R(\theta)$ *for some* $-1 \ne \lambda < 0$ *and* $\theta \in (\pi, 2\pi)$, *and ELS is linearly unstable;*

(II-3) If $i_1(z^2_{\beta,e}) = 2$ *and* $\exists k \ge 3$ *such that* $i_1(z^k_{\beta,e}) > 2(k-1)$, *then* $\gamma_{\beta,e}(2\pi) \approx R(2\pi - \theta_1)\diamond R(\theta_2)$ *holds with* $0 < \theta_1 < \theta_2 < \pi$, *and ELS is linearly stable;*

(II-4) If $i_1(z^k_{\beta,e}) = 2(k-1)$ *for all* $k \in \mathbf{N}$, *then* $\gamma_{\beta,e}(2\pi)$ *and ELS are hyperbolic or spectrally stable and linearly unstable.*

As usual, $z^k_{\beta,e}(t) = z_{\beta,e}(kt)$ is used for all $k \in \mathbf{N}$.

K.Meyer and D.Schmidt (2005): Using the central configuration coordinates, they decomposed the linearized Hamiltonian system at ELS:

$$\psi_{m,e}(\tau) = P^{-1}\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \diamond \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \diamond \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \diamond I_2 \diamond M(\beta)\right] P.$$

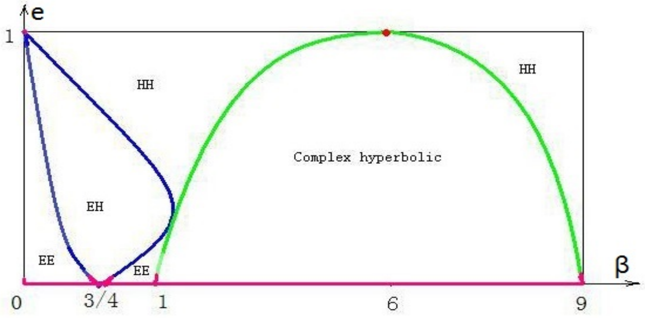
(i) the 8 eigenvalue 1 according to first integrals stays always for all $(m, e) \in (\mathbf{R}^+)^3 \times [0, 1)$;

(ii) the essential part $M(\beta)$ determines the linear stability:

$$\bar{B}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & \frac{2e\cos\theta - 1 - \sqrt{9-\beta}}{2(1+e\cos\theta)} & 0 \\ 1 & 0 & 0 & \frac{2e\cos\theta - 1 + \sqrt{9-\beta}}{2(1+e\cos\theta)} \end{pmatrix},$$

where $t \in [0, \tau]$ is transformed to the true anomaly $\theta \in [0, 2\pi]$, and

$$\beta = \frac{27(m_1 m_2 + m_1 m_3 + m_2 m_3)}{(m_1 + m_2 + m_3)^2} \in [0, 9], \quad e \in [0, 1).$$



R.Martínez, A.Samà and C.Simó (2004-2006) Perturbation method for $e \sim 0$ or $e \sim 1$ + numerical method:

EE: $\sigma(\gamma_{\beta,e}(2\pi)) = \{\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2\}$ with $\omega_i \in \mathbf{U} \setminus \mathbf{R}$ for $i = 1, 2$;

EH: $\sigma(\gamma_{\beta,e}(2\pi)) = \{\lambda, \lambda^{-1}, \omega, \overline{\omega}\}$ for some $-1 \neq \lambda < 0$ and $\omega \in \mathbf{U} \setminus \mathbf{R}$;

HH: $\sigma(\gamma_{\beta,e}(2\pi)) = \{\lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}\}$ for some $\lambda_i \in \mathbf{R} \setminus \{0, \pm 1\}$ with $i = 1, 2$;

Complex hyperbolic: $\sigma(\gamma_{\beta,e}(2\pi)) \subset \mathbf{C} \setminus (\mathbf{U} \cup \mathbf{R})$.

Difficulty: due to the substantial dependence of the coefficients on t when $0 < e < 1$:

$$\begin{aligned} \dot{y}(t) &= J\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & \frac{2e\cos(t)-1-\sqrt{9-\beta}}{2(1+e\cos(t))} & 0 \\ 1 & 0 & 0 & \frac{2e\cos(t)-1+\sqrt{9-\beta}}{2(1+e\cos(t))} \end{pmatrix} y(t), \\ y(2\pi) &= y(0). \end{aligned}$$

Denote the fundamental solution of this system by $\gamma_{\beta,e}(t) \in \mathrm{Sp}(4)$, which satisfies $\gamma_{\beta,e}(0) = I_4$. The linear stability of $z_{\beta,e} \equiv z_{m,e}(t)$ is determined by $\gamma_{\beta,e}(2\pi) \in \mathrm{Sp}(4)$.

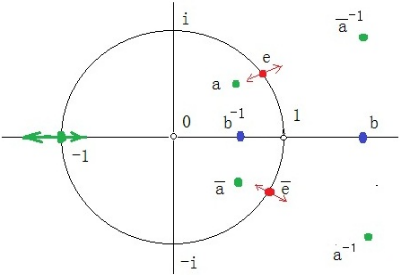
Looking for analytical method to be used for this problem.

Main results of Hu-Long-Sun (2012), ARMA(2014):

Main Theorem 1. (X.Hu-Y.Long-S.Sun) *The ELS is* 1*-nondegenerate when* $(\beta, e) \in (0, 9] \times [0, 1)$*. Specially we have*

$$i_1(\gamma_{\beta,e}) = 0 \quad \text{and} \quad \nu_1(\gamma_{\beta,e}) = \begin{cases} 3, & \text{if } \beta = 0, \\ 0, & \text{if } \beta \in (0, 9], \end{cases} \qquad e \in [0, 1).$$

Thus no eigenvalues of $\gamma_{\beta,e}(2\pi)$ can escape from **U** at 1 as $\beta > 0$!

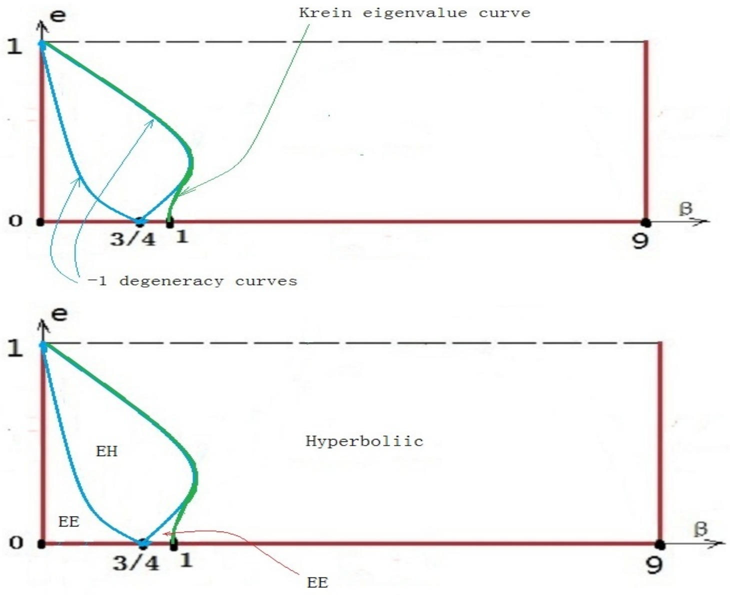


Main results of Hu-Long-Sun, 2012-14:

Main Theorem 2. (X.Hu-Y.Long-S.Sun) *In the* (β, e) *rectangle* $(0,9] \times [0,1)$ *there exist three distinct continuous curves from left to right: two* -1*-degeneracy curves* Γ_s *and* Γ_m *going up from* $(3/4, 0)$ *with tangents* $-\sqrt{33}/4$ *and* $\sqrt{33}/4$ *respectively and converges to* $(0,1)$*, and the Krein collision eigenvalue curve* Γ_k *going up from* $(1,0)$ *and converges to* $(0,1)$ *as* e *increases from* 0 *to* 1*; each of them intersects every horizontal segment* $e = \mathrm{constant} \in [0,1)$ *only once.*

Moreover the linear stability pattern of $\gamma_{\beta,e}(2\pi)$ *as well as that of the ELS* $z_{\beta,e}$ *changes if and only if* (β, e) *passes through one of these three curves* Γ_s*,* Γ_m *and* Γ_k*.*

Three separating curves and linear stability subregions



Well-known fact: For periodic solutions with period $\tau > 0$, the system is the Euler-Lagrange equation of the action functional

$$\mathcal{A}_\tau(q) = \int_0^\tau \left[\sum_{i=1}^{3} \frac{m_i |\dot{q}_i(t)|^2}{2} + U(q(t)) \right] dt$$

defined on the loop space $W^{1,2}(\mathbf{R}/\tau\mathbf{Z}, X)$, where

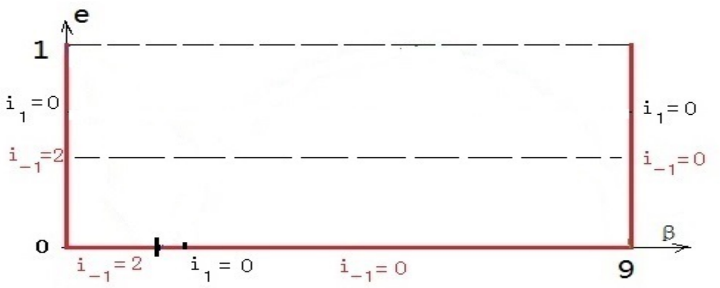
$$X \equiv \left\{ q = (q_1, q_2, q_3) \in (\mathbf{R}^2)^3 \;\middle|\; \sum_{i=1}^{3} m_i q_i = 0,\; q_i \neq q_j,\; \forall i \neq j \right\}.$$

Each τ-periodic solution of (1) appears to be a critical point of the action functional $\mathcal{A}_\tau$.

Venturelli (2001), Zhang-Zhou (2001), Viterbo (1989), An-Long (1998), Hu-Sun (2010) *ELS is a global minimizer of the action $\mathcal{A}(q)$ on the loops in the non-trivial homology class of $W^{1,2}(\mathbf{R}/\tau\mathbf{Z}, X)$. Specially for all $(\beta, e) \in [0, 9] \times [0, 1)$, its indices satisfy*

$$i_1(\gamma_{\beta,e}) = i_1(\mathrm{ELS}) = 0, \quad i_\omega(\gamma_{\beta,e}) = i_\omega(\mathrm{ELS}) \quad \forall\, \omega \in \mathbf{U} \setminus \{1\}.$$

New observations and ideas (I) Studies on the three boundary segments of $[0,9]\times[0,1)$ ($i_1(\gamma_{\beta,e})=0$ for all (β,e)):



On $\{0\}\times[0,1)$: $N_1(1,1)I_2$, $\nu_1(\gamma_{0,e})=3$, $i_{-1}(\gamma_{0,e})=2$, $\nu_{-1}(\gamma_{0,e})=0$,

On $(0,3/4]\times\{0\}$: st. elliptic, $i_{-1}(\gamma_{\beta,0})=2$, $\nu_{\pm1}(\gamma_{\beta,0})=0$,

On $(3/4,0)$: $(-I_2)R(\sqrt{3}\pi)$, $\nu_1(\gamma_{3/4,0})=0$, $i_{-1}(\gamma_{3/4,0})=0$, $\nu_{-1}(\gamma_{3/4,0})=2$.

On $(3/4,1]\times\{0\}$: st. elliptic, $\nu_{\pm1}(\gamma_{\beta,0})=i_{-1}(\gamma_{\beta,0})=0$.

On $(1,9]\times\{0\}$: CS hyperbolic, $\nu_{\pm1}(\gamma_{\beta,0})=i_{-1}(\gamma_{\beta,0})=0$.

On $\{9\}\times[0,1)$: real hyperbolic, $\nu_{\pm1}(\gamma_{\beta,0})=i_{-1}(\gamma_{\beta,0})=0$.

Let $N_1(1,1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ for $\lambda \in \mathbf{R}$. Recall that we have $i_1(\gamma_{\beta,e}) = 0$ for all $(\beta, e) \in [0,9] \times [0,1)$.

Then for any $e \in [0,1)$, $\gamma_{0,e}(2\pi) \approx I_2 \diamond N_1(1,1)$ $\Longrightarrow$

$$\begin{aligned} i_\omega(\gamma_{0,e}) &= i_1(\gamma_{0,e}) + S^+_{\gamma_{0,e}(2\pi)}(1) - S^-_{\gamma_{0,e}(2\pi)}(\omega) \\ &= i_1(\gamma_{0,e}) + S^+_{I_2}(1) + S^+_{N_1(1,1)}(1) - S^-_{I_2\diamond N_1(1,1)}(\omega) \\ &= 0 + 1 + 1 - 0 \\ &= 2, \qquad \forall\, \omega \in \mathbf{U} \setminus \{1\}. \end{aligned}$$

And for any $e \in [0,1)$, $\gamma_{9,e}(2\pi) \approx D(\lambda_1) \diamond D(\lambda_2)$ for $e \in [0,1)$ and some $\lambda_i \in\in \mathbf{R}^+ \setminus \{1\}$ $\Longrightarrow$

$$\begin{aligned} i_\omega(\gamma_{9,e}) &= i_1(\gamma_{9,e}) + S^+_{\gamma_{9,e}(2\pi)}(1) - S^-_{\gamma_{9,e}(2\pi)}(\omega) \\ &= i_1(\gamma_{9,e}) + S^+_{D(\lambda_1)}(1) + S^+_{D(\lambda_2)}(1) - S^-_{D(\lambda_1)\diamond D(\lambda_2)}(\omega) \\ &= 0 + 0 + 0 - 0 \\ &= 0, \qquad \forall\, \omega \in \mathbf{U} \setminus \{1\}. \end{aligned}$$

New observations and ideas (II) Reduction to a 2nd order OD operator.

Let

$$\xi_{\beta,e}(t) = \begin{pmatrix} R(t) & 0 \\ 0 & R(t) \end{pmatrix} \gamma_{\beta,e}(t), \;\; R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

for all $t \in [0, 2\pi]$. Then $\xi_{\beta,e}(2\pi) = \gamma_{\beta,e}(2\pi)$, $\xi_{\beta,e} \sim \gamma_{\beta,e}$, and it is the fundamental solution of:

$$\dot{y}(t) = J\overline{B}_{\beta,e}(t)y(t), \quad y(2\pi) = y(0),$$

where $$\overline{B}_{\beta,e}(t) = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 - R(t)K_{\beta,e}(t)R(t)^T \end{pmatrix},$$

$$K_{\beta,e}(t) = \begin{pmatrix} \frac{3-\sqrt{9-\beta}}{2(1+e\cos t)} & 0 \\ 0 & \frac{3+\sqrt{9-\beta}}{2(1+e\cos t)} \end{pmatrix}.$$

For $\omega \in \mathbf{U}$, $\overline{B}_{\beta,e}$ corresponds to a 2nd order self-adjoint linear operator:

$$A(\beta, e) = -\frac{d^2}{dt^2} I_2 - I_2 + R(t)K_{\beta,e}(t)R(t)^T, \qquad \text{defined on}$$

$$\overline{D}(\omega) = \{y \in W^{2,2}([0, 2\pi], \mathbf{C}^2) \mid y(2\pi) = \omega y(0), \; \dot{y}(2\pi) = \omega \dot{y}(0)\}.$$

New observations and ideas (III) Index monotonicity.

Fix $e \in [0,1)$ and $\omega \in \mathbf{U}$. On $\overline{D}(\omega)$ we have:

$$\begin{aligned} A(\beta, e) &= -\frac{d^2}{dt^2} I_2 - I_2 + R(t) K_{\beta,e}(t) R(t)^T \\ &= -\frac{d^2}{dt^2} I_2 - I_2 + \frac{1}{2(1+e\cos t)} (3I_2 + \sqrt{9-\beta} S(t)) \\ &\equiv \sqrt{9-\beta}\, \hat{A}(\beta, e), \end{aligned}$$

where for $\beta \in [0,9)$,

$$\hat{A}(\beta, e) = \frac{A(9,e)}{\sqrt{9-\beta}} + \frac{S(t)}{2(1+e\cos t)}, \quad S(t) = \begin{pmatrix} \cos 2t & \sin 2t \\ \sin 2t & -\cos 2t \end{pmatrix},$$

and $A(9,e) > 0$.

New observations and ideas (III) Index monotonicity.

Main Lemma 1. For β near β_0, the eigenvalues $\lambda(\beta)$ near $\lambda(\beta_0) = 0$ of $\hat{A}(\beta, e)$ satisfies

$$\frac{d}{d\beta}\lambda(\beta)|_{\beta=\beta_0} > 0.$$

In fact, we have

$$\lambda(\beta) = \lambda(\beta)\xi(\beta)\cdot\xi(\beta) = \hat{A}(\beta, e)\xi(\beta)\cdot\xi(\beta).$$

From $\hat{A}(\beta, e) = \frac{A(9,e)}{\sqrt{9-\beta}} + \frac{S(t)}{2(1+e\cos t)}$, differentiating w.r.t. β yields

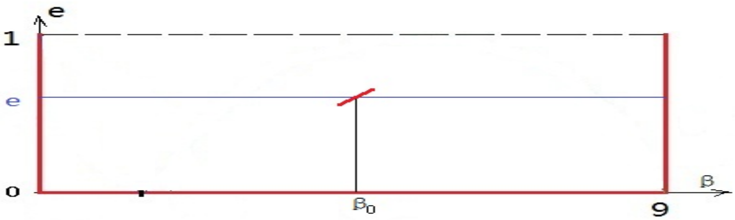
$$\begin{aligned}
\frac{d}{d\beta}\lambda(\beta)|_{\beta=\beta_0} &= (\frac{\partial}{\partial\beta}\hat{A}(\beta, e))\xi(\beta)\cdot\xi(\beta)|_{\beta=\beta_0} \\
&\quad + 2\hat{A}(\beta, e)\xi(\beta)\cdot(\frac{d}{d\beta}\xi(\beta))|_{\beta=\beta_0} \\
&= \frac{A(9, e)\xi(\beta)\cdot\xi(\beta)}{2(9-\beta)^{3/2}}|_{\beta=\beta_0} > 0.
\end{aligned}$$

Main Lemma 2. Fix $e \in [0, 1)$. For any $\omega \in \mathbf{U}$, when β increases in $(0, 9]$, the index $i_\omega(\gamma_{\beta,e})$ is non-increasing, i.e.,

$$\#\{\text{negative eigenvalues of } A(\beta, e)\} \text{ is non-increasing.}$$

Here $i_\omega(\gamma_{\beta,e}) = i_\omega(A(\beta, e)) = i_\omega(\hat{A}(\beta, e))$

$$= \#\{\text{negative eigenvalues of } \hat{A}(\beta, e)|_{\overline{D}(\omega)}\}.$$



Main new results

Main Theorem 1 (Hu-Long-Sun, 2012).

$$
i_1(\gamma_{\beta,e}) = 0, \qquad \forall\, (\beta, e) \in [0,9] \times [0,1), \quad (\textit{by minimization})
$$

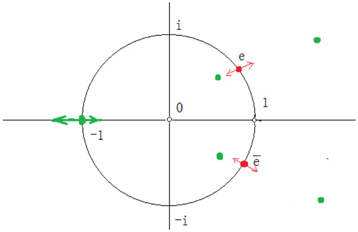
$$
\nu_1(\gamma_{\beta,e}) = \begin{cases} 3, & \text{if } \ \beta = 0, \\ 0, & \text{if } \ \beta \in (0,9], \end{cases} \qquad e \in [0,1).
$$

That is, the ELS is non-degenerate when $\beta > 0$ for all $e \in [0,1)$.

Idea of the proof. By Main Lemma 1,

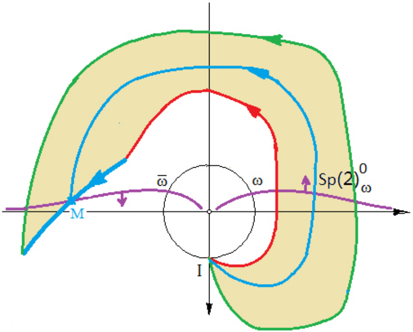
$\exists$ a "0" eigenvalue for some $\beta > 0 \Rightarrow \exists$ negative eigenvalue

$\Rightarrow i_1(\gamma_{\beta,e}) > 0$, contradiction !



Because $1 \notin \sigma(\gamma_{\beta,e}(2\pi))$ for $\beta > 0$, there are only 2 possible ways for eigenvalues to escape from $\mathbf{U}$ as shown in the Figure, i.e., from -1 or from Krein collision eigenvalues.

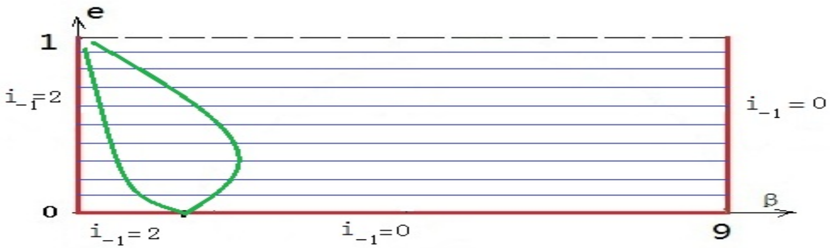
Important observation:



ω-index change implies the existence of some eigenvalue ω

$$i_\omega(\xi) - i_\omega(\gamma) \neq 0 \Rightarrow \omega \in \sigma(\gamma_{\beta,e}(2\pi))$$

for some point (β, e) on the end point curve, where $M = \gamma_{\beta,e}(2\pi)$.



Theorem 2.1 (Hu-Long-Sun). *Fix $e \in [0,1)$. the -1 index $i_{-1}(\gamma_{\beta,e})$ is non-increasing in β, and strictly decreasing precisely on two values of $\beta = \beta_1(e)$ and $\beta = \beta_2(e) \in (0,9)$, at which $-1 \in \sigma(\gamma_{\beta,e}(2\pi))$ holds. Let*

$$\beta_s(e) = \min\{\beta_1(e), \beta_2(e)\}, \quad \beta_m(e) = \max\{\beta_1(e), \beta_2(e)\},$$
$$\Gamma_s = \{(\beta_s(e), e) \mid e \in [0,1)\}, \quad \Gamma_m = \{(\beta_m(e), e) \mid e \in [0,1)\}.$$

They form the two -1-degeneracy curves in $[0,9] \times [0,1)$.

Idea of the proof. $i_{-1}(\gamma_{0,e}) = 2$ and $i_{-1}(\gamma_{9,e}) = 0$ + Main Lemma 2.

Operator theory $\Rightarrow$ smoothness of the two curves.

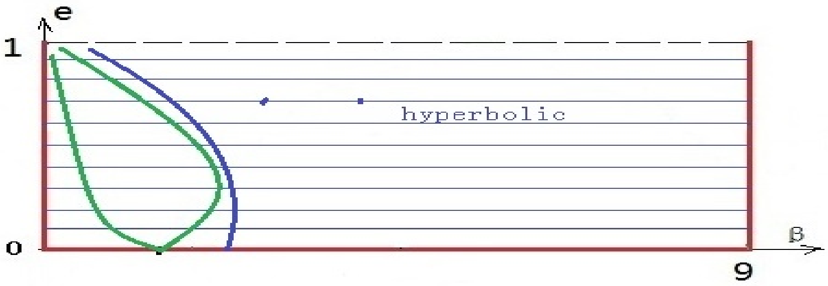
Theorem 2.2 (Hu-Long-Sun). *For every $e \in [0,1)$ we define*

$$\beta_k(e) = \inf\{\beta \in [0,9] \mid \sigma(\gamma_{\beta,e}(2\pi)) \cap \mathbf{U} = \emptyset\},$$
$$\Gamma_k = \{(\beta_k(e), e) \in [0,9] \times [0,1) \mid e \in [0,1)\}.$$

Then (i) $\beta_s(e) \le \beta_m(e) \le \beta_k(e) < 9$ holds for all $e \in [0,1)$;

(ii) Γ_k is the boundary curve of the hyperbolic region of $\gamma_{\beta,e}(2\pi)$ in the (β, e) rectangle $[0,9] \times [0,1)$;

(iii) Γ_k is continuous in $e \in [0,1)$, starts from $(1,0)$ and goes up, $\lim_{e\to 1} \beta_k(e) = 0$, and Γ_k is distinct from Γ_m.

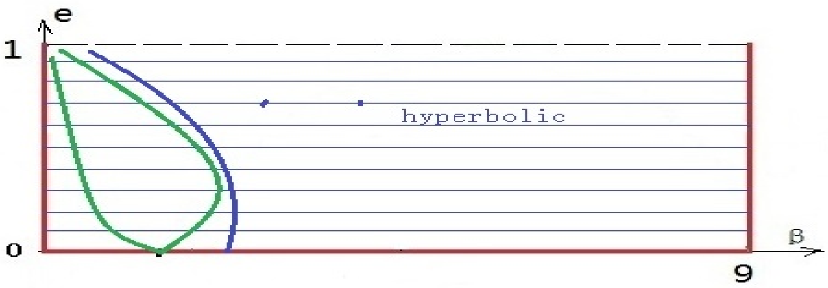


Idea of the proof. (A) $\gamma_{\beta_1,e}(2\pi)$ is hyperbolic $\Rightarrow$ $i_\omega(\gamma_{\beta_1,e}) = 0\ \forall\,\omega \in \mathbf{U}$.
Main Lemma 2 $\Rightarrow$ $i_\omega(\gamma_{\beta,e}) = 0\ \forall\,\omega \in \mathbf{U}$ and $\beta \in (\beta_k, 9]$
Main Lemma 1 $\Rightarrow$ $\nu_\omega(\gamma_{\beta,e}(2\pi)) = 0\ \forall\,\omega \in \mathbf{U}$ and $\beta \in (\beta_k, 9]$,

i.e., $\gamma_{\beta,e}(2\pi)$ is hyperbolic,

i.e., the hyperbolic subregion of $\gamma_{\beta,e}(2\pi)$ is connected. Then Γ_k is well-defined as a set and contains one point on each $\{e = \mathrm{const.}\}$.

(B) Other hard parts in the proof: to prove the continuity of Γ_k, and $\beta_k(e) \to 0$ as $e \to 1$. ∎

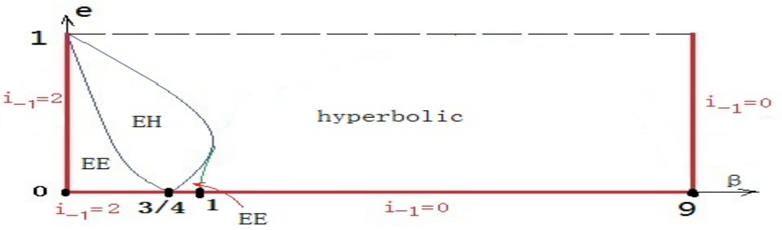


Theorem 3-(I) (Hu-Long-Sun). *Let $e \in [0,1)$. We have*

$$\text{(i)} \qquad i_{-1}(\gamma_{\beta,e}) = \begin{cases} 2, & \text{if } 0 \le \beta < \beta_s(e), \\ 1, & \text{if } \beta_s(e) \le \beta < \beta_m(e), \\ 0, & \text{if } \beta_m(e) \le \beta \le 9, \end{cases}$$

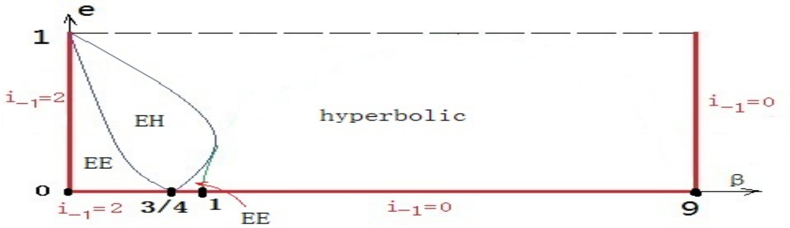
(ii) $\gamma_{\beta,e}(2\pi) \approx R(\theta_1) \diamond R(\theta_2)$ *for some* θ_1 *and* $\theta_2 \in (\pi, 2\pi)$, *and thus is strongly linearly stable, when* $0 < \beta < \beta_s(e)$;

(iii) $\gamma_{\beta,e}(2\pi) \approx D(\lambda) \diamond R(\theta))$ *for some* $0 > \lambda \neq -1$ *and* $\theta \in (\pi, 2\pi)$, *and it is hyperbolic-elliptic and thus linearly unstable, when* $\beta_s(e) < \beta < \beta_m(e)$.



Theorem 3-(II) (Hu-Long-Sun). *Let $e \in [0,1)$. We have*

(iv) $\gamma_{\beta,e}(2\pi) \approx R(\theta_1) \diamond R(\theta_2)$ for some $\theta_1 \in (0,\pi)$ and $\theta_2 \in (\pi, 2\pi)$ with $2\pi - \theta_2 < \theta_1$, and thus is strongly linearly stable, when $\beta_m(e) < \beta < \beta_k(e)$.



Theorem 4 (Hu-Long-Sun). *Let $e \in [0, 1)$.*

(i) If $\beta_s(e) < \beta_m(e)$, $\gamma_{\beta_s(e),e}(2\pi) \approx N_1(-1, 1)\diamond R(\theta)$ for some $\theta \in (\pi, 2\pi)$, and is spectrally stable and linearly unstable;

(ii) If $\beta_s(e) = \beta_m(e) < \beta_k(e)$, $\gamma_{\beta_s(e),e}(2\pi) \approx -I_2\diamond R(\theta)$ for some $\theta \in (\pi, 2\pi)$, and is inearly stable, but not strongly linearly stable;

(iii) If $\beta_s(e) < \beta_m(e) < \beta_k(e)$, $\gamma_{\beta_m(e),e}(2\pi) \approx N_1(-1, -1)\diamond R(\theta)$ for some $\theta \in (\pi, 2\pi)$, and is spectrally stable and linearly unstable;

(iv) If $\beta_s(e) \leq \beta_m(e) < \beta_k(e)$, $\gamma_{\beta_k(e),e}(2\pi) \approx N_2(e^{\sqrt{-1}\theta}, b)$ for some $\theta \in (0, \pi)$ and $(b_2 - b_3)\sin\theta > 0$, and is spectrally stable and linearly unstable;

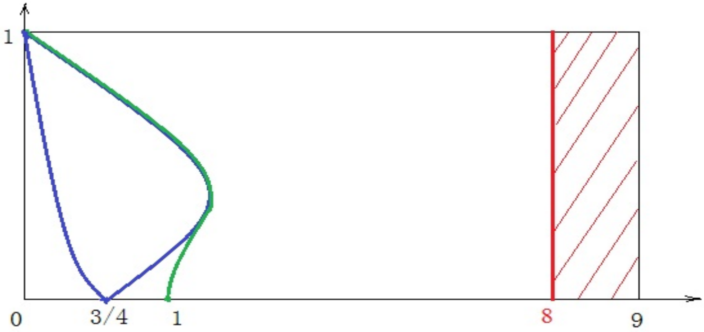
(v) If $\beta_s(e) < \beta_m(e) = \beta_k(e)$, either $\gamma_{\beta_k(e),e}(2\pi) \approx N_1(-1, 1)\diamond D(\lambda)$ for some $-1 \neq \lambda < 0$ and is linearly unstable; or $\gamma_{\beta_k(e),e}(2\pi) \approx N_2(-1, c)$ with $c_1, c_2 \in \mathbf{R}$ and $c_2 \neq 0$, and is spectrally stable and linearly unstable;

(vi) If $\beta_s(e) = \beta_m(e) = \beta_k(e)$, either $\gamma_{\beta_k(e),e}(2\pi) \approx M_2(-1, c)$ with $c_1 \in \mathbf{R}$ and $c_2 = 0$ which possesses basic normal form $N_1(-1, 1)\diamond N_1(-1, 1)$, or $\gamma_{\beta_k(e),e}(2\pi) \approx N_1(-1, 1)\diamond N_1(-1, 1)$; and thus is spectrally stable and linearly unstable.

New estimate of Yuwei Ou, 2013:

Theorem. (Y. Ou, 2013) $\gamma_{\beta,e}(2\pi)$ *is hyperbolic for all* (β, e) *in rectangle* $(8, 9] \times [0, 1)$, *i.e.,*

$$\sigma(\gamma_{\beta,e}(2\pi)) \subset \mathbf{C} \setminus \mathbf{U}, \qquad \forall\, (\beta, e) \in (8, 9] \times [0, 1).$$



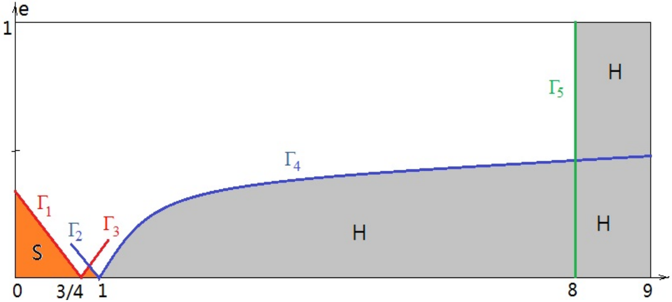
New estimate of X.Hu, Y.Ou and P.Wang, arXiv:1308.4745:

Theorem. *There exists a real function* $f(\beta,\omega)$ *such that* $\gamma_{\beta,e}(2\pi)$ *is* *linearly stable, if*

$$e < \frac{1}{1+f(\beta,-1)^{1/2}}, \qquad \text{for } \beta \in [0,3/4), \quad \text{or}$$

$$e < \min\left\{\frac{1}{\sqrt{f(\beta,-1)}}, \frac{1}{\sqrt{1+f(\beta,e^{i\sqrt{2}\pi})}}\right\}, \text{ for } \beta \in (3/4,1).$$

$\gamma_{\beta,e}(2\pi)$ *is hyperbolic, if* $e < (\sup\{f(\beta,\omega) \mid \omega \in \mathbf{U}\})^{-1/2}$,



The function $f(\beta, \omega)$ is defined via the trace function by

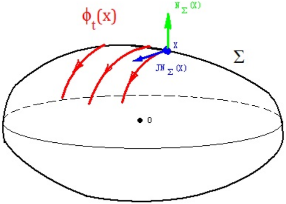
$$
\begin{aligned}
f(\beta, \omega) &= \mathrm{Tr}\left[(K_\beta^- (-J\frac{d}{dt} - \nu J - B_{\beta,0})^{-1})^2 \right] \\
&= \mathrm{Tr}\left[(K_\beta^+ (-J\frac{d}{dt} - \nu J - B_{\beta,0})^{-1})^2 \right],
\end{aligned}
$$

where $\omega = e^{2\pi\nu}$, $K_\beta^\pm = \frac{\cos(t) \pm |\cos(t)|}{2} K_\beta$,

$$
\begin{aligned}
K_\beta &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3+\sqrt{9-\beta}}{2} & 0 \\ 0 & 0 & 0 & \frac{3-\sqrt{9-\beta}}{2} \end{pmatrix}, \\
B_{\beta,0} &= \begin{pmatrix} I & -J \\ J & \hat{K}_{\beta,0} \end{pmatrix}, \quad \hat{K}_{\beta,0} = \begin{pmatrix} \frac{3+\sqrt{9-\beta}}{2} & 0 \\ 0 & \frac{3-\sqrt{9-\beta}}{2} \end{pmatrix},
\end{aligned}
$$

and $-J\frac{d}{dt} - \nu J - B_{\beta,0}$ is invertible for $\nu = i/2$ and $\nu = i/\sqrt{2}$.

Applications to the multiplicity of closed characteristics on prescribed energy hypersurfaces in $\mathbf{R}^{2n}$



$\Sigma \subset \mathbf{R}^{2n}$— a compact (strictly) convex smooth (C^3) hypersurface.

$N_\Sigma(x)$ — the outward normal vector of Σ at $x \in \Sigma$ such that

$$\langle N_\Sigma(x), v\rangle = 0, \quad \langle N_\Sigma(x), x\rangle = 1, \quad \text{for all} \;\; v \in T_x\Sigma, \;\; x \in \Sigma.$$

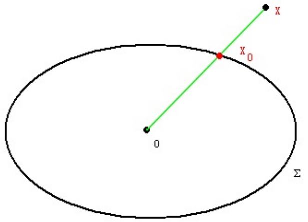
$JN_\Sigma(x)$ — a tangential vector field on Σ.

Look for solution (τ, x) (i.e., closed characteristic, τ-minimal period) of:

$$\begin{cases} \dot{x}(t) = JN_\Sigma(x(t)), \qquad x(t) \in \Sigma, \quad \forall t \in \mathbf{R}, \\ \quad x(\tau) = x(0). \end{cases}$$

Here $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ is the standard symplectic matrix on $\mathbf{R}^{2n}$.

$\mathrm{CC}(\Sigma)$–set of all geometrically distinct $(x(\mathbf{R}) \neq y(\mathbf{R}))$ closed characteristics on Σ.



$j_\Sigma(x) = \lambda(x)$ if $x = \lambda(x)x_0$ for some $x_0 \in \Sigma$ and $\lambda(x) > 0$, $j_\Sigma(0) = 0$.
Fix an α with $1 < \alpha < 2$. Define a Hamiltonian function H for Σ:

$$H(x) = j_\Sigma(x)^\alpha, \qquad \forall\, x \in \mathbf{R}^{2n}.$$

$\Longrightarrow$ $H \in C^1(\mathbf{R}^{2n}, \mathbf{R}) \cap C^3(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R})$, $\Sigma = H^{-1}(1)$,
$H'(x_0) = \lambda(x_0) N_\Sigma(x_0)$ for all $x_0 \in \Sigma$, where $\lambda(x_0)$ is smooth in $x_0 \in \Sigma$.

Periodic motions with prescribed energy of Hamiltonian systems:

$$\begin{cases} \dot{x}(t) &= JH'(x(t)),\ \forall t \in \mathbf{R}, \\ H(x(t)) &= 1, \qquad \forall t \in \mathbf{R}, \\ x(\tau) &= x(0), \end{cases}$$

Looking for (τ, x)– Closed characteristics on $\Sigma \equiv H^{-1}(1) \subset \mathbf{R}^{2n}$.

Two long standing important conjectures in Hamiltonian analysis:

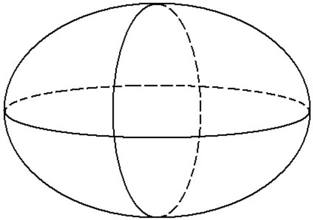
Multiplicity conjecture:

$^{\#}\mathrm{CC}(\Sigma) \geq n$, for every compact convex hypersurface $\Sigma \subset \mathbf{R}^{2n}$.

Stability conjecture:

$\exists \geq 1$ elliptic CC, for every compact convex hypersurface $\Sigma \subset \mathbf{R}^{2n}$.

Example: Weakly non-resonant ellipsoid



For $r_1, \ldots, r_n > 0$, define $\Sigma = H^{-1}(1)$, where

$$H(x) = \frac{1}{2} \sum_{j=1}^{n} \frac{x_j^2 + x_{j+n}^2}{r_j^2}, \quad \text{for all } x = (x_1, \ldots, x_{2n}) \in \mathbf{R}^{2n},$$

Then $r_i / r_j \notin \mathbf{Q}$ for all $i \neq j$ $\Longrightarrow$ $^{\#}\mathrm{CC}(\Sigma) = n$.

Known results on local multiplicity problem:

A. Liapunov (1892), J. Horn (1903)

$H : \mathbf{R}^{2n} \to \mathbf{R}$ is analytic, $\sigma(JH''(0)) = \{\pm\sqrt{-1}\omega_1, \ldots, \sqrt{-1}\omega_n\}$ are purely imaginary, and satisfy $\frac{\omega_i}{\omega_j} \notin \mathbf{Z}$ for all $i \neq j$.

$\Longrightarrow \quad {}^{\#}\mathrm{CC}(H^{-1}(\epsilon)) \geq n, \quad \forall \ 0 < \epsilon << 1.$

A. Weinstein (1973):

H is C^2 near 0 in $\mathbf{R}^{2n}$, $H''(0) > 0$

$\Longrightarrow \quad {}^{\#}\mathrm{CC}(H^{-1}(\epsilon)) \geq n, \quad \forall \ 0 < \epsilon << 1.$

J. Moser (1976), T. Bartsch (1997).

Known results on global multiplicity problem:

P. Rabinowitz, A. Weinstein (1978-79),

$^{\#}\mathrm{CC}(\Sigma) \geq 1$, for $\Sigma \subset \mathbf{R}^{2n}$ compact convex (star-shaped) hypersurface.

Results under pinching conditions

I.Ekeland-J.-M.Lasry (1980), A.Ambrosetti-G.Mancini (1981), H.Hofer (1982), M.Girardi (1984), H.Berestycki-J.M.Lasry-B.Ruf-G.Mancini (1985), Y.Dong-Y.Long (2004)

Results for free case (without pinching conditions)

I.Ekeland-L.Lassoud (1987), I.Ekeland-H.Hofer (1987), A. Szulkin (1988), ${}^{\#}\mathrm{CC}(\Sigma) \geq 2$, for compact convex hypersurface $\Sigma \subset \mathbf{R}^{2n}$.

H.Hofer-K.Wysocki-E.Zehnder (1998), ${}^{\#}\mathrm{CC}(\Sigma) = 2$ or $+\infty$, for compact convex hypersurface $\Sigma \subset \mathbf{R}^{4}$.

Y.Long-C.Zhu (2002) ($[a] = \max\{k \in \mathbf{Z} \mid k \leq a\}$ for all $a \in \mathbf{R}$)

(i) ${}^{\#}\mathrm{CC}(\Sigma) \geq [n/2] + 1$, for compact convex hypersurface $\Sigma \subset \mathbf{R}^{2n}$.

(ii) ${}^{\#}\mathrm{CC}(\Sigma) \geq n$, for compact convex hypersurface $\Sigma \subset \mathbf{R}^{2n}$, if all CCs are non-degenerate.

C.Liu-Y.Long-C.Zhu (2002)

${}^{\#}\mathrm{CC}(\Sigma) \geq n$, for compact convex hypersurface $\Sigma = -\Sigma \subset \mathbf{R}^{2n}$.

Results for free case (continued)

Wei Wang–Xijun Hu–Y. Long, (2007)

$^{\#}\mathrm{CC}(\Sigma) \geq 3$, $\forall$ convex compact smooth hypersurface $\Sigma \subset \mathbf{R}^6$.

Recent new results for free case (without pinching conditions)

Wei Wang, (2013-14)

$^{\#}\mathrm{CC}(\Sigma) \geq 4$, $\forall$ convex compact smooth hypersurface $\Sigma \subset \mathbf{R}^8$.

$^{\#}\mathrm{CC}(\Sigma) \geq \left[\frac{n+1}{2}\right]$, $\forall$ convex compact smooth hypersurface $\Sigma \subset \mathbf{R}^{2n}$.

Main difficulties toward proofs for multiplicity results

1. 1 CC $\leftrightarrow$ ∞ many critical values of f on E.

For every $m \in \mathbf{N}$, $x^m(t) = x(mt)$ satisfy $f(x^m) \to +\infty$!

2. To get ${}^{\#}\mathrm{CC}(\Sigma) \geq 2$, one needs a contradiction if ${}^{\#}\mathrm{CC}(\Sigma) = 1$.

3. To get ${}^{\#}\mathrm{CC}(\Sigma) \geq 3$, one needs certain structures.

4. It is not clear whether Hofer-Wysocki-Zehnder method works for $\mathbf{R}^{2n}$ when $n \geq 3$.

The Clarke-Ekeland dual action principal:

$$f(u) = \int_0^1 \left\{ \langle \frac{1}{2} Ju, \Pi u \rangle + H^*(-Ju) \right\} dt,$$

$$\forall\ u \in E = \left\{ v \in L^{(\alpha-1)/\alpha}(\mathbf{R}/\mathbf{Z}, \mathbf{R}^{2n}) \ \Big|\ \int_0^1 u dt = 0 \right\},$$

where $H(y) = j(y)^\alpha$ with $\alpha \in (1,2)$, $H^*(x) = \sup_{y \in \mathbf{R}^{2n}} \{\langle x, y \rangle - H(y)\}$, and Πu is defined by

$$\frac{d}{dt} \Pi u = u \quad \text{and} \quad \int_0^1 \Pi u dt = 0.$$

Let $u \in E \setminus \{0\}$ satisfy $f'(u) = 0$. Then $\exists \xi_u \in \mathbf{R}^{2n}$ s.t. $z_u(t) = \Pi u(t) + \xi_u$ is a 1-periodic solution of

$$\dot{z}(t) = JH'(z(t)), \quad z(1) = z(0).$$

Let $h = H(z_u(t))$ and $1/m$ be the minimal period of z_u for some $m \in \mathbf{N}$. Then $(\tau, x_u) \in \mathrm{CC}(\Sigma)$, where

$$\tau = \frac{1}{m} h^{(\alpha-2)/\alpha}, \qquad x_u(t) = h^{-1/\alpha} z_u(h^{(2-aa)/\alpha} t).$$

Using Fadell-Rabinowitz index theory (Lyusternic-Schnirelmann type argument), [Ekeland-Hofer, 1987]

$\Longrightarrow$ f possesses a sequence of critical values $\{c_k\}_{k\geq 1}$:

$$-\infty < c_1 = \min\{f(u) \mid u \in E\} \leq c_2 \leq \cdots \leq c_k \leq c_{k+1} \leq \cdots < 0,$$

$$^{\#}\mathrm{CC}(\Sigma) = +\infty, \qquad \text{if} \;\; c_k = c_{k+1} \;\; \text{for some} \;\; k \in \mathbf{N},$$

and for every $k \in \mathbf{N}$, $\exists(\tau, x) \in \mathrm{CC}(\Sigma)$ and $m \in \mathbf{N}$ such that

$$f'(u_m^x) = 0, \qquad f(u_m^x) = c_k,$$

$$i_1(x^m) \leq 2k - 2 + n \leq i(x^m) + \nu(x^m) - 1,$$

where $u_m^x(t) = (m\tau)^{(\alpha-1)/(2-\alpha)}\dot{x}(m\tau t)$.

Suppose $q \equiv {}^{\#}\mathrm{CC}(\Sigma) < +\infty$. Specially we observed

$$2\mathbf{N} - 2 + n \subseteq \bigcup_{j=1}^{q} \bigcup_{m \in \mathbf{N}} [i(u_m^{x_j}), i(u_m^{x_j}) + \nu(u_m^{x_j}) - 1].$$

Important property: for a fixed $(\tau, x) \in \mathrm{CC}(\Sigma)$, for every $m \in \mathbf{N}$ we have

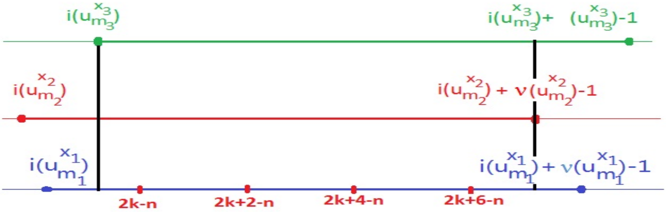
$$i(u_m^x) \ \le \ i(u_m^x) + \nu(u_m^x) - 1 < i(u_{m+1}^x) \ \le \ i(u_{m+1}^x) + \nu(u_{m+1}^x) - 1,$$

$i.e.,$

$$[i(u_m^x), i(u_m^x) + \nu(u_m^x) - 1] \cap [i(u_{m+1}^x), i(u_{m+1}^x) + \nu(u_{m+1}^x) - 1] = \emptyset.$$

Then we obtain

$$q \ \ge \ ^{\#}\left((2\mathbf{N} - 2 + n) \bigcap \bigcap_{j=1}^{q} [i(u_{m_j}^{x_j}), i(u_{m_j}^{x_j}) + \nu(u_{m_j}^{x_j}) - 1] \right) \ \equiv \ p.$$



Otherwise, $p > q \Rightarrow \exists k \in \mathbf{N}$ and j such that $c_k = f(u_{m_j}^{x_j}) = c_{k+1}$

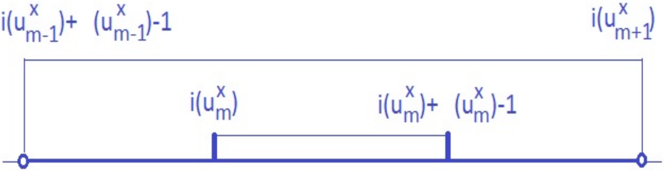
$\Rightarrow \ ^{\#}\mathrm{CC}(\Sigma) = +\infty$, contradiction !

Next observation:

We enlarge the index interval $[i(u_m^x), i(u_m^x) + \nu(u_m^x) - 1]$ to

the index jump interval

$(i(u_{m-1}^x) + \nu(u_{m-1}^x) - 1, i(u_{m+1}^x))$.



Then by the same reason of $q \geq p$, we obtain

$$q \geq {}^{\#}\left((2\mathbf{N} - 2 + n) \bigcap \bigcap_{j=1}^{q} \left(i(u_{m_j-1}^{x_j}) + \nu(u_{m_j-1}^{x_j}) - 1, i(u_{m_j+1}^{x_j})\right)\right).$$

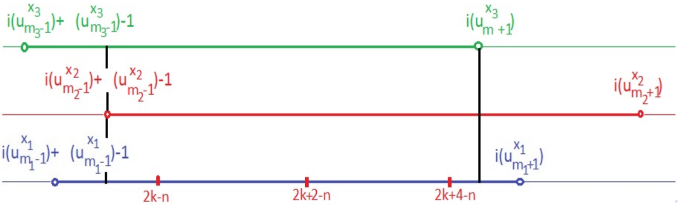
Common index jump theorem (Long-Zhu, 2002)

Suppose $\mathrm{CC}(\Sigma) = \{(\tau_j, x_j) \mid 1 \le j \le q\}$. Let

$\kappa_1 = \min_{1 \le j \le q}(i_1(x_j) + 2S^+(x_j) - \nu_1(x_j))$ and $\kappa_2 = \min_{1 \le j \le q}(i_1(x_j) - 1)$.

Then there exist $N, m_1, \ldots, m_q \in \mathbf{N}$ such that

$$\begin{aligned} q &\ge \# \left((2\mathbf{N} - 2 + n) \bigcap \bigcap_{j=1}^{q} \left(i_1(u^{x_j}_{m_j - 1}) + \nu_1(u^{x_j}_{m_j - 1}) - 1, i_1(u^{x_j}_{m_j + 1}) \right) \right) \\ &\ge \# ((2\mathbf{N} - 2 + n) \cap [2N - \kappa_1, 2N + \kappa_2]) \\ &\ge \min \left\{ \left[\frac{i_1(u^{x_j}_1) + S^+(x_j) - \nu_1(u^{x_j}_1)}{2} \right] \mid 1 \le j \le q \right\} \equiv \varrho_n(\Sigma) \\ &\ge [n/2] + 1. \end{aligned}$$



Thank you !