Arnold diffusion in the restricted planar three body problem

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Tere M. Seara[1]

(A. Delshams, V. Kaloshin, A. de la Rosa)

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The restricted planar three body problem (RP3BP)

- We consider the motion of three bodies q_1, q_2 and q, of masses m_1, m_2, and 0 under the effects of the Newtonian gravitational force. Usually one works with the mass ratio $\mu = \frac{m_2}{m_1+m_2}$, $m_1 \geq m_2$, and one considers he masses $1-\mu$, μ $(0 \leq \mu \leq 1/2)$ and 0.
- The bodies with mass (primaries) q_1, q_2 are not influenced by the massless one q.
- q_1 and q_2 form a two body problem. Therefore their motion is governed by Kepler laws.
- We will assume the two primaries q_1, q_2 move on ellipses (elliptic case): a particular case is when they move in circles (circular case)
- Goal: understand the motion of the massless body q under the influence of the other two.

The restricted planar three body problem (RP3BP)

The particle q with zero mass moves under the effects of the Newtonian gravitational force exerted by the two primaries q_1 and q_2 of masses $1-\mu$ and μ evolving in elliptic orbits around their center of mass. The circular case is a particular case of the elliptic one, where the primaries move in elliptic orbits.

Typical models in the elliptic case with eccentricity e_0:

- Sun–Jupiter–asteroid or comet: $e_0 = 0.048$
- Sun–Earth–Moon systems: $e_0 = 0.016$

The equations of the RP3BP

- The motion of the massless particle $q = (q^1, q^2) \in \mathbb{R}^2$ (planar problem) is described by Newton laws. After normalizing:

$$\frac{d^2q}{dt^2} = \frac{(1-\mu)(q_1(t)-q)}{||q_1(t)-q||^3} + \frac{\mu(q_2(t)-q)}{||q_2(t)-q||^3},$$

  where $q_1(t)$, $q_2(t)$ are the position of the primaries, which move in an elliptic orbit of excentricity e_0.
- This is a 2π-periodic in time Hamiltonian system (2 and 1/2 degrees of freedom) with Hamiltonian

$$\mathcal{H}(q,p,t;\mu,e_0) = \frac{p^2}{2} - \frac{(1-\mu)}{|q-q_1(t)|} - \frac{\mu}{|q-q_2(t)|}.$$

  $p = (p^1, p^2) = \frac{dq}{dt}$, $q = (q^1, q^2)$.
- Parameters: $0 < e_0 < 1$ the excentricity of the ellipse ($q_1(t)$ and $q_2(t)$ depend on e_0) and $\mu \in [0, 1/2]$.

The equations of the RP3BP

- In the elliptic case $e_0 > 0$, one has:

$$q_1(t) = -\mu r_0(t) q_0(t), \quad q_2(t) = (1-\mu) r_0(t) q_0(t)$$

where

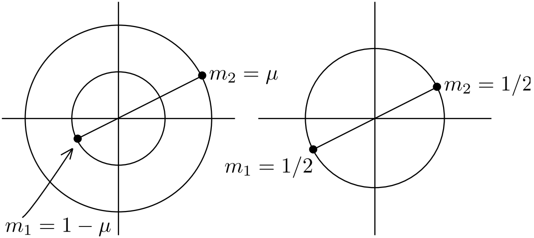
$$r_0 = r_0(t; e_0) = \frac{1 - e_0^2}{1 + e_0 \cos f(t)}, \qquad \frac{df}{dt} = \frac{(1 + e_0 \cos f)^2}{(1 - e_0^2)^{3/2}}.$$

and $f(t) = f(t; e_0)$ is the true anomaly, and

$$q_0(t) = (\cos f(t), \sin f(t))$$

- In the circular case $e_0 = 0$, one has: $q_1(t) = -\mu q_0(t)$, $q_2(t) = (1-\mu) q_0(t)$ and $q_0(t) = (\cos t, \sin t)$ correspond to the circular motion of the primaries.

Motion of the primaries in the circular case: $e_0 = 0$



- In general, the Hamiltonian is 2π-periodic in time, therefore it is NOT a first integral of the problem.
- The RPC3BP has a first integral called Jacobi constant

$$\mathcal{J}(q, p, t; \mu) = \mathcal{H}(q, p, t; \mu, 0) - (q_1 p_2 - q_2 p_1).$$

Circular RP3BP in rotating polar coordinates

When $e_0 = 0$, we can make some classical changes of variables to simplify the Hamiltonian:

- Fix the primaries at the x axis taking a reference system which moves periodically with time (periodic in time change of variables): sinodic coordinates

$$q_1 = (\mu, 0), \quad q_2 = (1 - \mu, 0).$$

- Polar coordinates for the third body: $q = (r \cos \phi, r \sin \phi)$.
  - y symplectic conjugate to r (radial velocity).
  - G symplectic conjugate to ϕ (angular momentum).
- We get an autonomous Hamiltonian of two degrees of freedom:

$$H(r, \phi, y, G; \mu) = \frac{y^2}{2} + \frac{G^2}{2r^2} - G - U(r, \phi; \mu),$$

- $U(r, \phi; \mu)$ is the Newtonian potential, which satisfies $U(r, \phi; \mu) \simeq \frac{1}{r}$.
- H is a first integral (Is the Jacobi constant $\mathcal{J}$).

Elliptic RP3BP in rotating polar coordinates

If we perform the same changes of variables in the elliptic case (remember that there is no an extra first integral in this case):
We get an non-autonomous Hamiltonian:

$$H(r,\phi,y,G,t;\mu,e_0)=\frac{y^2}{2}+\frac{G^2}{2r^2}-G-U(r,\phi,t;\mu,e_0),$$

$U(r,\phi,t;\mu,e_0)$ also satisfies $U(r,\phi,t;\mu,e_0)\simeq\frac{1}{r}$.
The system has two and a half degrees of freedom.
We will work in the extended phase space:
$((r,\phi,y,G,s)\in\mathbb{R}\times\mathbb{T}\times\mathbb{R}^2\times\mathbb{T}$

$$H(r,\phi,y,G,s;\mu,e_0)=\frac{y^2}{2}+\frac{G^2}{2r^2}-G-U(r,\phi,s;\mu,e_0),$$

and $\dot{s}=1$

Limiting case $\mu \to 0$

For $\mu = 0$ and for any e_0:

- The massless body q is only influenced by one body q_1 (q_2 has also zero mass!).
- Its motion is governed by Kepler laws (the central force problem).
- It moves on conic sections.

Limiting case $\mu \to 0$

For $\mu = 0$ and for any e_0:

- As $U(r, \phi, s; 0, e_0) = \frac{1}{r}$, the Hamiltonian for $\mu = 0$, becomes, both in the elliptic and circular case:

$$H(r, \phi, y, G, s; 0, e_0) = H_0(r, y, G) - G = \frac{y^2}{2} + \frac{G^2}{2r^2} - \frac{1}{r} - G,$$

  $h = H_0$ is the energy.

- Possible types of motion:
  - $H^{\pm}$ (hyperbolic): motion on hyperbolas: $h > 0$
  - $P^{\pm}$ (parabolic): motion on parabolas: $h = 0$
  - $B^{\pm}$ (bounded): motion on ellipses $h < 0$
- The angular momentum G is preserved.

Purposes for $\mu > 0$

We want to see:

- There is another possible type of motion: Oscillatory motion :

$$\limsup_{t\to\pm\infty} \|q\| = +\infty \quad \text{and} \quad \liminf_{t\to\pm\infty} \|q\| < +\infty$$

  Proved for any $0 < \mu \leq 1/2$ (Guardia-Martin-S) and $e_0 = 0$ in:
  Oscillatory motions for the restricted planar circular three body problem
  Preprint at
  *http : //arxiv.org/abs/*1207.6531
  Future project $e_0 > 0$.
- For $e_0 > 0$, the angular momentum G can have changes of $O(1)$ even if μ is very small: **Arnold diffusion**

$\mu > 0$, The elliptic case: Increassing the angular momentum

Final goal: in the elliptic restricted three body (ERTBP) problem we want to see that the angular momentum of the third body $G(t)$ can have *large changes*

We have partial results when the eccentricity $e_0 > 0$ and $\mu > 0$ are small enough:

Given any $G_1, G_2 \gg 1$, there exist heteroclinic trajectories of the ERTBP whose angular momentum satisfies, for some $T > 0$:

$$G(0) < G_1 \qquad G(T) > G_2$$

Proven for $0 < \mu \ll e_0 \ll 1$ and any $1 \ll G_1, G_2 \leq 1/e_0$.

Previous results for oscillatory motions or diffusion close to parabolic orbits

- Sitnikov 1960 (later Moser) considered the restricted spatial elliptic three body problem with a specific configuration.
- Llibre-Simó 1980 (oscillatory motions in the RPC3BP for $0 < \mu \ll 1$)
- Moeckel 1984: extended the result of Sitnikov to the case of three bodies with positive masses, two of them equal, in an isosceles configuration.
- Xia 1992 (for RPC3BP oscillatory motions for every $\mu \in (0, 1/2]$ except a finite number of values)
- Galante-Kaloshin 2011(orbits initially bounded and which become oscillatory: $\mu = 10^{-3}$, realistic for the Jupiter-Sun)
- Kaloshin and Gorodetski 2011 (results about the Hausdorff dimension of oscillatory motions for both the Sitnikov problem and the RPC3BP)
- Xia 1993 (local diffusion in the ERTBP)
- Martínez-Pinyol 1994 (Massive computations in the ERTBP)

Previous results: other types of oscillatory motions or diffusion:

- Llibre-Martínez-Simó 1985 (oscillatory motions close to L_2 in the CRTBP)
- Bolotin 2006 (close to collision in the ERTBP)
- Capiñski-Zgliczyñski 2011 (close to L_2 in the ERTBP)
- Féjoz-Guàrdia-Kaloshin-Roldán 2012 (close to resonances in the ERTBP)

Limiting case $\mu \to 0$: Infinity

- Equations

$$
\begin{aligned}
\dot{r} &= y \\
\dot{y} &= \frac{G^2}{r^3} - \frac{1}{r^2} \\
\dot{\phi} &= -1 + \frac{G}{r^2} \\
\dot{G} &= 0 \\
\dot{s} &= 1
\end{aligned}
$$

- For any value of (ϕ_0, G_0, s_0), the “infinity”:

$$(r, y, \phi, G, s) = (\infty, 0, \phi_0 - t, G_0, s_0 + t), t \in \mathbb{T}$$

  is a periodic solution.

- At infinity, H coincides with angular momentum: $H(\infty, \phi, 0, G_0, s; 0, e_0) = -G_0$.

Limiting case $\mu \to 0$: McGehee coordinates

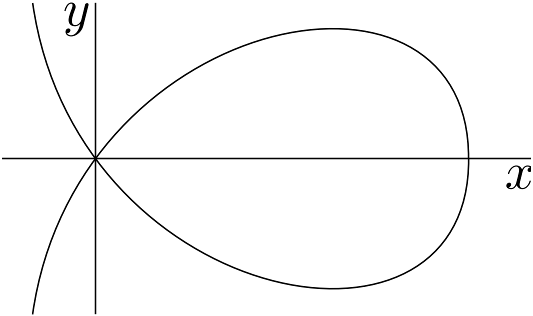
$x^2 := 1/r$: Gives a better geometrical understanding of the problem

- $$\begin{aligned} \dot{x} &= -\frac{x^3}{2}\frac{\partial K_0}{\partial y} \\ \dot{y} &= \frac{x^3}{2}\frac{\partial K_0}{\partial x} \\ \dot{\phi} &= \frac{\partial K_0}{\partial G} - 1 \\ \dot{G} &= 0 \\ \dot{s} &= 1 \end{aligned}$$

- $K_0(x, y, G) = H_0(\frac{1}{x^2}, y, G) = \frac{y^2}{2} + \frac{G^2x^4}{2} - x^2$, is a first intergral.
- For any value of (ϕ_0, G_0, s_0):
  $\Lambda_{\phi_0,G_0,s_0} = \{(x, y, \phi, G, s) = (0, 0, \phi_0 - t, G_0, s_0 + t), t \in \mathbb{T}\}$ is a periodic solution.

Limiting case $\mu \to 0$: McGehee coordinates

Homoclinic manifold: For any fixed G_0, in the (x, y) plane; $(0, 0)$ is a (parabolic) critical point which has the separatrix loop $\gamma_{G_0} = \{K_0(x, y, G_0) = 0\}$.



In the whole extended phase space this will give rise to an homolinic manifold $\gamma_{\phi_0, G_0, s_0}$ to the periodic orbit $\Lambda_{\phi_0, G_0, s_0}$

Limiting case $\mu \to 0$: A priori unstable structure, inner dynamics

Main features we will use:

- The 3 dimensional manifold:

$$\Lambda_\infty = \{x = y = 0,\ (\phi, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\}$$

  is invariant.
- $\Lambda_\infty = \bigcup_{\phi,G,s} \Lambda_{\phi,G,s}$, being $\Lambda_{\phi,G,s}$ periodic orbits.
- The inner dynamics on Λ_∞ is trivial:

$$(\phi, G, s) \to (\phi - t, G, s + t)$$

- Λ_∞ has stable and unstable manifolds.

Limiting case $\mu \to 0$: A priori unstable structure, outer dynamics

An invariant 4-dimensional homoclinic manifold to Λ_∞.

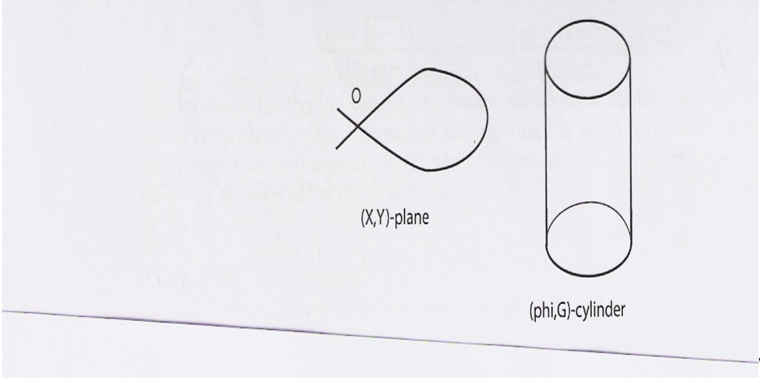
$$\begin{aligned} \gamma &= W_0^s(\Lambda_\infty) = W_0^u(\Lambda_\infty) \\ &= \{K_0(x, y, G) = 0,\ (\phi, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\} \end{aligned}$$

This makes Λ_∞ a normally parabolic invariant manifold

γ can be seen as a union of parabolic homoclinic orbits to $\Lambda_{\phi,G,s}$.

$$\gamma = \bigcup_{(\phi,G,s)} \gamma_{\phi,G,s}$$

Limiting case $\mu \to 0$: A priori unstable structure, outer dynamcis



We can parameterize the homoclinic manifold as:

$$\gamma = \{z := (x_G(\tau), y_G(\tau), \phi_G(\tau) + \phi, G, s), \tau \in \mathbb{R}, G \in \mathbb{R}_+, (\phi, s) \in \mathbb{T}^2\}$$

Limiting case $\mu \to 0$: A priori unstable structure, outer dynamics

We can define the scattering map (Delshams-Llave-S. 2000) in Λ_∞. Its is associated to the homoclinic manifold γ

$$S_0 : \Lambda_\infty \to \Lambda_\infty$$

by $z_+ = S_0(z_-)$ iff $\exists z \in \gamma$ such that

$$\mathrm{d}(\varphi(t; z), \varphi(t; z_\pm)) \to 0 \text{ as } t \to \pm\infty.$$

The orbit through z is a heteroclinic connection between the orbits through $z_\pm \in \Lambda_\infty$.

Limiting case $\mu \to 0$: A priori unstable structure, outer dynamics

Using the point of $z \in \Gamma$ given by:

$$z = (x_G(\tau), y_G(\tau), \phi_G(\tau) + \phi, G, s)$$

one can compute S_0 in coordinates:

$$S_0(\phi, G, s) = (\phi, G, s)$$

As $S_0 = Id$, the unperturbed periodic orbits $\Lambda_{\phi,G,s}$ only have homoclinic connections.

Dynamics of infinity for $\mu > 0$

In McGehee variables (x, y, ϕ, G, s), the Hamiltonian is:

$$K(x, y, \phi, G, s; \mu, e_0) = \frac{y^2}{2} + \frac{G^2 x^4}{2} - V(x, \phi, s; \mu, e_0)$$

with $V(x, \phi, s; \mu, e_0) = x^2 \tilde{V}(x, \phi, s; \mu, e_0)$

Implications:

- $\Lambda_\infty = \{x = y = 0,\ (\phi, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\}$ is still invariant.
- The periodic orbits $\Lambda_{\phi,G,s}$ persist.
- The inner dynamics on Λ_∞ is still trivial:

$$(\phi, G, s) \to (\phi - t, G, s + t)$$

$\mu > 0$: The elliptic case: increasing the angular momentum

Main goal:

- For $\mu > 0$ (and $e_0 > 0$) we want to see that we that the stable and unstable manifolds of Λ_∞ intersect transversaly.
- For $\mu > 0$ (and $e_0 > 0$) we want to see that we that the unstable manifold of the periodic orbits $\Lambda_{\phi_1, G_1, s}$ intersect transversaly the stable manifold of other periodic orbit $\Lambda_{\phi_2, G_2, s}$ with $G_2 > G_1$.

Equivalently:

For $\mu > 0$ (and $e_0 > 0$) we want to see that we can define a scattering map in Λ_∞ such that the image of one periodic orbit intersects other periodic orbits with larger angular momentum G:

$$S_\mu(\Lambda_{\phi_1, G_1, s}) \cap \Lambda_{\phi_2, G_2, s} \neq \emptyset$$

with $G_2 > G_1$.

Then we will have heteroclinic connections between periodic orbits.

$\mu > 0$, The elliptic case: increasing the angular momentum

Remember the Hamiltonian in the elliptic case is NOT autonomous:

$$H(r, \phi, y, G, s; \mu, e_0) = \frac{y^2}{2} + \frac{G^2}{2r^2} - G - U(r, \phi, s; \mu, e_0),$$

The phase space is 5 dimensional.

For $\mu > 0$ (and $e_0 > 0$) it is possible to have heteroclinic connections between periodic orbits with different angular momentum.

$\mu > 0$, The circular case $e_0 = 0$

In the circular case the Hamiltonian is autonomous:

$$H(r, \phi, y, G; \mu, 0) = \frac{y^2}{2} + \frac{G^2}{2r^2} - G - U(r, \phi; \mu, 0)$$

The energy of $\Lambda_{\phi,G,s}$ is $H = -G$.
As the energy is preserved heteroclinic orbits between periodic orbits with different angular momentum are not possible!
Arnold diffusion is not possible but the transversal intersection between the invariant manifolds og Λ_∞ will give rise to another important phenomenum: The existence of oscillatory motions.

The invariant manifolds of Λ_∞ for $e_0 > 0$, $\mu > 0$: Melnikov approach

For $\mu > 0$, we want to see that the manifolds $W^s_\mu(\Lambda_\infty)$ and $W^u_\mu(\Lambda_\infty)$ intersect transversally.
This result is based on a Melnikov type computation (see A. de la Rosa's talk)
Classical Melnikov potential:

$$\mathcal{L}(\phi, G, s; e_0) = \int_{-\infty}^{\infty} \overline{\Delta V}(x_G(t), \phi_G(t) + \phi, s + t; e_0)\, dt.$$

where $V(x, \phi, s; \mu, e_0) = x^2 + \mu\overline{\Delta V}(x, \phi, s; e_0) + O(\mu^2)$
Intersection property: If the function

$$\tau \mapsto \mathcal{L}(\phi, G, s - \tau; e_0)$$

has a *non-degenerate critical point* $\tau^*(\phi, G, s; e_0)$, then there is a transversal intersection between $W^u(\tilde{\Lambda}_\infty)$ and $W^s(\tilde{\Lambda}_\infty)$ close to $\tilde{z}_0 = (x_G(\tau^*), y_G(\tau^*), \phi_G(\tau^*) + \phi, G, s)$.

The invariant manifolds of $\tilde{\Lambda}_\infty$ for $e_0 > 0$, $\mu > 0$: the reduced Poincaré function

For any fixed (ϕ, G, e_0), we just need to find a non-degenerate critical point $s^*(\phi, G; e_0)$ of $s \mapsto \mathcal{L}(\phi, G, s; e_0)$, that is, a solution $s^*(\phi, G; e_0)$ of the equation

$$\frac{\partial \mathcal{L}}{\partial s}(\phi, G, s; e_0) = 0, \quad \frac{\partial^2 \mathcal{L}}{\partial s^2}(\phi, G, s; e_0) \neq 0$$

and we recover $\tau^*(\phi, G, s; e_0) = s - s^*(\phi, G; e_0)$
Once we have $\tau^*(\phi, G, s; e_0)$ we can consider the *reduced Poincaré function*

$$\mathcal{L}^*(\phi, G; e_0) = \mathcal{L}(\phi, G, s - \tau^*(\phi, G, 0; e_0); e_0) = \mathcal{L}(\phi, G, s^*(\phi, G; e_0); e_0)$$

The scattering map for $e_0 > 0$, $\mu > 0$

The scattering map S given by the homoclinic intersection associated to the critical point $s^*(\phi, G; e_0)$ is given as:

$$(\phi, G, s) \mapsto (\phi - \mu \frac{\partial \mathcal{L}^*}{\partial G}(\phi, G; e_0) + O(\mu^2), G + \mu \frac{\partial \mathcal{L}^*}{\partial \phi}(\phi, G; e_0) + O(\mu^2), s)$$

S is given, up to first order in μ, as the time $-\mu$ Hamiltonian flow of the autonomous Hamiltonian $\mathcal{L}^*(\phi, G; e_0)$!
Then, looking at the level curves of $\mathcal{L}^*(\phi, G; e_0)$ we get the images under the scattering map.

The scattering map for $e_0 > 0$, $\mu > 0$

The inner dynamics in $\tilde{\Lambda}_\infty$ is trivial:

$$(\phi, G, s) \mapsto (\phi, G, s + t)$$

The classical geometric mechanism to obtain diffusion does not work: there is no possibility of combining the inner and the outer dynamics to obtain large changes of G.

The time 2π-Poincaré map $P(\phi, G, s) = (\phi, G, s)$, therefore $S \circ P = S$

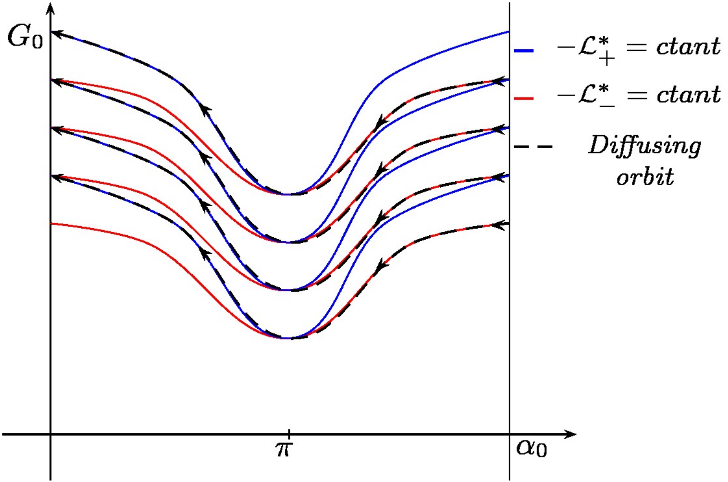
Only with one scattering map we cannot get large changes in G.

Combining two scattering maps for $e_0 > 0$, $\mu > 0$

One can see that the function $\mathcal{L}(\phi, G, s; e_0)$ has two non-degenerate critical points $s^*_\pm(\phi, G; e_0)$ which give rise to two different reduced Poincaré functions $\mathcal{L}^*_\pm$. The *scattering maps* $S_\pm$ are given by

$$(\phi, G) \mapsto \left(\phi - \mu \frac{\partial \mathcal{L}^*_\pm}{\partial G}(\phi, G; e_0) + O(\mu^2), G + \mu \frac{\partial \mathcal{L}^*_\pm}{\partial \phi}(\phi, G; e_0) + O(\mu^2)\right).$$

- $S_\pm$ are given, except for $O(\mu^2)$, as the time μ Hamiltonian flow of the autonomous Hamiltonians $-\mathcal{L}^*_\pm(\phi, G)$.
- The iterates under $S_\pm$ follow closely the level curves of $\mathcal{L}^*_\pm$.
- One can see that $\{\mathcal{L}^*_+, \mathcal{L}^*_-\}$ only vanishes on $\phi = 0, \pi$, therefore, we can choose alternatively $S_\pm$ to get diffusing pseudo-orbits and get diffusion along $1 \ll G \leq 1/e_0$.


G_0
$-\mathcal{L}_+^* = ctant$
$-\mathcal{L}_-^* = ctant$
Diffusing orbit
π
α_0

The foliations of their level curves are transversal.

We can construct heteroclinic chains of periodic orbits with increasing angular momentum choosing the right scattering map any time

Computation of the Melnikov potential $\mathcal{L}$ for $e_0 G \ll 1$ and big G

Computation of the Melnikov potential is delicate.

We have rigourous computations and bounds of the errors for $e_0 G \leq 1$.

Main idea:

- $\mathcal{L}$ is periodic in s and ϕ.
- The k-th Fourier coefficient in the angle s is of order $O(e^{-k\frac{G^3}{3}})$. This is difficult to prove.
- One needs to compute the asymptotic of the first Fourier coefficients and bound the rest.

Arnold diffusion: $e_0 > 0$, $\mu > 0$

- We have rigourous results for the existence of heteroclinic orbits with increasing angular momentum if $e_0 G \leq 1$ and $\mu e^{\frac{G^3}{3}} << \ll 1$
- A rigourous λ-lemma is needed to get true orbits.

How can improve the range of the parameters with the same results?

- A priori chaotic: In a recent work (Guardia-Martin-S) we have proved that $W^u(\tilde{\Lambda}_\infty)$ and $W^s(\tilde{\Lambda}_\infty)$ intersect transversally for $e_0 = 0$. Then, the circular restricted theree body problem becomes a priori chaotic for any value of μ, and we get results for $|e_0 e^{\frac{G^3}{3}}| << 1$

Arnold diffusion: $e_0 > 0$, any $\mu > 0$

One can see that this problem is a perturbation of the two body problem without assuming μ small, nor e_0 small.

Take ε small and perform the following changes of variables

$$r = \frac{1}{\varepsilon^2}\widetilde{r}, \quad y = \varepsilon\widetilde{y}, \quad \alpha = \widetilde{\alpha} \quad \text{and} \quad G = \frac{1}{\varepsilon}\widetilde{G}$$

and we rescale time as

$$t = \frac{1}{\varepsilon^3}s.$$

The rescaled system is Hamiltonian with respect

$$\widetilde{H}\left(\widetilde{r}, \widetilde{y}, \alpha, \widetilde{G}, \frac{s}{\varepsilon^3}; \mu, e_0\right) = \frac{\widetilde{y}^2}{2} + \frac{\widetilde{G}^2}{2\widetilde{r}^2} - \widetilde{V}\left(\widetilde{r}, \alpha, \frac{s}{\varepsilon^3}; \varepsilon, e_0, \mu\right),$$

The equations in scaled variables for small ε

where

$$\widetilde{V}(\widetilde{r},\alpha,\tfrac{s}{\varepsilon^3};\varepsilon,e_0,\mu) = \frac{1-\mu}{(\widetilde{r}^2-2(\mu\varepsilon^2)\widetilde{r}\cos\alpha+(\mu\varepsilon^2)^2)^{1/2}} + \frac{\mu}{(\widetilde{r}^2+2((1-\mu)\varepsilon^2)\widetilde{r}\cos\alpha+((1-\mu)\varepsilon^2)^2)^{1/2}}.$$

where $\alpha = \phi + f(t_0 + \frac{s}{\varepsilon^3}; e_0)$.

Note that, for any μ, and e_0, $\widetilde{V} = \frac{1}{\widetilde{r}} + O(\varepsilon^2)$ and its dependence on time is through $\phi = \phi + f(t_0 + \frac{s}{\varepsilon^3}; e_0)$,

In this way one can see:

- The exponentially small splitting comes from the fact that the restricted three body problem is a small and fast perturbation of the two body problem for ε small and any e_0 and μ.
- One can expect the diffusion phenomenon if we are able to deal with these exponentially small phenomena (done for $e_0 = 0$).
- The first step will be the case $e_0 G$ small without assumptions in μ